MULTIPLICITIES

PRAMATHANATH SASTRY

This short note is meant to prove that if C is a curve, $x \in C$ a point, M a very ample line bundle on C, then

$$\deg_C(M) \geq \mathrm{mult}_x(C). \tag{1}$$

This is an important step in proving that Seshadri's criterion is necessary for a line bundle to be ample. We do need a scheme theoretic understanding of the degree of a line bundle M which has a non-zero section on a curve C. Such a non-zero section can be regarded as a copy of C in M, and it meets the zero section (canonically identified with C) in a zero-dimensional scheme which can be regarded as the spectrum of an algebra $\prod_i A_i$ where the A_i are Artin local rings. Then $\deg_C M = \sum_i \dim_k A_i$.[1]

We work over an algebraically closed field k.

The order of a non-zero rational function. Let X be a variety, $k(X)$ its field of rational functions (also known as the "function field" of X) and V a subvariety of X of *codimension one*. Define $A_V(= A)$ to be the stalk of $\mathscr{O}_X$ at the generic point of V. Then A is a local k-algebra, of Krull dimension one, which is integral, whose residue field is $k(V)$, and whose field of fractions is $k(X)$. For $f \in A \smallsetminus \{0\}$, we see that $\dim\,[A/(f)] = 0$ if f is not a unit, and $A/(f) = 0$ if f is a unit, whence $A/(f)$ is a finite length A-module. Let $l_A(M)$ denote the length of an A-module M. Define the *order* $\mathrm{ord}_x(f)$ *of* f *at* x by the formula

$$\mathrm{ord}_V(f) := l_A(A/(f)).$$

It is a non-negative integer which is zero precisely when f is a unit in A. If $k(X)^*$ is the multiplicative group of non-zero elements of $k(X)$, then any $r \in k(X)^*$ can be written as $r = f/g$ with f and g non-zero elements of A, and one defines

$$\mathrm{ord}_V(r) = \mathrm{ord}_V(f) - \mathrm{ord}_V(g).$$

It is not hard to see that

$$\mathrm{ord}_V \colon k(X)^* \longrightarrow \mathbb{Z}$$

is a homomorphism of groups.

Date: October 25, 2016.

[1] This is just a way of saying that the $\deg_C M$ is the number of points on the zero-dimensional scheme mentioned "counted properly". Over the complex numbers this agrees with the notion of degree as $c_1(M) \cap [C]$. The idea is to reduce to the non-singular case by going to the normalization of C, and in the non-singular case appealing to Poincaré Duality and/or a version of the Hopf index theorem. Other proofs using Čech cohomology also exist.

General Formula. If $X' \to X$ is a finite birational map, so that $k(X)^* = k(X')^*$, one has the formula (for subvarieties V of X of codimension one)

$$\operatorname{ord}_V(r) = \sum \operatorname{ord}_{V'}(r) \cdot [k(V') : k(V)] \qquad (r \in k(X)^* = k(X')^*) \tag{2}$$

where the sum is over subvarieties V' of X' which map onto V and where the symbol $[k(V') : k(V)]$ denotes the degree of the field extension $k(V')$ of $k(V)$. The proof is not very difficult. To begin with note that it is enough to prove (2) when $X' \to X$ is the *normalization* of X. This case is dealt with in [F, Appendix A, Example A.3.1, p.412] using only basic commutative algebra.

The case of curves. Now suppose C is a curve, that is, an integral scheme of dimension one of finite type over k. Let $\pi \colon C' \to C$ be the blow-up of C at a closed point x of C and denote by $\mathfrak{m}_x$ the maximal ideal of the localring $\mathscr{O}_{C,x}$. By the definition given in class of $\operatorname{mult}_x(C)$ we have

$$\operatorname{mult}_x(C) = \sum_{y \in \pi^{-1}(x)} \dim_k [\mathscr{O}_{C',y}/\mathfrak{m}_x \mathscr{O}_{C',y}]. \tag{3}$$

Let $t \in \mathfrak{m}_x$ be a non-zero element. Then $t \in \mathfrak{m}_x \mathscr{O}_{C'y}$ for every $y \in \pi^{-1}(x)$, whence for such y, $\mathscr{O}_{C',y}/(t)$ surjects onto $\mathscr{O}_{C',y}/\mathfrak{m}_x \mathscr{O}_{C',y}$. This gives $\operatorname{ord}_y(t) = \dim_k [\mathscr{O}_{C',y}/(t)] \ge \dim_k [\mathscr{O}_{C',y}/\mathfrak{m}_x \mathscr{O}_{C',y}]$. Thus, by (3),

$$\sum_{y \in \pi^{-1}(x)} \operatorname{ord}_y(t) \ge \operatorname{mult}_x(C). \tag{4}$$

According to (2), $\operatorname{ord}_x(t) = \sum_{y \in \pi^{-1}(x)} \operatorname{ord}_y(t) \cdot [k(y) : k(x)] = \sum_{y \in \pi^{-1}(x)} \operatorname{ord}_y(t)$, whence by (4) we have

$$\operatorname{ord}_x(t) \ge \operatorname{mult}_x(C). \tag{5}$$

Now suppose M is a line bundle on C with $h^0(M) > 0$ and $s \in \Gamma(C, M) \smallsetminus \{0\}$. Then locally around any point $p \in C$, s can be regarded (up to a multiplication by a unit) as a non-zero function and hence $\operatorname{ord}_p(s)$ makes sense, for multiplication by a unit does not affect the order of an element in $\mathscr{O}^*_{C,p}$. It is well-known that

$$\deg_C M = \sum_{p \in C} \operatorname{ord}_p(s).$$

Note that $\operatorname{ord}_p(s) \ne 0$ if and only if $s(p) = 0$ and in this case $\operatorname{ord}_p(s) \ge 1$. Now suppose the zero scheme of s contains our point of interest x. Then by the above formula and (5) we have

$$\deg_C M \ge \operatorname{ord}_x(s) \ge \operatorname{mult}_x(C).$$

If M is very ample, then we can always find a non-zero section of M which vanishes at our point of interest x. This observation gives (1).

References

[F] W. Fulton, *Intersection Theory*, Springer-Verlag, New-York, 1984.

[H] R. Hartshorne, *Algebraic Geometry*, Graduate Texts in Mathematics **52**, Springer, New York, 1977.