SELF-INTERSECTION VIA NOETHER NORMALISATION

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We give a very simple geometric proof of Noether Normalisation. The proof is less than a page (see Section 2). Using that we give another way of understanding $\int_V c_1(L)^d$ for a line bundle L on a d-dimensional projective variety V, which avoids Bertini's theorem (see Theorem 3.1.1). After that we discuss, essentially as an appendix, properties of étale maps, and prove the ones we used.

In what follows k is an algebraically closed field. A variety is a finite type separated reduced irreducible scheme over k. The symbol $\mathbb{P}^n$ will denote projective space of dimension n over k. The function field of a variety V is denoted $k(V)$. If V is a variety and v a point of V, then $\kappa(v)$ denotes the *residue field at* v, i.e., the residue field of the local ring $\mathscr{O}_{V,v}$. If v is the generic point of V then note that $\kappa(v) = k(V)$.

1. Tangent spaces as actual spaces in projective space

1.1. Let V first be an affine variety of dimension d, say V is a closed subvariety of $\mathbb{A}^N$ (whose co-ordinate ring is $k[X_1, \ldots, X_N]$) and let $v \in V$ be a closed non-singular point of V. Recall that if I is the ideal of V in $k[X_1, \ldots, X_N]$ then the tangent space of V at v has traditionally been regarded as a linear subvariety of $\mathbb{A}^N$, given by $A + v$ where A defined by the equations:

$$\sum_{i=1}^{N} \frac{\partial f}{\partial X_i}(v) X_i = 0, \qquad f \in I. \tag{1.1.1}$$

Inspite of appearances, the above is a finite set of conditions, for the f can be made to vary over any finite set of generators of I to give A. In other words, if $f_1, \ldots, f_r$ are generators of I, then our tangent space T_v of V at v is given by

$$\sum_{i=1}^{N} \frac{\partial f_j}{\partial X_i}(v) X_i = \sum_{i=1}^{N} \frac{\partial f_j}{\partial X_i}(v) X_i(v), \qquad j = 1, \ldots, r. \tag{1.1.2}$$

Smoothness (non-singularity) of V at v ensures that T_v so defined is of dimension d, for the rank of the matrix $(\frac{\partial f_j}{\partial X_i}(v))_{ij}$ is then $N - d$.

Next if V is projective, say $V \hookrightarrow \mathbb{P}^N$, and v is a non-singular point of V, then picking a hyperplane H in $\mathbb{P}^N$ which does not contain v and removing H and we are back to the situation of the last paragraph. Thus we have a linear space T_v in $\mathbb{P}^N \smallsetminus H = \mathbb{A}^N$. This can be completed within $\mathbb{P}^N$ to give a linear subspace $P_v \cong \mathbb{P}^d$ of $\mathbb{P}^N$. We call P_v the projective tangent space at v of V in $\mathbb{P}^N$.

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2. Projective Noether Normalisation

2.1. The main result is that a projective variety of dimension d has a finite map to $\mathbb{P}^d$. There are some bells and whistles we add which are useful for the intersection theory part later.

Theorem 2.1.1. *Let V be a projective variety over k with* $\dim V = d$ *and let $v \in V$ a closed non-singular point of V. Then there exists a finite map*

$$h\colon V \to \mathbb{P}^d$$

which is étale at v.

Remark: A map is *finite* if it is proper and all its fibres are finite. As for the notion of *étale*, for our purposes the following definition of étale suffices for most situations. A map of varieties $f\colon V \to W$ is *étale* at a closed point $v \in V$ if the induced map of complete local rings (with $w = f(v)$)

$$\widehat{\mathscr{O}}_{W,w} \to \widehat{\mathscr{O}}_{V,v}$$

is an isomorphism. If $v \in V$ is non-singular, this is *equivalent* to saying that the resulting map of tangent spaces $T_v \to T_w$ is an isomorphism. The étale locus of f is open in V. More precisely the set of closed points on which f is étale together with all the generalisations of these points is an open subset of V. For a more detailed discussion and some proofs see Section 4 below.

Proof. The case $d = 0$ is trivial. So we assume $d \geq 1$. Pick an embedding $V \hookrightarrow \mathbb{P}^N$ of V. One checks that most linear subvarieties of $\mathbb{P}^N$ (i.e., intersections of hyperplanes in $\mathbb{P}^N$) of dimension $N - d - 1$ do not meet V. Pick one, call it L, which does not meet the (projective) tangent space P of V at v either. Check that such an L exists. Note that L can be identified with $\mathbb{P}^{N-d-1}$ and P with $\mathbb{P}^d$.

If S is a linear subspace of $\mathbb{P}^N$ of dimension $N-d$ containing L then every positive dimensional subvariety of S must meet L. Indeed, since $S = \mathbb{P}^{N-d}$ every positive dimensional subvariety W of it must meet the very ample divisor L, otherwise, one has $\mathscr{O}_S(L)|_W$ is very ample as well as trivial, contradicting the fact that $\dim W > 0$.

Now pick a closed point $x \in \mathbb{P}^N \smallsetminus L$. There is exactly one linear subvariety S_x of $\mathbb{P}^N$ of dimension $N - d$ which contains x and L (namely the union of all lines passing through x which intersect L). Since S_x contains L it cannot contain P from our argument in the previous paragraph (since $\dim P = d \geq 1$ and $P \cap L = \emptyset$). It follows that P meets S_x in exactly one point which we denote $g(x)$. It is well-known and easy to see that the map

$$\begin{aligned} \mathbb{P}^N \smallsetminus L &\to P \\ x &\mapsto g(x) \end{aligned}$$

is a map of varieties. In fact, if we take a hyperplane H containing L then on the affine space $\mathbb{A}^N = \mathbb{P}^N \smallsetminus H$, the map g can be identified with the usual projection $\mathbb{A}^N \to \mathbb{A}^d$. Varying H, we can cover $\mathbb{P}^N \smallsetminus L$. We set $h = g|_V$. More pedantically, the map

$$h\colon V \to P$$

is defined as the composite $V \hookrightarrow \mathbb{P}^N \smallsetminus L \xrightarrow{g} P$.

Clearly for $p \in P$, $g^{-1}(p) = S_p \smallsetminus L$. Thus $h^{-1}(p) = S_p \cap V$, since V does not meet L. Now $h^{-1}(p)$ does not meet L (since V does not) and hence $h^{-1}(p)$ is finite from the argument we gave above (namely that every positive dimensional subvariety of

S_p must meet L). A map between projective varieties is always proper, and hence if it has finite fibres it must be a finite map. Thus h is a finite map. We make the identification $P = \mathbb{P}^d$. It is clear that the tangent space of $v \in V$ (namely an affine part of P) is isomorphic to the tangent space of P at $h(v) = v$, which again is an affine part of P, under the map induced by h. Hence h is étale at v. □

Remarks 2.1.2. 1) The affine case (done in commutative algebra classes) is an easy consequence. Indeed, if Y is an affine variety of dimension d, it can be embedded in $\mathbb{A}^N$ for suitable N. Since k is perfect (it is algebraically closed!), the smooth locus of Y is non-empty. Pick a closed point $v \in Y$ which is non-singular (i.e., v is a smooth point of Y). Let V be the projective completion of Y in $\mathbb{P}^N$. We have a finite map $h\colon V \to \mathbb{P}^d = P$ from the above proof, where P is the projective tangent space to $v \in V$. The hyperplane at infinity H_∞ in $\mathbb{P}^N$ intersects $P = \mathbb{P}^d$ in a hyperplane of $\mathbb{P}^d$ and it is clear that for the map $g\colon \mathbb{P}^N \smallsetminus L \to P$, fibres of g over $H_\infty \cap P$ lie in $H_\infty \smallsetminus L$. In particular $h^{-1}(P \smallsetminus H_\infty) = Y$. Note $P \smallsetminus H_\infty = \mathbb{A}^d$. Thus we get a finite map

$$g\colon Y \to \mathbb{A}^d$$

which is étale at v. The last bit (g being étale at v) is a bonus which you don't see in most commutative algebra books.

2) The map h in the theorem is necessarily surjective since it is a finite map between varieties of the same dimension. In greater detail since h is finite $\dim h(V) = \dim V$. On the other $h(V)$ is closed in $\mathbb{P}^d$. Since $\dim h(V) = d = \dim \mathbb{P}^d$ we conclude that $h(V) = \mathbb{P}^d$.

3. The integer $\int_V c_1(L)^{\dim(V)}$ when L is ample

3.1. Suppose V is a projective variety of dimension d and L is an ample line bundle on it. We wish to show that $\int_V c_1(L)^d$ is a positive integer. As was pointed out in class, we may, without loss of generality assume L is very ample. Suppose $\dim \mathrm{H}^d(V, L) = N + 1$. Then L induces an embedding $V \hookrightarrow \mathbb{P}^N$ and the restriction of $\mathscr{O}_{\mathbb{P}^N}(1)$ to V is L. Let D be an *effective* Cartier divisor on V in the linear system determined by L. In plain terms, let $D = H \cap V$ where H is a hyperplane in $\mathbb{P}^N$, where the scheme structure on D is determined by its ideal sheaf $\mathscr{I}_D = \mathscr{I}_V + \mathscr{I}_H$ in $\mathscr{O}_{\mathbb{P}^N}$, where $\mathscr{I}_V$ and $\mathscr{I}_H$ are the ideal sheaves of V and H. Note that $\int_V D^d = \int_V D_1 \cdot \ldots \cdot D_d$ where D_i are effective Cartier divisors each of them hyperplane sections of hyperplanes in $\mathbb{P}^N$. By Noether normalisation we have a finite map, generically étale (since étale is an open condition)

$$h\colon V \to \mathbb{P}^d$$

where the target space $\mathbb{P}^d$ can in fact be regarded as a subspace of $\mathbb{P}^N$. In fact, from the proof of Noether normalisation, if $p \in \mathbb{P}^d$ then $h^{-1}(p) = S_p \cap V$ where S_p is a linear sub-variety of $\mathbb{P}^N$ of dimension $N - d$. Now the étale locus U of h is open (see Remark after the statement of Theorem 2.1.1 above or Subsection 4.3 below). Let Z be the complement of U. Since h is a proper map therefore $F = h(Z)$ is closed in $\mathbb{P}^d$. Now $\dim Z < d$ and hence $\dim F < d$. Let W be the complement of F in $\mathbb{P}^d$. It is non-empty (for $\dim F < d$) and

$$h^{-1}(W) \xrightarrow{\text{via } h} W$$

is finite and étale. Let $p \in W$ be any point. Since h is étale on each point of $h^{-1}(p)$, each of these points is a smooth (i.e., non-singular) point of V (look at completions at the various local rings and the definition of étale). On the other hand, using the notation in the proof of Noether normalisation, $h^{-1}(p) = S_p \cap V$ where S_p is a linear subspace of $\mathbb{P}^N$ of dimension $N-d$. Basic projective geometry tells us that $S_p = H_1 \cap \cdots \cap H_d$ where the H_i are hyperplanes in $\mathbb{P}^N$. If $D_i = H_i \cap V$ for $i = 1, \ldots, d$ then

$$h^{-1}(p) = D_1 \cap \cdots \cap D_d.$$

In particular the intersection $D_1 \cap \cdots \cap D_d$ is a finite collection of smooth points of V. Thus $\int_V c_1(L)^d = \#(D_1 \cap \cdots \cap D_d) > 0$. We point out that we have already seen in 2) of Remarks 2.1.2 that h is surjective, so these fibres are non-empty.

In summary, we have the following (and where we remind ourselves that the scheme theoretic intersection of two or more closed subschemes is the one whose ideal sheaf is the sum of the various ideal sheaves of the closed subschemes and that the empty scheme is not zero-dimensional):

Theorem 3.1.1. *Let L be very ample on a d-dimensional projective variety V. Then we can find effective Cartier divisors $D_1, \ldots, D_d$ in the linear system determined by L such that the scheme theoretic intersection $D_1 \cap \cdots \cap D_d$ is smooth, zero-dimensional, and lies in the smooth locus of V. In particular $\int_V c_1(L)^d > 0$ for L ample, since it is so for L very ample.*

4. More on étale maps and tangent spaces

4.1. Intrinsic and extrinsic tangent spaces. Suppose $Y \hookrightarrow \mathbb{A}^N$ is a closed immersion with Y a d-dimensional variety and v a closed non-singular point of Y. Let $\mathfrak{M}$ be the maximal ideal of $k[\mathbf{X}] = k[X_1, \ldots, X_N]$ which defines the point v. The *cotangent space* of $\mathbb{A}^N$ at v is (as is well-known) defined to be $\mathfrak{M}/\mathfrak{M}^2$. For $f \in k[\mathbf{X}]$ we write $\mathrm{d}f|_v$, or simply $\mathrm{d}f$ if it is understood that our calculations are at v, for the image of $f - f(v)$ in $\mathfrak{M}/\mathfrak{M}^2$. We know from Taylor's expansion that

$$f = f(v) + \sum_{i=1}^{N} \frac{\partial f}{\partial X_i}(v)(X_i - X_i(v)) + \Phi$$

where $\Phi \in \mathfrak{M}^2$. It follows that $\mathrm{d}f = \sum_{i=1}^N \frac{\partial f}{\partial X_i}(v)\mathrm{d}X_i$.

Suppose I is the ideal of $k[\mathbf{X}]$ which defines Y. Let $\mathfrak{m}$ be the maximal ideal of the co-ordinate ring $A = k[\mathbf{X}]/I$ of Y. As is well-known the cotangent space of Y at v is then $\mathfrak{m}/\mathfrak{m}^2$. For $\varphi \in A$ let $\delta\varphi|_v$, or simply $\delta\varphi$ when it is clear we are working at v, be the image of $\varphi - \varphi(v)$ in $\mathfrak{m}/\mathfrak{m}^2$. For $f \in k[\mathbf{X}]$ write $\bar{f}$ for its image in A. Then we have a map (clearly surjective)

$$\mathfrak{M}/\mathfrak{M}^2 \longrightarrow \mathfrak{m}/\mathfrak{m}^2 \qquad (4.1.1)$$

given by $\mathrm{d}f \mapsto \delta\bar{f}$. If $f \in I$ then $\bar{f} = 0$ whence $\delta\bar{f} = 0$. Thus the subspace $\mathscr{N}^*$ of $\mathfrak{M}/\mathfrak{M}^2$ given by $\mathscr{N}^* = \{\mathrm{d}f \mid f \in I\}$ lies in the kernel of the surjective map (4.1.1). It is in fact exactly the kernel as the following argument shows. First suppose $\mathbf{f} = (f_1, \ldots, f_r)$ generates the ideal I. If $f = \sum_{i=1}^r a_i f_i$, with $a_i \in k[\mathbf{X}]$, then as $f_i(v) = 0$ we have

$$\mathrm{d}f = \sum_{i=1}^{r} a_i(v)\mathrm{d}f_i + \sum_{i=1}^{r} f_i(v)\mathrm{d}a_i = \sum_{i=1}^{r} a_i(v)\mathrm{d}f_i.$$

Hence $\mathscr{N}^*$ is spanned by the vectors $\mathrm{d}f_j = \sum_{i=1}^N \frac{\partial f_j}{\partial X_i}(v)\mathrm{d}X_i$, $j = 1, \ldots, r$. Since v is a non-singular point of Y, the rank of the matrix $(\frac{\partial f_j}{\partial X_i}(v))_{i,j}$ is $N - d$ (see [H,

p. 31]). Moreover $\{\mathrm{d}X_i\}_{i=1}^N$ is a basis for $\mathfrak{M}/\mathfrak{M}^2$ (as is easy to see). Thus

$$\dim \mathscr{N}^* = N - d.$$

On the other hand, since $\mathscr{O}_{Y,v} = A_\mathfrak{m}$ is a regular local ring of Krull dimension d [H, p. 32, Theorem 5.1], $\dim_k \mathfrak{m}/\mathfrak{m}^2 = d$. In fact, if $t_1, \ldots, t_d$ is a system of parameters for $A_\mathfrak{m}$, and regarding t_i as elements of A (by shrinking Y around v if necessary), then $\{\delta t_i\}_{i=1}^d$, is a basis of $\mathfrak{m}/\mathfrak{m}^2$. It follows that $\mathscr{N}^*$ is indeed the kernel of (4.1.1). We thus have an exact sequence of finite dimensional k-vector spaces

$$0 \longrightarrow N^* \longrightarrow \mathfrak{M}/\mathfrak{M}^2 \longrightarrow \mathfrak{m}/\mathfrak{m}^2 \longrightarrow 0.$$

Writing $\mathscr{N}$ for the dual of $\mathscr{N}^*$ we get an exact sequence

$$0 \longrightarrow T \longrightarrow \widehat{T} \longrightarrow \mathscr{N} \longrightarrow 0$$

where T and $\widehat{T}$ are the (intrinsic) tangent spaces of Y and $\mathbb{P}^N$ at v respectively. Let $\frac{\partial}{\partial X_i}$, (or more precisely $\frac{\partial}{\partial X_i}|_v$), $i = 1, \ldots, N$ be the basis of $\widehat{T}$ dual to the basis $\mathrm{d}X_j$, $j = 1, \ldots, N$. From the description of $\mathscr{N}^*$ and $\mathscr{N}$, it is clear that

$$T = \{\sum_{i=1}^N a_i \frac{\partial}{\partial X_i} \mid a_i \in k \text{ and } \sum_{i=1}^N a_i \frac{\partial f_j}{\partial X_i}(v) = 0 \text{ for } j = 1, \ldots, r\}.$$

In other words T is determined by

$$\mathbf{a} = \begin{pmatrix} a_1 \\ \vdots \\ a_N \end{pmatrix} \in k^N$$

such that $\mathbf{a}$ satisfies (1.1.1). This gives the connection with the embedded tangent space we discussed in Section 1.

4.2. Étale maps and tangent spaces. Suppose $h\colon V \to W$ is étale at $v \in V$ and v is a smooth closed point of V. According to the definition we have given, this means the completions of the local rings at v and w are the same. If $\mathfrak{m}$ and $\mathfrak{n}$ are the maximal ideals of local rings at v and w respetively, it follows that $\mathfrak{n}/\mathfrak{n}^2 \to \mathfrak{m}/\mathfrak{m}^2$ is an isomorphism (we are dealing with two isomorphic power series rings, where the analytic variables correspond to each other under the isomorphism). This means the map at the level of tangent spaces is an isomorphism. Conversely, since we are dealing with regular local rings, if the tangent spaces are isomorphic under the natural map, then so are the co-tangent spaces $\mathfrak{n}/\mathfrak{n}^2$ and $\mathfrak{m}/\mathfrak{m}^2$. If $\mathbf{t} = (t_1, \ldots, t_d)$ is a regular system of parameters for $\mathscr{O}_{W,w}$, then the basis $t_i + \mathfrak{n}^2$, $i = 1, \ldots, d$ of $\mathfrak{n}/\mathfrak{n}^2$ maps to a basis of $\mathfrak{m}/\mathfrak{m}^2$. It follows that the images ts_i of the t_i are a regular system of parameters for $\mathscr{O}_{V,v}$, whence the map $k|t_1, \ldots, t_d|] = \widehat{\mathscr{O}}_{W,w} \to \widehat{\mathscr{O}}_{V,v} = k[|s_1, \ldots, s_d|]$ is an isomorphism.

4.3. Étale locus is open. Let $g\colon Y \to \mathbb{A}^d$ be the affine Noether Normalisation discussed in 1) of Remarks 2.1.2. From the proof there, we have a commutative diagram

$$\begin{array}{ccc} Y & \hookrightarrow & \mathbb{A}^N \\ & {\scriptstyle g}\searrow & \downarrow {\scriptstyle \text{natural projection}} \\ & & \mathbb{A}^d \end{array}$$

with the hooked arrow being a closed immersion. In terms of co-ordinate rings, the downward vertical arrow is the one induced by the natural inclusion $k[X_1, \ldots, X_d] \to k[X_1, \ldots, X_d, \ldots, X_N]$. Since $\mathbb{A}^d$ is smooth over k, the étale locus U of g is in the smooth locus of Y. Indeed if $v \in U$, and $p = g(v)$, then $\widehat{\mathscr{O}}_{U,v} = \widehat{\mathscr{O}}_{\mathbb{A}^d,p}$ and the latter is a regular local ring. With this v and p let T and $\widehat{T}$ be as in Subsection 4.1, and let T' be the tangent space of $\mathbb{A}^d$ at p. The natural map $T \to T'$ is seen to be the composite

$$T \hookrightarrow \widehat{T} \xrightarrow{\pi} T'$$

where π is the map which $\mathrm{d}X_i \mapsto \mathrm{d}X_i$ for $i = 1, \ldots, d$ and $\mathrm{d}X_i \mapsto 0$ for $i > d$. The image of T in T' is clearly the linear span of the vectors $\sum i = 1^d \frac{\partial f_j}{\partial X_i}(v)$, $j = 1, \ldots, r$. Thus $T \to T'$ is an isomorphism if and only if

$$\mathrm{rk} \begin{pmatrix} \frac{\partial f_1}{\partial X_1}(v) & \cdots & \frac{\partial f_1}{\partial X_d}(v) \\ \vdots & & \vdots \\ \frac{\partial f_r}{\partial X_1}(v) & \cdots & \frac{\partial f_r}{\partial X_d}(v) \end{pmatrix} = d$$

It follows that the *non-étale* locus of g is precisely the locus where the $d \times d$ minors of the matrix

$$\begin{pmatrix} \frac{\partial f_1}{\partial X_1} & \cdots & \frac{\partial f_1}{\partial X_d} \\ \vdots & & \vdots \\ \frac{\partial f_r}{\partial X_1} & \cdots & \frac{\partial f_r}{\partial X_d} \end{pmatrix}$$

vanish. This is a closed subscheme of Y, and hence the étale locus is open.

4.4. Generalities. The following is not really needed for the proof of Theorem 2.1.1 or Theorem 3.1.1. I have brought it up to help you with read more from any of the numerous excellent books on the topic. Let $h \colon V \to W$ be a map of varieties, v a closed point of V and $w = f(v)$. It is not hard to show that f is étale at v if and only if the following three conditions are satisfied.

(a) $\mathscr{O}_{V,v}$ is flat over $\mathscr{O}_{W,w}$,
(b) the point v is isolated in the fibre $h^{-1}(w) = V \times_W \mathrm{Spec}\kappa(w)$; and,
(c) $\kappa(w) \to \kappa(v)$ is separable.

It is not hard to show that in this case, since v is isolated in its fibre over w, there is an open neighbourhood U of v in V such that the map $U \to W$ induced by h has finite fibres (i.e., $U \to W$ is *quasi-finite*). The advantage of the necessary and sufficient conditions (a)—(c) is that we can generalise the notion of being étale at v to the case where v is not necessarily closed in V. Thus in this case one can define h to be étale at v if conditions (a)—(c) are satisfied. One checks that the étale locus is an open locus. Note that h is étale at the generic point of V if and only if the image of the generic point of V is the generic point of W and $k(W) \to k(V)$ is separable.

References

[H] R. Hartshorne, *Algebraic Geometry*, Graduate Texts in Mathematics **52**, Springer, New York, 1977.

[L1] J. Lipman, *Dualizing sheaves, differentials and residues on algebraic varieties*, Asterisque **117**, Société Mathématique de France, 1984.