PROJECTION FORMULA

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1. **Preliminaries**

Throughout, $f\colon X \to Y$ is a finite-type map of noetherian schemes. Recall that f_* is a right adjoint to f^*, with f_* regarded as a functor from $\mathscr{O}_X$-modules to $\mathscr{O}_Y$-modules and f^* regarded as a functor from $\mathscr{O}_Y$-modules to $\mathscr{O}_X$-modules. Thus for $\mathscr{F}$ an $\mathscr{O}_X$-module and $\mathscr{G}$ an $\mathscr{O}_Y$-module we have an isomorphism

$$\mathrm{Hom}_{\mathscr{O}_X}(f^*\mathscr{G},\, \mathscr{F}) \xrightarrow{\ \sim\ } \mathrm{Hom}_{\mathscr{O}_Y}(\mathscr{G},\, f_*\mathscr{F}) \tag{1}$$

which is bifunctorial in $\mathscr{F}$ and $\mathscr{G}$.

If you wish, you may restrict yourself to quasi-coherent sheaves on X and Y, but the general case is no more difficult than the quasi-coherent case. However, the latter has the advantage that when $X = \mathrm{Spec}\, B$ and $Y = \mathrm{Spec}\, A$, then this adjointess reduces to a special case of the familiar Hom-$\otimes$ adjointness. Indeed, via this adjointness we have:

$$\mathrm{Hom}_B(M \otimes_A B, N) \xrightarrow{\ \sim\ } \mathrm{Hom}_A(M, \mathrm{Hom}_B(B,\, N)) = \mathrm{Hom}_A(M,\, N)$$

for A-modules M and B-modules N.

Note that one deduces natural maps

$$\mathscr{G} \longrightarrow f_*f^*\mathscr{G} \tag{2}$$

and

$$f^*f_*\mathscr{F} \longrightarrow \mathscr{F}. \tag{3}$$

For (2), set $\mathscr{F} = f^*\mathscr{G}$ in (1) and consider the image of the identity map on $f^*\mathscr{G}$. For (3), set $\mathscr{G} = f_*\mathscr{F}$ in (1) and consider the element on the left side of (1) corresponding the the identity map on the right side.

2. **Basics for projection formula**

We are trying to prove that if $\mathscr{F}$ is quasi-coherent on X and $\mathscr{V}$ is locally free on Y (of finite rank, of course), then for $i \geq 0$ we have a natural isomorphism

$$\mathbf{R}^i f_*\mathscr{F} \otimes_{\mathscr{O}_Y} \mathscr{V} \xrightarrow{\ \sim\ } \mathbf{R}^i f_*(\mathscr{F} \otimes_{\mathscr{O}_X} f^*\mathscr{V}). \tag{4_i}$$

(See [H, Chap III, p. 253, Exercise 8.3].) If X and Y are affine, and f is given by the ring map $A \to B$, then we can drop the assumption that $\mathscr{V}$ is locally free. In fact (4_i) is trivially an isomorphism if $i > 0$ since both sides are zero, and for $i = 0$, then are talking about the well-known identification

$$M \otimes_A N = M \otimes_B (B \otimes_A N) \tag{$*$}$$

for a B-module M and an A-module N.

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In the general case too, it pays to consider the case when $\mathscr{V}$ is not locally free. In fact if $\mathscr{G}$ is quasi-coherent (or just simply a $\mathscr{O}_Y$-module) we have a comparison map which is bifunctorial in $\mathscr{F}$ and $\mathscr{G}$

$$\theta_Y(\mathscr{F},\mathscr{G})\colon f_*\mathscr{F}\otimes_{\mathscr{O}_Y}\mathscr{G}\longrightarrow f_*(\mathscr{F}\otimes_{\mathscr{O}_X}f^*\mathscr{G}) \tag{5}$$

defined as the adjoint to the map

$$f^*(f_*\mathscr{F}\otimes_{\mathscr{O}_Y}\mathscr{G}) = f^*f_*\mathscr{F}\otimes_{\mathscr{O}_X}f^*\mathscr{G}\xrightarrow{(3)\otimes f^*\mathscr{G}}\mathscr{F}\otimes_{\mathscr{O}_X}f^*\mathscr{G}.$$

It is not hard to show that for an open subscheme $U\subset Y$, $\theta_X|_U=\theta_U$. In greater detail, if $f'\colon f^{-1}U\to U$ is the "restriction"[1] of f to $f^{-1}U$ then the following diagram commutes:

$$\begin{array}{ccc}
(f_*\mathscr{F}\otimes_{\mathscr{O}_Y}\mathscr{G})|_U & \xrightarrow{\theta_Y|_U} & (f_*(\mathscr{F}\otimes_{\mathscr{O}_X}f^*\mathscr{G}))|_U \\
\Vert & & \Vert \\
f'_*(\mathscr{F}|_{f^{-1}(U)})\otimes_{\mathscr{O}_U}\mathscr{G}|_U & \xrightarrow[\theta_U]{} & \mathscr{F}|_{f^{-1}(U)}\otimes_{\mathscr{O}_{f^{-1}(U)}}(f')^*(\mathscr{G}|_U)
\end{array}$$

It is clear that if $\mathscr{G}=\mathscr{O}_Y$, then θ_Y is an isomorphism. It follows that it is an isomorphism when $\mathscr{G}$ is a trivial vector bundle. Now suppose $\mathscr{G}=\mathscr{V}$, our locally free $\mathscr{O}_Y$-module. Then we can find an open cover $\mathscr{U}$ of Y such that $\mathscr{V}$ is trivial on each member of $\mathscr{U}$. Since θ_Y is local on Y, we see that $(\theta_Y)(\mathscr{F},\mathscr{V})|_U$ is an isomorphism for every $U\in\mathscr{U}$. Hence (4_0) is true.

More generally (again with $\mathscr{V}$ locally free), if $\mathscr{F}\otimes\mathscr{I}^\bullet$ is an injective resolution of $\mathscr{F}$ (in the category of quasi-coherent sheaves), then $\mathscr{I}^\bullet\otimes f^*\mathscr{V}$ is an injective resolution of $\mathscr{F}\otimes f^*\mathscr{V}$. From what we have proved, we have

$$f_*(\mathscr{I}^\bullet\otimes_{\mathscr{O}_X}f^*\mathscr{V})\xrightarrow{\sim}f_*(\mathscr{I}^\bullet)\otimes_{\mathscr{O}_Y}\mathscr{V}.$$

We thus get (using the fact—in the third line below—that $(-)\otimes_{\mathscr{O}_Y}\mathscr{V}$ is exact)

$$\begin{aligned}
\mathbf{R}^if_*(\mathscr{F}\otimes f^*\mathscr{V}) &= \mathrm{H}^i(f_*(\mathscr{I}^\bullet\otimes_{\mathscr{O}_X}f^*\mathscr{V}))\\
&\xrightarrow{\sim}\mathrm{H}^i(f_*(\mathscr{I}^\bullet)\otimes_{\mathscr{O}_Y}\mathscr{V})\\
&= \mathrm{H}^i(f_*\mathscr{I}^\bullet)\otimes_{\mathscr{O}_Y}\mathscr{V}\\
&= \mathbf{R}^if_*\mathscr{F}\otimes\mathscr{V}.
\end{aligned}$$

2.1. **An important variant.** Suppose we know that there is an integer $d\geq 0$ such that

$$\mathbf{R}^if_*\mathscr{F}=0,\qquad i\geq d.$$

This happens, for example, when f is *proper* and all fibres have dimension less than or equal to d. In this case we have an isomorphism

$$\mathbf{R}^df_*\mathscr{F}\otimes_{\mathscr{O}_Y}\mathscr{G}\xrightarrow{\sim}\mathbf{R}^df_*(\mathscr{F}\otimes_{\mathscr{O}_X}f^*\mathscr{G})$$

with $\mathscr{G}$ quasi-coherent, and not merely locally free. Without getting into too many details, the map

$$\mathbf{R}^df_*\mathscr{F}\otimes_{\mathscr{O}_Y}\mathscr{G}\longrightarrow\mathbf{R}^df_*(\mathscr{F}\otimes_{\mathscr{O}_X}f^*\mathscr{G}) \tag{5}$$

can be defined even when $\mathscr{G}$ is not locally free. (Details are left to the reader, but do take a look at the Remarks below for an idea.) This map is local on Y. So we may shrink Y if necessary and assume that it is affine. Since direct images and

[1] or as the pedants would have it, the "base change" of f under the open immersion $U\hookrightarrow Y$

tensor products commute with direct limits, and since every quasi-coherent sheaf on an affine scheme is the direct limit of coherent schemes, we can assume $\mathscr{G}$ is coherent. Since Y is affine, we then have a presentation

$$\mathscr{O}_Y^m \longrightarrow \mathscr{O}_Y^n \longrightarrow \mathscr{G} \to 0$$

of $\mathscr{G}$. Apply the 3-lemma to the commutative diagram with exact rows where T is the right exact functor $\mathbf{R}^d f_*$

$$\begin{array}{ccccccc}
T(\mathscr{F}) \otimes \mathscr{O}_Y^m & \longrightarrow & T(\mathscr{F}) \otimes \mathscr{O}_Y^n & \longrightarrow & T(\mathscr{F}) \otimes \mathscr{G} & \longrightarrow & 0 \\
\simeq \downarrow & & \simeq \downarrow & & \downarrow (5) & & \\
T(\mathscr{F} \otimes f^*\mathscr{O}_Y^m) & \longrightarrow & T(\mathscr{F} \otimes f^*\mathscr{O}_Y^n) & \longrightarrow & T(\mathscr{F} \otimes f^*\mathscr{G}) & \longrightarrow & 0
\end{array}$$

Remarks 2.1.1. 1) If f is an affine map, then with $d = 0$ we have $\mathbf{R}^i f_*\mathscr{F} = 0$ for $i > d$ ($\mathscr{F}$ quasi-coherent). For this d we are essentially back to the identification $(*)$ we had earlier since we can cover Y by affine open sets, and clearly the various isomorphisms induced by $(*)$ glue. Recall also that if $\mathbf{R}^i f_*\mathscr{F} = 0$ for $i > d$ and for every quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$, then f is necessarily affine. Important example: closed immersions are affine maps.

2) Here is an alternate way of getting at the maps (4_i), which has the advantage that one does not need $\mathscr{V}$ to be locally free. We can no longer assert that (4_i) is an isomorphism however (except when $i = d$ and d is as in the discussion above). The idea is this. Using Čech cohomology one can get a cup-product like map

$$\mathbf{R}^p f_*\mathscr{A} \otimes_{\mathscr{O}_Y} \mathbf{R}^q f_*\mathscr{B} \longrightarrow \mathbf{R}^{p+q} f_*(\mathscr{A} \otimes_{\mathscr{O}_X} \mathscr{B}) \qquad (\mathscr{A}, \mathscr{B}\ \mathscr{O}_X\text{-modules}).$$

For $\mathscr{F}$ quasi-coherent on X and $\mathscr{G}$ quasi-coherent on Y, there results a composite

$$(6_i) \qquad \mathbf{R}^i f_*\mathscr{F} \otimes_{\mathscr{O}_Y} \mathscr{G} \xrightarrow{\mathbf{R}^i f_*(\mathscr{F}) \otimes (2)} \mathbf{R}^i f_*\mathscr{F} \otimes_{\mathscr{O}_Y} f_*f^*\mathscr{G} \longrightarrow \mathbf{R}^i f_*(\mathscr{F} \otimes_{\mathscr{O}_X} f^*\mathscr{G})$$

where the second arrow is from the cup-product like map mentioned. When $\mathscr{G}$ is locally free, this agrees with (4_i). I did not wish to veer off towards cup-products, and hence restricted myself to $\mathscr{V}$ locally free in the definition of (4_i). However, the alternate and more general definition in this Remark does get used when $i = d$, the index of the "top" direct image. For d such as this, we have shown that (6_d) (which is the same as (5)) is an isomorphism when $\mathscr{G}$ is *quasi-coherent*.

3) Please see [EGA-III, Chapitre 0, §§ 12.2, pp. 57–58] especially (12.2.2.1), (12.2.2.2) and (12.2.2.3) for more on cup products and other matters that came up in the remark above. Be warned that EGA does all of this for locally ringed spaces, and for $\mathscr{O}_Z$-modules, where Z is one of X or Y. However the statement that (6_d) is an isomorphism is not proven there (!!) and that seems to require noetherian schemes since the notion of quasi-coherence is defined only for schemes, and it is not clear that quasi-coherents are direct limits of coherents in the non-noetherian case (the definition in [H] for quasi-coherence is incorrect when the underlying scheme is not noetherian).

References

[EGA-III] A Grothendieck and J. Dieudonne, *Élements de Géométrie Algébrique* **III**, Publications Math. IHES **11**, Paris, 1961

[H] R. Hartshorne, *Algebraic Geometry*, Graduate Texts in Mathematics **52**, Springer, New York, 1977.