DEGREE OF A LINE BUNDLE AND HYPERPLANE INTERSECTIONS

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Throughout k is an algebraically closed field, and C an integral[1] projective curve over k. The curve C *could be singular.* For a k-vector space V, V^* will denote its dual. A divisor on a k-variety will mean a *Cartier divisor*, i.e., a Weil divisor which is locally principal. On a scheme Z (not necessarily a k-scheme) an *effective divisor* will mean an effective Cartier divisor, i.e., a closed subscheme D of Z such that the ideal sheaf Z is invertible (i.e., a line bundle). We denote the ideal sheaf of such a D by $\mathscr{O}_Z(-D)$ or simply $\mathscr{O}(-D)$.

The idea is to define the degree of a line bundle on a possibly singular curve C and understand hyperplane intersections of an embedding of such a curve into a projective space. The proof of the main result, namely Theorem 1.1.1 is quite short. Just two pages in Section 2. Setting the context and general remarks giving links to other aspects (e.g. Bezout's theorem) take up a little space before the proof. We follow this with a long appendix which is not really necessary for the main goal. It gives the link between the very straightforward ideas in the proof of Theorem 1.1.1 and Grothendieck's beautiful construction of the relative Picard scheme.

In keeping with the conventions of the course, for a vector-bundle $\mathscr{E}$ on a scheme Y, $\mathbb{P}(\mathscr{E})$ will denote the classifiying space for line-bundle quotients of $\mathscr{E}$. Equivalently, $\mathbb{P}(\mathscr{E})$ is the space of sub-bundles $\mathscr{F}$ of $\mathscr{E}$ such that $\operatorname{rank}\mathscr{F} = \operatorname{rank}\mathscr{E} - 1$.

If $\mathscr{L}$ is line bundle on a scheme Y, and s a global section of $\mathscr{L}$, then $Z(s)$ will denote the zero-scheme of s. Note that $Z(s)$ is a closed subscheme of Y. If $\mathscr{L}$ is trivial, then s can be identified, via an isomorphism $\mathscr{L} \cong \mathscr{O}_Y$, with a global section of $\mathscr{O}_Y$, whence we have an ideal sheaf $\mathscr{I} := s\mathscr{O}_Y$ of $\mathscr{O}_Y$, and $Z(s)$ is the closed subscheme given by $\mathscr{I}$. The ideal $\mathscr{I}$ does not depend on the chosen trivialisation as is easy to verify. Since $\mathscr{L}$ is locally trivial, $Z(s)$ can be defined on open sunschemes where $\mathscr{L}$ is trivial, and these locally closed subschemes patch. Equivalently, the global section gives us a map $\mathscr{O}_Y \to \mathscr{L}$, whence a map $\mathscr{L}^{-1} \to \mathscr{O}_Y$. Then $\mathscr{I}$ is the image of $\mathscr{L}^{-1}$ in $\mathscr{O}_Y$.

1. The space of hyperplane intersections

1.1. **Aim.** Suppose $\mathscr{L}$ is a very ample line bundle on C and $C \hookrightarrow \mathbb{P}^N$ the projective embedding given by $\mathscr{L}$. We wish to show that for any hyperplane H of $\mathbb{P}^N$, the scheme $C \cap H$ is finite over k and if A_H is the co-ordinate ring of this finite scheme (so that $\dim_k A_H < \infty$), then $\dim_k A_H$ is independent of H. As is well known (see below for a slightly fuller discussion), $C \cap H$ is also the zero scheme $Z(s)$ of a non-zero section s (determined uniquely up to a non-zero scalar multiple by H) of $\mathscr{L}$. In other words, the number of zeros of s *counted properly* does not depend on

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[1]integral = reduced and irreducible.

the non-zero section s. Equivalently, the number of points of intersection of H and C *counted properly* is independent of H. This number is then clearly an invariant of L, and since by Bertini's theorem we can find an H which avoids the singular points of C and such that the points of the hyperplane section of C by H occur with multiplicity one, this is the what is classically called the degree of C (in the projective space it has been embedded into by L). It is natural to call it the degree of L. In formal terms

Theorem 1.1.1. *Let $\mathscr{L}$ be a very ample line bundle on C and $C \hookrightarrow \mathbb{P}^N$ the resulting closed immersion given by the complete linear system $|\mathscr{L}|$. If H is a hyperplane in $\mathbb{P}^N$, then the scheme $C \cap H$ is a finite k-scheme and the integer $\dim_k A_H$ does not depend upon H, where $A_H = \Gamma(C \cap H, \mathscr{O}_{C\cap H})$. Equivalently, if s is a non-zero section of $\mathscr{L}$, then the the zero scheme of s, $Z(s)$, is a finite scheme over k, and $\dim_k \Gamma(Z(s), \mathscr{O}_{Z(s)})$ does not depend upon the non-zero section s.*

1.2. **Consequences.** Using Theorem 1.1.1 we can define the degree of a line bundle on C, even when C is not smooth. When L is very ample, one defines $\deg L = \dim_k \Gamma(Z(s), \mathscr{O}_{Z(s)})$ for a non-zero section s of L. This is well-defined according to the theorem. Moreover if M and Q are two very ample bundles on C, then it is easy to see that $\deg(M \otimes Q) = \deg(M) + \deg(Q)$. In general, when L is not necessarily very ample, one can always write $L \cong M \otimes_{\mathscr{O}_C} N^{-1}$ where M and N are very ample. (For example, if $\mathscr{A}$ is very ample, then so is every tensor power of $\mathscr{A}$, and for $n \gg 0$, $L \otimes \mathscr{A}^n$ is very ample. Set $M = L \otimes \mathscr{A}^n$ and $N = \mathscr{A}^n$.)

Definition 1.2.1. Let L be a line bundle on C. The *degree of* L, $\deg(L)$ is defined by the formula $\deg(L) = \deg(M) - \deg(N)$ where $L \cong M \otimes_{\mathscr{O}_C} N^{-1}$ with M and N very ample on C.

The degree of L is well defined, for if $L \cong P \otimes Q^{-1}$ as another decomposition with P and Q very ample, then $M \otimes Q \cong P \otimes N$. Thus $\deg(M) + \deg(Q) = \deg(P) + \deg(N)$ and it follows that $\deg(L)$ is well-defined.

Another obvious consequence is this:

Proposition 1.2.2. *Let X be a smooth projective surface, C and D be distinct curves on X, sharing no irreducible component, such that C is integral and D is a very ample divisor. Let $L = \mathscr{O}(D)|_C$. If $Z = C \cap D$ is the scheme-theoretic intersection of C and D (necessarily zero-dimensional and hence affine) and $A = \Gamma(Z, \mathscr{O}_Z)$ so that $Z = \operatorname{Spec} A$, then $\dim_k A = \deg(L)$. In particular, if C and D intersect transversally, then*

$$\#(C \cap D) = \deg(L).$$

Proof. This follows from the fact that if $i\colon C \hookrightarrow X$ is the natural inclusion, then $i^{-1}(D) = Z$, whence $\mathscr{O}_X(D)|_C = \mathscr{O}_C(Z)$, i.e. $L = \mathscr{O}_C(Z)$. If s is the canonical section of $\mathscr{O}_C(Z)$, i.e., the section corresponding to $1 \in k(C)$, then $Z = Z(s)$ and hence we are done. □

Remark 1.2.3. Note that we did not require D to be integral. As an exercise, prove Bezout's theorem for projective plane curves using Proposition 1.2.2.

1.3. **Basics recalled.** Suppose $\mathscr{L}$ is a very ample line bundle on it. Let $V = \Gamma(C, \mathscr{L})$, $\mathbb{P} = \mathbb{P}(V^*)$, the projective space of hyperplanes through the origin in V^*, and $\mathbb{P}^* = \mathbb{P}(V)$ the dual projective space to $\mathbb{P}$, namely the projective space of hyperplanes in V. Note that points on $\mathbb{P}$ can be regarded as lines through the origin in V and points on $\mathbb{P}^*$ as lines through the origin in V^*.

The space $\mathbb{P}$ has another well-known interpretation. It is the complete linear system associated to L, i.e., its k-rational points "are" the effective divisors D on C such that $\mathscr{O}(D) \cong \mathscr{L}$. Indeed, given such a point $p \in \mathbb{P}$, it represents a one-dimensional subspace L_p of V, and if s is a non-zero vector in $L_p \subset \Gamma(C, \mathscr{L})$, the zero scheme of s, $Z(s)$, does not depend on the choice of s, and is an effective divisor in C. Conversely, if D is an effective divisor such that $\mathscr{O}(D) \cong L$, then there is a non-zero section s of L, corresponding to the non-zero section $1 \in \Gamma(C, \mathscr{O}(D))$, such that $Z(s) = D$. The section s, being non-zero, defines a one-dimensional subspace L of V, whence a point of $\mathbb{P}$. The two processes are easily seen to be inverses of each other. (See also [H, p. 145, Proposition 6.15] and note that C is an integral scheme.)

Since $\mathscr{L}$ is very ample on C, it gives rise to an embedding $C \hookrightarrow \mathbb{P}^*$. Since a hyperplane H in $\mathbb{P}^*$ is the same as a point p in $\mathbb{P}$, such a hyperplane gives rise to an effective divisor D_H in C in the linear system determined by $\mathscr{L}$. It is well-known, and easy to see that $D_H = H \cap C$, where the right side is the scheme theoretic intersection of H with C. Note that this shows that C does not lie entirely in any hyperplace H of $\mathbb{P}^*$ and that every effective divisor D in the complete linear system given by $\mathscr{L}$ is obtained by intersecting C with a hyperplane H in $\mathbb{P}^*$, with H uniquely determined by D.

1.4. **The main idea of proof of Theorem 1.1.1.** The idea is to make the correspondence above—between effective divisors in the complete linear system $|\mathscr{L}|$ and k-rational points in $\mathbb{P}$—more geometric. There are other ways of proving Theorem 1.1.1 but I am taking this route because it is a kinder gentler introduction to one of the main ideas behind Grothendieck's construction of the Picard scheme. As before $\mathscr{L}$ is very ample on C, and V, $\mathbb{P}$, $\mathbb{P}^*$ are as above.

We will show that there exists a commutative diagram of schemes

$$\begin{array}{ccc} \mathbf{D} & \xhookrightarrow{\;i\;} & C \times_k \mathbb{P} \\ & {\scriptstyle q}\searrow & \downarrow{\scriptstyle \pi} \\ & & \mathbb{P} \end{array} \tag{1.4.1}$$

with i a closed immersion, q finite and flat, and π the usual projection, such that if $p \in \mathbb{P}$ is a closed point (so that the residue field at p, $k(p)$, is k), and H is the corresponding hyperplane in $\mathbb{P}^*$, then $q^{-1}(p) \hookrightarrow \pi^{-1}(p) = C$ is the divisor $D_H = H \cap C$. In fact, $(\mathbb{P}, \mathbf{D})$ enjoys a universal property, where essentially the k-valued point p in the above discussion can be replaced by a T-valued point of $\mathbb{P}$, for a k-scheme T. We will say more on that later. However note that our aim is achieved if we can indeed find $\mathbf{D}$ embedded in a diagram such as the above. Indeed, since $\mathbf{D} \to \mathbb{P}$ is finite and flat, it is affine, and over an affine open suscheme $U = \operatorname{Spec} A$ of $\mathbb{P}$, $q^{-1}(U) = \operatorname{Spec} B$. B is therefore a finite flat A-algebra, in particular it is a projective A-module. Since U is connected, this means $\dim_{k(\mathfrak{p})}(B \otimes_A k(\mathfrak{p}))$ is constant as $\mathfrak{p}$ varies over the prime ideals of A. If $\mathfrak{p}$ is a maximal ideal, and H the hyperplane in $\mathbb{P}^*$ correspondingto the closed point $\mathfrak{p}$ of $\mathbb{P}$, then $B \otimes_A k(\mathfrak{p}) = A_H$, where A_H is as in Subsection 1.1. Thus $\dim_k A_H = \dim_{k(\mathfrak{p})} A_H$ does not depend on H.

We now discuss the universal property alluded to above. First, for any k-scheme W, write C_W for $C \times_k W$. If $W = \operatorname{Spec} A$, then we will often write C_A instead of C_W. Write $\mathscr{L}_W$ for the pull-back of $\mathscr{L}$ to C_W, and when $W = \operatorname{Spec} A$, $\mathscr{L}_A := \mathscr{L}_W$. Now suppose W is a k-scheme and $\mathscr{D}$ is an effective divisor on C_W such that the induced map $q_W \colon \mathscr{D} \to W$ is flat, and such that $\mathscr{O}(\mathscr{D}) \cong \mathscr{L}_W$. Then there exists a unique map $\gamma \colon W \to \mathbb{P}$ such that

$$(\mathbf{1} \times \gamma)^{-1}(\mathbf{D}) = \mathscr{D}.$$

However, note that the universal property is not essential for the proof of Theorem 1.1.1.

2. **Construction of D**

Throughout $\mathscr{L}$ is a very ample line bundle on C and the notations are as above.

2.1. We have the following cartesian diagram

(2.1.1)
$$\begin{array}{ccc} C \times_k \mathbb{P} & \xrightarrow{\varpi} & C \\ {\scriptstyle\pi}\downarrow & \square & \downarrow{\scriptstyle v} \\ \mathbb{P} & \xrightarrow[u]{} & \operatorname{Spec} k \end{array}$$

where u and v are the structural morphisms, and π and ϖ the projections. Since u and v are flat, the flat base change theorem tells us that

(2.1.2) $$u^* v_* \mathscr{G} = \pi_* \varpi^* \mathscr{G}$$

for a quasi-coherent $\mathscr{O}_C$-modules $\mathscr{G}$. Applying (2.1.2) to $\mathscr{G} = \mathscr{L}$ we see that

(2.1.3) $$V \otimes_k \mathscr{O}_{\mathbb{P}} = \pi_* \varpi^* \mathscr{L}.$$

Consider the universal exact sequence $0 \to K \to V^* \otimes_k \mathscr{O}_{\mathbb{P}} \to \mathscr{O}(1) \to 0$. Dualizing we get an exact sequence of vector bundles (with $Q = K^\vee$)

(2.1.4) $$0 \longrightarrow \mathscr{O}(-1) \xrightarrow{\sigma} V \otimes_k \mathscr{O}_{\mathbb{P}} \longrightarrow Q \longrightarrow 0.$$

Since $V \otimes_k \mathscr{O}_{\mathbb{P}} = \pi_* \varpi^* \mathscr{L}$, the map σ can be re-written as

$$\sigma \colon \mathscr{O}(-1) \to \pi_* \varpi^* \mathscr{L}.$$

Since π^* is right adjoint to π_* this induces a map of line bundles on $C \times_k \mathbb{P}$ stepcounterthm

(2.1.4) $$s \colon \pi^* \mathscr{O}(-1) \to \varpi^* \mathscr{L}$$

whence a non-zero section (also denoted s) of $\varpi^* \mathscr{L} \otimes \pi^* \mathscr{O}(1)$. To cut a long story short, our scheme $\mathbf{D}$ is the effective divisor on $C \times_k \mathbb{P}$ defined by this section, i.e.,

(2.1.5) $$\mathbf{D} = Z(s).$$

It remains to show (a) the flatness of $\mathbf{D}$ over $\mathbb{P}$, (b) that fibre of $\mathbf{D} \to \mathbb{P}$ over a closed point $p \in \mathbb{P}$ is $C \cap H$ where H is the hyperplane in $\mathbb{P}^*$ corresponding to p, and (c) the universal property of ($\mathbb{P}$, $\mathbf{D}$). We will show (a) and (b) in the next sub-section. This will finish the proof Theorem 1.1.1. We prove (c) in the appendix.

2.2. Proof of Theorem 1.1.1. Let $\mathbf{D}$ be as in (2.1.5). Let $i\colon \mathbf{D} \hookrightarrow C_{\mathbb{P}}$ be the obvious closed immersion, and $q\colon \mathbf{D} \to \mathbb{P}$ the composite $q = \pi \circ i$. We first show that q is *flat*. For this we need make a few observations. First, recall that by the *local flatness criterion* [M, Thm. 22.3, p.174] or [SGA1, Thm. 5.6, p.98], if $(A, \mathfrak{m})$ and $(B, \mathfrak{n})$ are noetherian local rings, $A \to B$ a *local map* (i.e., the image of $\mathfrak{m}$ in B is contained in $\mathfrak{n}$), and M is a finitely generated B-module, then M is flat *over* A if $\mathrm{Tor}_1^A(k, M) = 0$, where $k = A/\mathfrak{m}$. The second useful observation is this. If A is an integral domain and $\varphi\colon A \to A$ an A-module map, then φ is injective unless it is zero. Indeed φ is given by $a \mapsto \varphi(1)a$, and the assertion follows since A is a domain. We have the following immediate consequence. Suppose Z is an *integral scheme* and $\varphi\colon \mathscr{L}_1 \to \mathscr{L}_2$ is an $\mathscr{O}_Z$-module map between two *invertible sheaves*, then φ is injective if $\varphi \neq 0$.

By definition of $\mathbf{D}$ as $Z(s)$, where s is as in (2.1.4), and the above observation concerning maps between invertible sheaves on integral schemes, we have an exact sequence[2]

$$0 \longrightarrow \pi^*\mathscr{O}(-1) \xrightarrow{s} \mathscr{L}_{\mathbb{P}} \longrightarrow \mathscr{L}_{\mathbb{P}}|_{\mathbf{D}} \longrightarrow 0. \tag{2.2.1}$$

Let $p \in \mathbb{P}$. Consider the composite diagram of cartesian squares

$$\begin{array}{ccccc} C_{k(p)} & \xrightarrow{\beta} & C_{\mathbb{P}} & \xrightarrow{\varpi} & C \\ {\scriptstyle v'}\big\downarrow & \square & {\scriptstyle \pi}\big\downarrow & \square & \big\downarrow{\scriptstyle v} \\ \operatorname{Spec} k(p) & \xrightarrow[\alpha]{} & \mathbb{P} & \xrightarrow[u]{} & \operatorname{Spec} k \end{array}$$

where α is the natural map (i.e., α is "equal" to p), v' is the structural map, and β the base change of α. Now $\beta^*\mathscr{L}_{\mathbb{P}} = \beta^*\varpi^*\mathscr{L} = (\varpi \circ \beta)^*\mathscr{L} = \mathscr{L}_{k(p)}$, and since $k \to k(p)$ is flat (i.e., $u \circ \alpha$ is flat), we get $\pi'_*\beta^*\mathscr{L}_{\mathbb{P}} = \pi'_*\mathscr{L}_{k(p)} = V \otimes_k k(p)$.

Now, the universal exact sequence $0 \longrightarrow \mathscr{O}(-1) \xrightarrow{\sigma} V \otimes_k \mathscr{O}_{\mathbb{P}} \longrightarrow Q \longrightarrow 0$ on $\mathbb{P}$ splits Zariksi locally, since Q is a vector-bundle. It follows that for any $p \in \mathbb{P}$ we get the induced exact sequence

$$0 \longrightarrow k(p) \xrightarrow{\sigma \otimes k(p)} V \otimes_k k(p) \longrightarrow Q \otimes_k k(p) \longrightarrow 0.$$

Since $V \otimes_k k(p) = \pi'_*\beta^*\mathscr{L}_{\mathbb{P}} = \pi'_*\mathscr{L}_{k(p)} = \Gamma(C_{k(p)}, \mathscr{L}_{k(p)})$, $\sigma \otimes k(p)$ gives a section s' of $\mathscr{L}_{k(p)}$, and a little thought shows that $s' = s|_{C_{k(p)}}$. Moreover, since $\sigma \otimes k(p) \neq 0$, therefore $s' \neq 0$. Since $\mathbf{C}_{k(p)}$ is an integral curve and $s' \neq 0$, on tensoring (2.2.1) with $\mathscr{O}_{C_{k(p)}}$ we get an exact sequence

$$0 \longrightarrow \mathscr{O}_{C_{k(p)}} \xrightarrow{s'} \mathscr{L}_{k(p)} \longrightarrow \mathscr{L}_{k(p)}|_{\mathbf{D} \cap C_{k(p)}} \longrightarrow 0. \tag{2.2.2}$$

Now suppose $x \in \mathbf{D}$ and $\pi(x) = p$. Let $A = \mathscr{O}_{\mathbb{P},\,p}$, and $B = \mathscr{O}_{\mathbf{D},x}$. From (2.2.1) and (2.2.2) we see that $\mathrm{Tor}_1^A(k(p), (\mathscr{L}_{\mathbb{P}}|_{\mathbf{D}})_x) = 0$. Since $(\mathscr{L}_{\mathbb{P}}|_{\mathbf{D}})_x \cong B$, this means

[2] There is a work-around which avoids the use of the fact that $C_{\mathbb{P}}$ is integral in showing (2.2.1) is exact. It is certainly exact without the zero on the left. The way around this is the following. Note that the sequence (2.2.2) is exact because $C_{k(p)}$ is integral and $s' \neq 0$. Now check that if $A \to B$ is a flat local homomorphism of local rings, $k = A/\mathfrak{m}_A$, and $b \in B$ an element such that the image of b in $B/\mathfrak{m}_A B$ is a non-zero divisor in $B/\mathfrak{m}_A B$, then b is a non-zero divisor in B. A few pointers may help, let $I = (b)$. Then $B \xrightarrow{b} B$ factors through I. Check that (a) $(B \twoheadrightarrow I) \otimes_A k$ is an isomorphism and (b) $\mathrm{Tor}_1^A(k, B/I) = 0$. By the local criterion for flatness, from (b) conclude that B/I is A-flat. Show that the if $J = \ker(B \to I)$, then $J \otimes_A k = 0$. Conclude that $J/\mathfrak{m}_B J = 0$, whence $J = 0$, and $B \to I$ is an isomorphism.

that $\operatorname{Tor}_1^A(k(p),\, B) = 0$, and by the local flatness criterion we conclude that B is flat over A. Since $x \in \mathbf{D}$ was arbitrary, this means $\mathbf{D}$ is flat over $\mathbb{P}$.

Suppose p is a closed point of $\mathbb{P}$ so that $k = k(p)$ and $C_{k(p)} = C$, $\mathscr{L}_{k(p)} = \mathscr{L}$ etc. Let s' be as above, i.e. s' arises from $\sigma \otimes k(p)$. Then $s' \in \Gamma(C\,\mathscr{L}) = V$. Since $s' \neq 0$, it determines a line in V, and it is clear that this line corresponds to the point p, if we regard $\mathbb{P} = \mathbb{P}(V^*)$ as the space of lines through the origin in V. This means $Z(s') = C \cap H$ where H is the hyperplane in $\mathbb{P}^*$ corresponding to $p \in \mathbb{P}$. On the other hand, by definition of $\mathbf{D}$ as $Z(s)$, we have $q^{-1}(p) = Z(s')$. Thus $q^{-1}(p) = C \cap H$.

Remark 2.2.3. Everything we did above goes through if instead of the assumption that $\mathscr{L}$ is very ample, we instead make the weaker assumption that the complete linear system $|\mathscr{L}|$ is base point free, i.e., that $\mathscr{L}$ is generated by global sections. Then we have a map $\varphi\colon C \to \mathbb{P}^*$ and a closed point p on $\mathbb{P}$ has two interpretations. The first is as a member D of $|\mathscr{L}|$ and the second is as a hyperplane H in $\mathbb{P}^*$. Then the divisor $\mathbf{D}$ we constructed on $C_{\mathbb{P}}$ remains flat over $\mathbb{P}$ via $q\colon \mathbf{D} \to \mathbb{P}$. Then with p, D and H as the entities just introduced, we have $D = q^{-1}(p) = \varphi^{-1}(H)$. I will leave it to you to formulate the appropriate universal property. The proof of the universal property for this case is no different from the one offered below for $\mathscr{L}$ very ample.

Appendix A. Proof of the Universal Property

A.1. The basic property we need is the following. Suppose S is a k-scheme and $\mathscr{K}$ a line bundle on S (the example to keep in mind is $S = \mathbb{P}$ and $\mathscr{K} = \mathscr{O}(-1)$). Let $\pi_S\colon C_S \to S$ be the projection. Then the natural map from $\mathscr{K}$ to $(\pi_S)_*\pi_S^*\mathscr{K}$ is an isomorphism:

$$\mathscr{K} \xrightarrow{\ \sim\ } (\pi_S)_*\pi_S^*\mathscr{K} \tag{A.1.1}$$

We prove these statements later. The main hypotheses which make the assertion true are that C_S is flat over S, and all the fibres of π_S are geometrically integral. Moreover the isomorphism (A.1.1) is "universal". In greater detail, if $g\colon T \to S$ is a map of k-schemes, and $g_T\colon C_T \to C_S$ the induced map, then the base change map from $g^*(\pi_S)_*\pi_S^*\mathscr{K}$ to $(\pi_T)_*g_T^*\pi_S^*\mathscr{K}$ is an isomorphism

$$g^*(\pi_S)_*\pi_S^*\mathscr{K} \xrightarrow{\ \sim\ } (\pi_T)_*g_T^*\pi_S^*\mathscr{K} \tag{A.1.2}$$

and fits into the following commutative diagram

$$\begin{array}{ccc} g^*\mathscr{K} & \xrightarrow[g^*(\text{A.1.1})]{\sim} & g^*(\pi_S)_*\pi_S^*\mathscr{K} \\ {\scriptstyle (\text{A.1.1})}\big\downarrow\wr & & \wr\big\downarrow{\scriptstyle (\text{A.1.2})} \\ (\pi_T)_*\pi_T^*g^*\mathscr{K} & = & (\pi_T)_*g_T^*\pi_S^*\mathscr{K} \end{array} \tag{A.1.3}$$

In fact (A.1.2) follows from (and is equivalent to) the following functorial isomorphism for quasi-coherent $\mathscr{O}_W$-modules $\mathscr{G}$

$$(\pi_{S*}\pi_S^*\mathscr{K}) \otimes_{\mathscr{O}_S} \mathscr{G} \xrightarrow{\ \sim\ } \pi_{S*}(\pi_S^*\mathscr{K} \otimes_{\mathscr{O}_{C_S}} \pi_S^*\mathscr{G}). \tag{A.1.4}$$

One consequence all this is that *the natural map from* $\operatorname{Hom}_{C_S}(\pi_S^*\mathscr{K},\, \mathscr{L}_S)$ *to* $\operatorname{Hom}_S(\pi_{S*}\pi_S^*\mathscr{K},\, \pi_{S*}\mathscr{L}_S)$ *given by* $\varphi \mapsto \pi_{S*}\varphi$ *is an isomorphism.* This is so because

(A.1.1) gives us an isomorphism $\mathrm{Hom}_S(\pi_{S*}\pi_S^*\mathscr{K}\,,\,\pi'_*\mathscr{L}_S) \xrightarrow{\sim} \mathrm{Hom}_S(\mathscr{K}\,,\,\pi_{S*}\mathscr{L}_S)$ and the composite

$$\mathrm{Hom}_{C_S}(\pi_S^*\mathscr{K}\,,\,\mathscr{L}_S) \xrightarrow{\text{natural}} \mathrm{Hom}_S(\pi_{S*}\pi_S^*\mathscr{K}\,,\,\pi_{S*}\mathscr{L}_S) \xrightarrow{\sim} \mathrm{Hom}_S(\mathscr{K}\,,\,\pi_{S*}\mathscr{L}_S)$$

is the usual adjoint isomorphism $\mathrm{Hom}_{C_S}(\pi_S^*\mathscr{K}\,,\,\mathscr{L}_S) \xrightarrow{\sim} \mathrm{Hom}_S(\mathscr{K}\,,\,\pi_{S*}\mathscr{L}_S)$. We record this:

$$\text{(A.1.5)} \qquad \mathrm{Hom}_{C_S}(\pi_S^*\mathscr{K}\,,\,\mathscr{L}_S) \xrightarrow[\text{natural}]{\sim} \mathrm{Hom}_S(\pi_{S*}{\pi_S}^*\mathscr{K}\,,\,\pi_{S*}\mathscr{L}_S).$$

We will prove (A.1.1), (A.1.4) (whence (A.1.2)) and the commutativity of (A.1.3) later. As for (A.1.2), note that the base-change map is well defined, and therefore to show it is an isomorphism we may assume S and T are affine. Setting $\mathscr{G} = g_*\mathscr{O}_T$ in (A.1.4) one sees easily that base change map in (A.1.2) is an isomorphism if (A.1.4) is an isomorphism for all quasi-coherent $\mathscr{G}$.

A.2. So suppose we have a commutative diagram of k-schemes

$$\begin{array}{ccc} \mathscr{D} & \overset{j}{\hookrightarrow} & C_W \\ & \underset{q_W}{\searrow} & \downarrow{\scriptstyle \pi_W} \\ & & W \end{array}$$

with q_W flat and $\mathscr{D}$ an effective divisor in C_W (and j the natural inclusion of a closed subscheme) such that $\mathscr{O}(\mathscr{D}) \cong \mathscr{L}_W \otimes {\pi_W}^*\mathscr{M}$ where $\mathscr{M}$ is a line bundle on W. With $\mathscr{K} := \mathscr{M}^{-1}$ we have an exact sequence

$$(*) \qquad 0 \longrightarrow {\pi_W}^*\mathscr{K} \xrightarrow{\alpha} \mathscr{L}_W \longrightarrow \mathscr{L}_W|_{\mathscr{D}} \longrightarrow 0,$$

since $\mathscr{O}(-\mathscr{D})$ is the ideal sheaf of $\mathscr{D}$. Since $\mathscr{D}$ is flat over W, the sequence

$$(**) \qquad 0 \longrightarrow \pi_W^*\mathscr{K} \otimes \pi_W^*\mathscr{G} \xrightarrow{\alpha\otimes\pi_W^*\mathscr{G}} \mathscr{L}_W \otimes \pi_W^*\mathscr{G} \longrightarrow (\mathscr{L}_W|_{\mathscr{D}}) \otimes \pi_W^*\mathscr{G} \longrightarrow 0.$$

is exact for every quasi-coherent $\mathscr{O}_W$-module $\mathscr{G}$. If we regard α as a section of $\mathscr{L}_W \otimes \pi_W^*\mathscr{M}$, then

$$\mathscr{D} = Z(\alpha).$$

Taking direct image and using (A.1.1) we get an injective map of $\mathscr{O}_W$-modules

$$\sigma_W \colon \mathscr{K} \hookrightarrow V \otimes_k \mathscr{O}_W.$$

We claim that in fact s_W identifies $\mathscr{K}$ as a sub-bundle of $V \otimes_k \mathscr{O}_W$. To that end let $w \in W$. Applying $(**)$ to $\mathscr{G} = k(w)$ we get an exact sequence

$$(\dagger) \qquad 0 \longrightarrow \pi_{k(w)}^*(\mathscr{K} \otimes k(w)) \xrightarrow{\alpha_w} \mathscr{L}_{k(w)} \longrightarrow \mathscr{L}_{k(w)}|_{\mathscr{D}\cap\mathbf{C}_{k(w)}} \longrightarrow 0.$$

where $\pi_{k(w)} := \pi_{\mathrm{Spec}\,k(w)}$. Taking global sections and using (A.1.1) for $\mathscr{K} \otimes k(w)$ on $\mathrm{Spec}\,k(w)$ we get an injective map

$$\sigma_w \colon \mathscr{K} \otimes k(w) \hookrightarrow V \otimes_k k(w).$$

It is easy to verify that $\sigma_w = \sigma_W \otimes k(w)$. Thus $\sigma_W \otimes k(w)$ is an injective map for every w in W. This means that if $Q_W = \mathrm{coker}\,(\alpha)$, then $\mathrm{Tor}_1^{\mathscr{O}_W}(Q_W, k(w)) = 0$ for all $w \in W$. Since Q_W is coherent, the vanishing of the Tor-modules means (via Nakayama) that Q_W is a flat $\mathscr{O}_W$-module, whence a vector bundle. It follows that $\mathscr{K}$ is a sub-bundle of $V \otimes_k \mathscr{O}_W$ via σ_W.

From the universal property of the exact sequence (2.1.4) we get a unique map $\gamma\colon W \to \mathbb{P}$ such that there there is an isomorphism

$$\theta\colon \gamma^*\mathscr{O}(-1) \xrightarrow{\ \sim\ } \mathscr{K}$$

fitting into a commutative diagram

$$\text{(A.2.1)}\qquad \begin{array}{ccc} \gamma^*\mathscr{O}(-1) & \xrightarrow{\gamma^*\sigma} & \gamma^*(V\otimes_k \mathscr{O}_{\mathbb{P}}) \\ \wr\big\downarrow & & \big\| \\ \mathscr{K} & \xrightarrow[\sigma_W]{} & V\otimes_k \mathscr{O}_W \end{array}$$

The above diagram can be expanded. Let $\widetilde{\gamma} = \mathbf{1}\times\gamma$. Let s be as in (2.2.1), i.e., s is the section of $\mathscr{O}(1)\otimes\mathscr{L}_{\mathbb{P}}$ such that $\mathbf{D} = Z(s)$. Then the commutative diagram (A.2.1) can be expanded to the diagram

$$\begin{array}{ccccccc} \gamma^*\mathscr{O}(-1) & \xrightarrow[\gamma^*(\text{A.1.1})]{\sim} & \gamma^*\pi_*\pi^*\mathscr{O}(-1) & \xrightarrow{\text{via } s} & \gamma^*\pi_*\mathscr{L}_{\mathbb{P}} & \xrightarrow{\sim} & \gamma^*(V\otimes_k\mathscr{O}_{\mathbb{P}}) \\ \big\| & & \wr\big\downarrow & \spadesuit & \wr\big\downarrow & & \big\| \\ \gamma^*\mathscr{O}(-1) & & \pi_{S*}\widetilde{\gamma}^*\pi^*\mathscr{O}(-1) & \xrightarrow{\text{via } s} & \pi_{S*}\widetilde{\gamma}^*\mathscr{L}_{\mathbb{P}} & & \big\| \\ \big\| & & \big\| & & \big\| & & \big\| \\ \gamma^*\mathscr{O}(-1) & \xrightarrow[(\text{A.1.1})]{\sim} & \pi_{S*}\pi_S^*\gamma^*\mathscr{O}(-1) & \clubsuit & \big\| & & \big\| \\ \theta\big\downarrow\wr & & \wr\big\downarrow \text{via } \theta & & \big\| & & \big\| \\ \mathscr{K} & \xrightarrow[(\text{A.1.1})]{\sim} & \pi_{S*}\pi_S^*\mathscr{K} & \xrightarrow[\text{via } \alpha]{} & \pi_{S*}\mathscr{L}_S & \xrightarrow{\sim} & V\otimes_k\mathscr{O}_W \end{array}$$

Here the downward arrows in the sub-rectangle $\spadesuit$ are the ones arising from the natural transformation $\gamma^*\pi_* \to \pi_{S*}\widetilde{\gamma}^*$ with the arrow on the left being an isomorphism by virtue of (A.1.2). It follows that $\spadesuit$ commutes. The sub-rectangle on the top left commutes by (A.1.3). The natural transformation $\mathbf{1}\to\pi_{S*}\pi_S^*$ ensures that the rectangle at the bottom left corner commutes. The rectangle on the right commutes since $\gamma^*(V\otimes_k\mathscr{O}_{\mathbb{P}}) = V\otimes_k\mathscr{O}_S$ is an expression of the base change isomorphism for the direct image of $\mathscr{L}$ under $C\to \operatorname{Spec} k$ and the compatibility of base change maps for a composite of cartesian diagram. The outer rectangle commutes since it is simply the expansion of (A.2.1). It follows that the hold out $\clubsuit$ also commutes. From $\clubsuit$ we get $\pi_{S*}(\widetilde{\gamma}^*(s)\circ\pi_S^*(\theta^{-1})) = \pi_{S*}(\alpha)$. By (A.1.5) it follows that $\widetilde{\gamma}^*(s) = \alpha\circ\pi_S^*(\theta)$. Thus $Z(\widetilde{\gamma}^*(s)) = Z(\alpha)$. This means $\widetilde{\gamma}^{-1}(\mathbf{D}) = \mathscr{D}$ as required.

A.3. Proofs of assertions in Subsection A.1. In order to prove (A.1.4) it is enough to assume $S = \operatorname{Spec} A$ and $\mathscr{K} = \mathscr{G}$, for the assertion is local on S, and $\mathscr{K}$ is locally trivial. We make these assumptions. For an A-module M, we have to show that

$$\varphi_M\colon \Gamma(C_A,\, \mathscr{O}_{C_A})\otimes_A M \longrightarrow \Gamma(C_A,\, \pi_S^*\widetilde{M})$$

is an isomorphism. By [H, Prop. 12.5, p.286]and [H, Prop. 12.10, p.289], it enough for us to prove $\varphi_{k(s)}$ is surjective for every s in S, where, as usual, $k(s)$ is the residue field of $\mathscr{O}_{S,s}$. Now $\Gamma(C_A,\, \pi_S^*\widetilde{k(s)}) = \Gamma(C_{k(s)},\, \mathscr{O}_{C_{k(s)}})$. Moreover, since k is algebraically closed, and C is proper and inetgral, $\Gamma(C, \mathscr{O}_C) = k$, and therefore by the flat base change theorem $\Gamma(C_{k(s)},\, \mathscr{O}_{C_{k(s)}}) = k(s)$. This means that

$\varphi_{k(s)}\colon \Gamma(C_A,\, \mathscr{O}_{C_A}) \otimes_A k(s) \to k(s)$ is surjective since it is a map of $k(s)$-algebras, and hence is a non-zero map. This proves (A.1.4) and hence (A.1.2).

By (A.1.4) we have an isomorphism of functors of quasi-coherent $\mathscr{O}_S$-modules

$$(\pi_{S*}\pi_S^*\mathscr{K}) \otimes_{\mathscr{O}_S} (-) \xrightarrow{\sim} \pi_{S*}(\pi_S^*\mathscr{K} \otimes_{\mathscr{O}_{C_S}} \pi_S^*(-)).$$

The functor on left is right-exact and the one on the right is left-exact (since π_S is flat, whence π_S^* is exact). It follows that both functors are exact. In particular $\pi_{S*}\pi_S^*\mathscr{K}$ is a flat $\mathscr{O}_S$-module, and since it is coherent, it is therefore a vector bundle. The calculations in the last paragraph in fact shows that it is a line bundle

To prove (A.1.1) it is again enough to assume $\mathscr{K} = \mathscr{O}_S$. Let $s \in S$. Consider the composite

$$k(s) = \mathscr{O}_S \otimes_{\mathscr{O}_S} k(s) \longrightarrow \pi_{S*}\mathscr{O}_{C_S} \otimes_{\mathscr{O}_S} k(s) \xrightarrow[\text{(A.1.2)}]{\sim} \Gamma(C_{k(s)},\, \mathscr{O}_{C_{k(s)}}) = k(s).$$

Once again the composite is a map of $k(s)$-algebras, and hence it the identity map. This means $\mathscr{O}_S \to \pi_{S*}\mathscr{O}_{C_S}$ is surjective by Nakayama's Lemma. It follows that it is an isomorphism, being a surjective map between line bundles. This establishes (A.1.1).

As for (A.1.3), once again we may assume $S = \operatorname{Spec} A$ and $\mathscr{K} = \mathscr{O}_S$. We may also assume, without loss of generality, that $T = \operatorname{Spec} A'$ for an A-algebra A'. Pick a finite affine open cover $\mathscr{U} = \{U_\alpha\}$, with $U_\alpha = \operatorname{Spec} B_\alpha$ (say) for each index α. Let $\mathscr{U}'$ be the pull-back of this cover to $C_T = C_{A'}$. If for each index α, $B'_\alpha = B_\alpha \otimes_A A'$ and $U'_\alpha := \operatorname{Spec} B'_\alpha$ then $\mathscr{U}' = \{U'_\alpha\}$. Let $D_A^\bullet$ (resp. $D_{A'}^\bullet$) be the Čech complex of $\mathscr{O}_{C_S}$ (resp. $\mathscr{O}_{C_T}$) with respect to $\mathscr{U}$ (resp. $\mathscr{U}'$). Then $D_{A'}^\bullet = D_A^\bullet \otimes_A A'$. We have a natural map $A \to D_A^0 = \oplus_\alpha B_\alpha$ given by the algebra map $A \to B_\alpha$ for each index α. Similarly we have a map $A' \to D_{A'}^0$. In fact, as can be readily checked, these maps take values in $Z^0(D_A^\bullet)$ and $Z^0(D_{A'}^\bullet)$, the modules of 0-cocycles of $D_A^\bullet$ and $D_{A'}^\bullet$ respectively. But the modules of 0-cocyles can be identified with $\Gamma(C_S,\, \mathscr{O}_{C_S})$ and $\Gamma(C_T,\, \mathscr{O}_{C_T})$ repsectively, and the resulting maps $A \to \Gamma(C_S,\, \mathscr{O}_{C_S})$ and $A' \to \Gamma(C_T,\, \mathscr{O}_{C_T})$ are the global sections of $\mathscr{O}_S \to \pi_{S*}\mathscr{O}_{C_S}$ and $\mathscr{O}_T \to \pi_{T*}\mathscr{O}_{C_T}$ respectively. Since the base change map $\mathrm{H}^i(C_S,\, \mathscr{O}_{C_S}) \otimes_A A' \to \mathrm{H}^i(C_T,\, \mathscr{O}_{C_T})$ can be identified with the natural map $H^j(D_A^\bullet) \otimes_A A' \to H^j(D^\bullet \otimes_A A')$, the commutativity of (A.1.3) follows from the fact that the ring homomorphism $A' \to B'_\alpha$ is induced by the ring homomorphism $A \to B_\alpha$, i.e., the former is the latter tensored over A with A'.

Remark A.3.1. What we have essentially proved is that if $f\colon X \to Y$ is a proper flat map of noetherian schemes such that the fibres of f are geometrically integral, then $\mathscr{K} \xrightarrow{\sim} f_*f^*\mathscr{K}$ for every line bundle $\mathscr{K}$ on Y, the map being the natural one. Moreover, the isomorphisms (A.1.2) and (A.1.4) continue to hold in this situation, and the isomorphism $\mathscr{K} \xrightarrow{\sim} f_*f^*\mathscr{K}$ is universal in the sense that diagram (A.1.3) commutes (with f, Y ... replacing π_S, S ...). No essential changes are needed in the proof we gave above. This is the essential content of [EGA-III, Chapitre 2, 7.8.6]

A.4. General comments I. The techniques used in this note can be used in a far more general situation. *Suppose $\pi\colon X \to S$ is a flat projective map whose fibres are geometrically reduced and irreducible, $\mathscr{L}$ a relatively very ample line bundle on X such that if $\mathscr{V} = \pi_*\mathscr{L}$, then the natural map $\varphi_{\mathscr{G}}\colon \mathscr{V} \otimes_{\mathscr{O}_S} \mathscr{G} \to \pi_*(\mathscr{L} \otimes_{\mathscr{O}_X} \pi^*\mathscr{G})$ is an isomorphism for all quasi-coherent $\mathscr{O}_S$-modules $\mathscr{G}$.* Note that Remark A.3.1 applies to $\pi\colon X \to S$ under these hypotheses. More can be said, namely:

- $\mathscr{V}$ is a vector-bundle for the right exact functor $(\pi_*\mathscr{V}) \otimes \mathscr{O}_S(-)$ of quasi-coherent $\mathscr{O}_S$-modules is actually exact, being isomorphic to the left exact functor $\pi_*(\mathscr{L} \otimes_{\mathscr{O}_X} \pi^*(-))$.
- X is a closed S-subscheme of $\mathbb{P}^* := \mathbb{P}(\mathscr{V})$ and $\mathscr{L}$ is canonically isomorphic to the restriction of $\mathscr{O}_{\mathbb{P}^*}(1)$ to X.
- $\mathbb{P} := \mathbb{P}(\mathscr{V}^\vee)$, the space of line sub-bundles of $\mathscr{V}$, parameterises all effective (Cartier) divisors D on X which are flat over S, and whose associated line bundle $\mathscr{O}(D)$ is isomorphic to $\mathscr{L}$.

In greater detail, if for an S-scheme T, $X_T := X \times_S T$, $\pi_T \colon X_T \to T$ is the base change of π, $\mathscr{L}_T$ the pull-back of $\mathscr{L}$ to X_T, $\mathscr{V}_T$ the pull-back of $\mathscr{V}$ to T, then the natural map $\mathscr{V}_T \to \pi_{T*}\mathscr{L}_T$ is an isomorphism (we will regard it as an equality). The line bundle $\mathscr{L}_\mathbb{P} \otimes_{\mathscr{O}_\mathbb{P}} \pi_\mathbb{P}^*\mathscr{O}_\mathbb{P}(-1)$ has a natural section s arising as the adjoint to the tautological injection $\mathscr{O}_\mathbb{P}(-1) \xrightarrow{\sigma} \mathscr{V}_\mathbb{P}$. Then the effective divisor$\mathbf{D} := Z(s)$ is flat over $\mathbb{P}$ and if T is an S-scheme such that there is an effective divisor $\mathscr{D}$ in X_T such that $\mathscr{O}(\mathscr{D}) \cong \mathscr{L}_T \otimes_{\mathscr{O}_{X_T}} \pi_T^*\mathscr{M}$ for some line bundle $\mathscr{M}$ on T, then there is a unique S-map $\gamma \colon T \to \mathbb{P}$ such that $(\mathbf{1} \times \gamma)^*\mathbf{D} = \mathscr{D}$. In such a case there is a canonical isomorphism $\gamma^*\mathscr{O}_\mathbb{P}(1) \xrightarrow{\sim} \mathscr{M}$.

The proofs are more or less the same the ones we gave in the special case we considered in this note.

A.5. General Comments II. One can actually get more out of this technique than what we have outlined. It is an essential technique in Grothendieck's proof (based on an earlier technique of Matsusaka) of the existence of the *relative Picard scheme* $\mathbf{Pic}_{X/S}$ for $X \to S$ as in the previous section. Here are some paint strokes made with a very broad brush.

Suppose $\Phi \in \mathbb{Q}[t]$ is a polynomial. Let $\pi \colon X \to S$ be as in the previous section with S noetherian and connected. Let $\mathscr{A}$ be a relative very ample line bundle for $X \to S$, i.e., $\pi_*\mathscr{A}$ is a vector bundle and the natural S-map $X \to \mathbb{P}(\pi_*\mathscr{A})$ (given by $\pi^*\pi_*\mathscr{A} \twoheadrightarrow \mathscr{A}$) is an embedding[3]. Standard theorems, not very difficult to prove, tell us that the Hilbert polynomial $\Psi(n) = \chi(X_{k(s)}\, \mathscr{A}^n_{k(s)})$ does not depend upon $s \in S$ [Mu, Cor. 3, p.52]. In fact, the proof is there on [H, p.262] also, where unfortunately, in the statement of Thm. 9.9, an integral hypothesis is made. However the proof given for (i) $\Rightarrow$ (ii) in loc.cit. (i.e., the "only if" part of the theorem) works without the integral hypothesis, and since it is barely a paragraph, the reader can check for herself/himself.

It turns out that one can find an integer m such that if K is an algebraically closed field and $\operatorname{Spec} K \to S$ a map (i.e., a "geometric point" of S), and L a line bundle on X_K with Hilbert polynomial (with respect to $\mathscr{A}_K$) equal to Φ, then L is m-regular. The integer m depends only on Φ and Ψ, and not on S or X, $\mathscr{A}$, or π. See Exercise 9.6.7 on p.295 of Kleiman's article in [FGA-ICTP]. Note that the very clever solution to the exercise on p.310 of loc.cit. has a small typo. On the fourth line from the bottom of that page, one should have $\mathcal{N} = \mathcal{L}(-a^2)$ and not $\mathcal{N} = \mathcal{L}(-a)$.

Suppose D is an effective divisor. Since divisor for us means Cartier divisor, the ideal sheaf I of D is invertible. We denote it $\mathscr{O}_X(-D)$ or simply $\mathscr{O}(-D)$. Note that this is consistent with the general notation.

[3]Often called *strongly very ample for* $X \to S$

For an S-scheme T if $\mathscr{F}$ is a coherent $\mathscr{O}_{X_T}$-module flat over T, we say say $\mathscr{F}$ has Hilbert polynomial P if $\mathscr{F}$ has Hilbert polynomial P on each fibre of π_T with respect to $\mathscr{A}$ restricted to that fibre. If T is connected then $\mathscr{F}$ has a Hilbert polynomial by [Mu, Cor. 3, p.52].

Let m be as above. To avoid annoying trivialities, let us assume the relative dimension of $\pi\colon X \to S$ is at least 1 so that all fibres are positive dimensional. Let Φ_m be the polynomial $\Phi_m(t) = \Phi(t+m)$. Via the theory of Quot schemes, the discussion above (the one referencing [FGA-ICTP, Exrercise 9.6.7, p.295]) says that the family of line bundles $\mathscr{L}$ on X with Hilbert polynomial Φ is bounded whence the family of line bundles with Hilbert polynomial Φ_m. It follows that if $\mathscr{M}$ is a line bundle on X such that $\mathscr{M}^{-1}$ has Hilbert polynomial Φ_m, then the possibilities for the Hilbert polynomial of $\mathscr{M}$ are finite in number. Let $\mathbf{Div} = \mathbf{Div}_\Phi^m$ be the open subscheme of the relative Hilbert scheme $\mathbf{Hilb}_{X/S}$ parameterising effective divisors D on X which are flat over S and such that $\mathscr{O}(D)$ has Hilbert polynomial Φ_m. Let $\mathbf{D}$ be the universal effective divisor on $X_{\mathbf{Div}}$. $\mathbf{Div}_\Phi^m$ is quasi-projective over S since $\mathscr{O}_{\mathbf{D}}$ (regarded as a sheaf which is flat over $\mathbf{Div}$) has only a finite number of possibilities for its Hilbert polynomial. For simplicity we will assume that $\Phi_m(0) \geq 2$ something that can be achieved by replacing m by $m+1$ for example. This means $\mathbf{D}$ is not a trivial divisor.

Let $\mathscr{L} = \mathscr{O}(\mathbf{D})$, $\mathscr{V} = \pi_{\mathbf{Div}*}\mathscr{L}$, and $\mathbb{P} := \mathbb{P}(\mathscr{V}^\vee)$. Let $p\colon \mathbb{P} \to \mathbf{Div}$ be the natural map. Now, on $X_\mathbb{P}$ we have a divisor $\mathbf{D}^*$ which is flat over $\mathbb{P}$ given by the natural section of $\mathscr{L}_\mathbb{P} \otimes \pi_\mathbb{P}^*\mathscr{O}(1)$. This gives a natural map $q\colon \mathbb{P} \to \mathbf{Div}$ such that the pull-back of $\mathbf{D}$ is $\mathbf{D}^*$. Note that $p \neq q$ unless $X = S$, for $p = q$ implies the following chain of equalities $\mathscr{L}_\mathbb{P} = (\mathbf{1}\times p)^*\mathscr{L} = (\mathbf{1}\times q)^*\mathscr{L} = (\mathbf{1}\times q)^*\mathscr{O}(\mathbf{D}) = \mathscr{O}(\mathbf{D}^*) = \mathscr{L}_\mathbb{P}\otimes\pi_\mathbb{P}^*\mathscr{O}(1)$. This implies, $\pi_\mathbb{P}^*\mathscr{O}(1) \cong \mathscr{O}_{X_\mathbb{P}}$, and on applying $\pi_{\mathbb{P}*}$ we get $\mathscr{O}_\mathbb{P}(1) = \mathscr{O}_\mathbb{P}$ which is absurd unless $\mathbb{P} = \mathbf{Div}$, whence $\mathscr{V} = \mathscr{O}_{\mathbf{Div}}$, i.e., $\mathbf{D}$ is trivial, a contradiction.

Suppose Z is an S-scheme and $f\colon Z \to \mathbf{Div}$ an S-map. Suppose we have an effective divisor $\mathscr{D}$ on X_Z, flat over Z, and a line bundle $\mathscr{M}$ such that

$$(\mathbf{1}\times f)^*\mathscr{L} \otimes \pi_Z^*\mathscr{M} \cong \mathscr{O}(\mathscr{D}).$$

The universal property of $(\mathbb{P}, \mathbf{D})$ (whose proof we have given in the special case of a curve C, but whose proof applies *mutatis mutandis* here) says that giving a pair $(\mathscr{D}, \mathscr{M})$ satisfying the above condition is equivalent to giving a map $\theta\colon Z \to \mathbb{P}$ such that $(\mathbf{1}\times\theta)^{-1}(\mathbf{D}^*) = \mathscr{D}$ and $p\circ\theta = f$.

Now suppose we have a pair of maps $f, g\colon Z \rightrightarrows \mathbf{Div}$ such that on X_Z we have $(\mathbf{1}\times f)^*\mathscr{L} \otimes \pi_Z^*\mathscr{M} \cong (\mathbf{1}\times g)^*\mathscr{L}$, i.e.,

$$\mathscr{O}_Z((\mathbf{1}\times g)^{-1}\mathbf{D}) \cong (\mathbf{1}\times f)^*\mathscr{L} \otimes \pi_Z^*\mathscr{M}$$

where $\mathscr{M}$ is a line bundle on Z. Then we get a unique S-map $\theta\colon Z \to \mathbb{P}$ such that $f = p\circ\theta$ and $g = q\circ\theta$ and $\mathscr{M} \cong \theta^*\mathscr{O}(1)$. In greater detail, θ is the unique map such that $p\circ\theta = f$ and $(\mathbf{1}\times\theta)^{-1}\mathbf{D}^* = (\mathbf{1}\times g)^{-1}\mathbf{D}$. The equality $g = q\circ\theta$ follows from the chain of equalities $(\mathbf{1}\times g)^{-1}\mathbf{D}) = (\mathbf{1}\times\theta)^{-1}\mathbf{D}^* = (\mathbf{1}\times\theta)\circ(\mathbf{1}\times q)^{-1}(\mathbf{D})$, the universal property of Hilbert schemes, and the fact that $\mathbf{D}$ is the universal divisor on $X_{\mathbf{Div}}$.

Here is a re-interpretation of the discussion above (with certain caveats having to do with sheafications with respect to the fppf topology on S). If we consider the relative Picard functor $\mathbf{Pic}_{X/S} = \mathbf{Pic}_{X/S}^\Phi$ parameterising line bundles on S wirh Hilbert polynomial Φ, then to say that $(\mathbf{1}\times f)^*\mathscr{L}\otimes\pi_Z^*\mathscr{M} \cong (\mathbf{1}\times g)^*\mathscr{L}$ for some line bundle $\mathscr{M}$ on Z is essentially equivalent[4] to saying that the maps $f, g\colon Z \rightrightarrows \mathbf{Div}$ fit into a commutative diagram of functors

$$\begin{array}{ccc} Z & \xrightarrow{\;g\;} & \mathbf{Div} \\ {\scriptstyle f}\downarrow & & \downarrow \\ \mathbf{Div} & \longrightarrow & \mathbf{Pic}_{X/S} \end{array}$$

where the map $\mathbf{Div} \to \mathbf{Pic}_{X/S}$ (occurring twice in the diagram) is the one arising from the line bundle $\mathscr{L}$ on $X_{\mathbf{Div}}$. The fact that such a diagram results in a unique map θ such that $p \circ \theta = f$ and $q \circ \theta = g$ means that $\mathbb{P} = \mathbf{Div} \times_{\mathbf{Pic}_{X/S}} \mathbf{Div}$ and p and q are the natural projections of this fibre product.

$$\begin{array}{ccc} \mathbb{P} & \xrightarrow{\;q\;} & \mathbf{Div} \\ {\scriptstyle p}\downarrow & \square & \downarrow \\ \mathbf{Div} & \longrightarrow & \mathbf{Pic}_{X/S} \end{array}$$

This means that $\mathbb{P}$ is a *scheme theoretic equivalence relation* on $\mathbf{Div}$, which is flat projective and surjective. Unpackaged, we are saying p and q are flat (in fact they are smooth), projective, surjective, and for each S-scheme T, the scheme map $\mathbb{P} \to \mathbf{Div} \times_S \mathbf{Div}$ yields an inclusion $\mathbb{P}(T) \hookrightarrow (\mathbf{Div} \times_S \mathbf{Div})(T) = \mathbf{Div}(T) \times \mathbf{Div}(T)$ which is an equivalent relation on $\mathbf{Div}(T)$. For notational convenience, when we think of $\mathbb{P}$ as a scheme-theoretic equivalence relation, we write R for it. From what we have said, essentially $\mathbf{Pic}_{X/S} = \mathbf{Div}/R$. Since R is flat and projective, and $\mathbf{Div}$ is quasi-projective, a theorem of Grothendieck says that $\mathbf{Div}/R$ exists as a scheme. Thus $\mathbf{Pic}_{X/S}$ is representable. This in a nutshell is the Grothendieck-Matsusaka method for constructing the relative Picard scheme under the hypotheses we have given. Details involve descent, fppf and étale topologies, sheafifications in these topologies, Hilbert schemes ....

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[4]"essentially equivalent" rather than "equivalent" because strictly speaking we need to say that $(\mathbf{1} \times f)^*\mathscr{L} \otimes \pi_Z^*\mathscr{M} \cong (\mathbf{1} \times g)^*\mathscr{L}$ holds after pulling back to an étale or fppf cover of Z.