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The Hilbert scheme of divisors on a curve

Let k be a field and X a complete smooth curve $/k$, geometrically irred. of genus g, s.t. $X(k) \neq \phi$. Fix $n \geq 1$ and a line bundle L_0 of deg $2g+n$.

$$\deg L_0 = 2g+n.$$

Note, $h^1(L_0) = 0$, and L_0 is generated by global sections. Since $n \geq 1$, L_0 is very ample. Note, such an L_0 exists since $X(k) \neq \phi$.

Let $V = H^0(X, L_0)$. Have $\dim V = g+n+1 =: N$ (say).

Set $G = G_n(V)$, the Grassmannian of n-dim'l quotients of V.

Have universal exact sequence of v.b.'s on G

$$0 \longrightarrow K_u \longrightarrow V \otimes_k \mathcal{O}_G \longrightarrow Q_u \longrightarrow 0 \qquad (\mathcal{E}_u)$$

where Q_u is a rank n vector bundle (the subscript u is of "universal").

Let $X_G = X \times_k G$, and $p_1 : X_G \longrightarrow X$, $p_2 : X_G \longrightarrow G$ be the two projections. Since L_0 is gen'd by global sections, we have a surjection:

$$\begin{array}{ccc} V \otimes_k \mathcal{O}_{X_G} & \xrightarrow{\varphi} & p_1^* L_0 , \\ \| & & \\ p_2^*(V \otimes_k \mathcal{O}_G) & & \end{array}$$

s.t. on applying p_{2*} to the surjection we get the identity map on $V \otimes_k \mathcal{O}_G$. In greater detail, we have

$$\begin{array}{ccc} p_{2*}(V \otimes_k \mathcal{O}_{X_G}) & \xrightarrow{p_{2*}\varphi} & p_{2*}(p_1^* L_0) \\ \| & & \| \text{ (flat base-change)} \\ V \otimes_k p_{2*}\mathcal{O}_{X_G} = V \otimes_k \mathcal{O}_G & & \pi_G^* \pi_{X*} L_0 = V \otimes_k \mathcal{O}_G . \end{array}$$

$$\begin{tikzcd} X_G \arrow[r, "p_1"] \arrow[d, "p_2"'] \arrow[dr, phantom, "\square"] & X \arrow[d, "\pi_X"] \\ G \arrow[r, "\pi_G"'] & \operatorname{Spec} k \end{tikzcd}$$

In fact, again using the fact that flat base change one has $p_{2*} p_1^* L_0 = \pi_G^* \pi_{X*} L_0$, we see that for any $z \in G$, the map $\Gamma(X \times \{z\}, \varphi|_{X \times \{z\}})$ yields the identity map on $V \otimes_k k(z)$.

We therefore have an $\mathcal{O}_{X_G}$-module map

$$p_2^* K_U \longrightarrow p_1^* L_0$$

given by the composite

$$p_2^* K_U \hookrightarrow V \otimes_k \mathcal{O}_{X_G} \xrightarrow{\varphi} p_1^* L_0.$$

Let

$$C := \operatorname{coker}(p_2^* K_U \longrightarrow p_1^* L_0).$$

The surjection $p_1^* L_0 \twoheadrightarrow C$ of $\mathcal{O}_{X_G}$-modules gives rise to a surjection of $\mathcal{O}_{X_G}$-modules

$$\mathcal{O}_{X_G} \twoheadrightarrow C \otimes_{\mathcal{O}_{X_G}} p_1^* L_0^{-1}$$

and hence we get a closed subscheme of X_G:

$$D' \hookrightarrow X_G$$

The ideal sheaf of D' is the kernel of $\mathcal{O}_{X_G} \twoheadrightarrow C \otimes_{\mathcal{O}_{X_G}} p_1^* L_0^{-1}$.

Claim: $D' \xrightarrow{\text{via } p_2} G$ is a finite map.

Proof of Claim: The map $D' \longrightarrow G$ is clearly proper since it is the composite of a closed immersion followed by p_2, which is proper. Therefore it is enough to show that $D' \longrightarrow G$ is quasi-finite. It is sufficient to show that the map $\mathcal{I}|_{X \times \{z\}} \longrightarrow \mathcal{O}_{X_G}|_{X \times \{z\}} = \mathcal{O}_{X \times \{z\}}$ is non-zero for every $z \in G$ where $\mathcal{I}$ is the ideal sheaf of D' in $\mathcal{O}_{X_G}$. This is where we need the fact that X is geometrically irreducible, for this implies that $X \times \{z\}$ $(= X \otimes k(z))$ is integral. Now $p_2^* K_U|_{X \times \{z\}}$ is free $\mathcal{O}_{X \times \{z\}}$-module of rank $N-n$, and the map $p_2^* K_n|_{X\times\{z\}} \longrightarrow p_1^* L_0|_{X\times\{z\}} = L_0 \otimes k(z)$, is non-zero, for upon taking global sections, one simply has the inclusion $\Gamma(p_2^* K_n|_{X\times\{z\}}) \hookrightarrow V\otimes_k k(z)$. This proves the claim.

Next, let $H^{(n)}$ be the stratum in G on which the coherent $\mathcal{O}_G$-module $\mathcal{F}' := p_{2*}\mathcal{O}_{\mathcal{D}'}$ is flat (whence locally free) of rank n. Such a stratum exists by elementary considerations (see proof given below). Let

$$\begin{aligned} \mathcal{D}^{(n)} &:= \text{inverse image } H^{(n)} \text{ in } \mathcal{D}' \text{ under } p_2 : \mathcal{D}' \longrightarrow G \\ &= H^{(n)} \times_G \mathcal{D}' \end{aligned}$$

If $\mathcal{F} := \mathcal{F}'|_{H^{(n)}}$, then a little thought shows that

$$\mathcal{F} = p_{2*}\mathcal{O}_{\mathcal{D}^{(n)}}$$

where we are writing p_2 for the map $\mathcal{D}^{(n)} \longrightarrow H^{(n)}$ in a (standard?) abuse of notation. We thus have the following commutative diagram

$$\begin{tikzcd} \mathcal{D}^{(n)} \arrow[r, hook] \arrow[dr, "\substack{\text{finite, flat, with} \\ \mathcal{F} = p_{2*}\mathcal{O}_{\mathcal{D}^{(n)}} \text{ of rank } n \\ \text{as an } \mathcal{O}_{H^{(n)}}\text{-module.}}"'] & X_{H^{(n)}} \arrow[d, "p_2"] \\ & H^{(n)} \end{tikzcd} \qquad \text{———— } (*)$$

Claim: $H^{(n)}$ is the Hilbert scheme of effective degree n divisors on X and diagram $(*)$ is the universal family of effective degree n divisors on X parametrised by $H^{(n)}$.

Proof:

Let T be a k-scheme and $D \hookrightarrow X_T$ a closed subscheme such that $D \longrightarrow T$ is flat, finite, with $\mathcal{O}_D \otimes_{\mathcal{O}_T} k(t)$ an n-dimensional $k(t)$-vector space for every $t\in T$. Consider the commutative diagram with the square being the standard cartesian square

$$\begin{tikzcd} \mathcal{D} \arrow[r, hook] \arrow[dr, "\substack{\text{finite flat of} \\ \text{``rank } n\text{''}}"'] & X_T \arrow[r, "q_1"] \arrow[d, "q_2"'] \arrow[dr, phantom, "\square"] & X \arrow[d] \\ & T \arrow[r] & \operatorname{Spec} k \end{tikzcd}$$

We have an exact sequence

$$0 \longrightarrow \mathcal{O}_{X_T}(-\mathcal{D}) \longrightarrow \mathcal{O}_{X_T} \longrightarrow \mathcal{O}_{\mathcal{D}} \longrightarrow 0,$$

whence an exact sequence

$$(\dagger) \cdots \quad 0 \longrightarrow \underset{\mathcal{M}}{\underset{\parallel}{\mathcal{O}_{X_T}(-\mathcal{D}) \otimes q_1^* L_0}} \longrightarrow q_1^* L_0 \longrightarrow q_1^* L_0 \big|_{\mathcal{D}} \longrightarrow 0.$$

For $t \in T$, let $\mathcal{D}_t := \mathcal{D} \otimes k(t) = \mathcal{D}\big|_{X \times \{t\}}$. Then $\mathcal{D}_t$ is an effective degree n divisor on $X_t := X \times \{t\}$. Moreover the line bundle

$$M_t = \mathcal{O}(-\mathcal{D}_t) \otimes_{\mathcal{O}_{X_t}} (L_0 \otimes k(t))$$

on X_t (obtained by restricting $\mathcal{M} = \mathcal{O}_{X_T}(-\mathcal{D}) \otimes q_1^* L_0$ to X_t) is of degree $2g$ (since L_0 has degree $2g+n$ and $-\mathcal{D}_t$ has degree $-n$). In particular $h^1(M_t) = 0$ and M_t is generated by global sections. These properties are of course also enjoyed by $L_0 \otimes k(t)$, $t \in T$. Standard semi-continuity then gives that $R^1 q_{2*} \mathcal{M} = R^1 q_{2*}(q_1^* L_0) = 0$, $q_{2*}\mathcal{M}$ is a vector bundle on T of rank $g+1$, and $q_{2*}(q_1^* L_0)$ is a vector bundle of rank $N = n+g+1$. In fact $q_{2*}(q_1^* L_0) = V \otimes_k \mathcal{O}_T$, by flat base change. By hypothesis $F = q_{2*}(q_1^* L_0 \big|_{\mathcal{D}})$ is a vector bundle of rank n on T (since $q_{2*}(\mathcal{O}_{\mathcal{D}})$ is such a vector bundle). We thus have an exact sequence of vector bundles on T:

$$0 \longrightarrow q_{2*}\mathcal{M} \longrightarrow V\otimes_k \mathcal{O}_T \longrightarrow F \longrightarrow 0 \qquad \text{———} \quad (\mathcal{E}_T)$$

and the ranks of these v.b.'s are such that the uni— property of $(G = G_n(V), (\mathcal{E}_n))$ applies and we obtain a (unique) classifying map

$$\gamma : T \longrightarrow G$$

such that $\gamma^*(\mathcal{E}_n) \cong (\mathcal{E}_T)$ (with $\gamma^*(V\otimes_k \mathcal{O}_G)$ being canonically identified with $V\otimes_k \mathcal{O}_T$). Moreover the argument for showing that the natural map $V\otimes_k \mathcal{O}_{X_G} = p_2^* p_{2*} p_1^* L_0 \twoheadrightarrow p_1^* L_0$ being a surjection applies <u>mutatis mutandis</u> and we have surjections $q_2^* q_{2*}\mathcal{M} \longrightarrow \mathcal{M}$, $V\otimes_k \mathcal{O}_{X_T} \twoheadrightarrow q_1^* L_0$. We therefore have a comm. diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & (1\times\gamma)^* p_2^* K_n \arrow[r] \arrow[d, equal] & (1\times\gamma)^* p_2^* V\otimes_k \mathcal{O}_G \arrow[r] \arrow[d, equal] & (1\times\gamma)^* p_2^* Q_n \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & q_2^* \gamma^* K_n \arrow[r] \arrow[d, "\wr"] & V\otimes_k \mathcal{O}_{X_T} \arrow[r] \arrow[d, equal] & q_2^* \gamma^* Q_n \arrow[r] \arrow[d, "\wr"] & 0 \\
0 \arrow[r] & q_2^* q_{2*}\mathcal{M} \arrow[r] \arrow[d] & V\otimes_k \mathcal{O}_{X_T} \arrow[r] \arrow[d, equal] & q_2^* q_{2*}\mathcal{O}_{\mathcal{D}(n)} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{M} \arrow[r] & V\otimes_k \mathcal{O}_{X_T} \arrow[r] & \mathcal{D} \arrow[r] & 0
\end{tikzcd}$$

(the downward arrow on the south-east corner being obviously surjective). In particular

$$\begin{aligned}
\mathcal{O}_{\mathcal{D}} &= \operatorname{coker}\left(q_2^* q_{2*}\mathcal{M} \longrightarrow V\otimes_k \mathcal{O}_{X_T}\right). \\
&= (1\times\gamma)^* \operatorname{coker}\left(p_2^* K_n \longrightarrow V\otimes_k \mathcal{O}_{X_G}\right) \\
&= (1\times\gamma)^* \mathcal{O}_{\mathcal{D}(n)}.
\end{aligned}$$

Thus

$$\mathcal{D} = (1\times\gamma)^{-1}\left(\mathcal{D}^{(n)}\right).$$

This proves the claim.

The Flattening Stratification business:

Let A be a (noetherian) ring and M a finite module. For each $\mathfrak{p}\in \operatorname{Spec} A$, let $e(\mathfrak{p}) := \dim_{k(\mathfrak{p})}(M\otimes k(\mathfrak{p}))$. Let $n\geq 0$ be an integer, and $\mathfrak{p}$ a prime ideal of A such that $e(\mathfrak{p})=n$. By Nakayama's lemma we have a surjection

$$A_{\mathfrak{p}}^{n} \twoheadrightarrow M_{\mathfrak{p}}.$$

It follows we have $f\in A\smallsetminus\mathfrak{p}$ and a surjection.

$$A_f^n \xrightarrow{\psi} M_f.$$

$\operatorname{Spec} A_f$ is an affine open neighborhood of $\mathfrak{p}$. Call it $U_{\mathfrak{p}}$. Clearly $e(\mathfrak{q})\leq n$ $\forall\ \mathfrak{q}\in U_{\mathfrak{p}}$. Moreover, since we are working with noetherian rings (otherwise assume M finitely presented) we have a presentation

$$A_f^m \xrightarrow{\theta} A_f^n \xrightarrow{\psi} M_f \longrightarrow 0 \qquad (**)$$

Now θ is an $n\times m$ matrix with entries from A_f. Let I be the A_f ideal generated by these entries. Let $B = A_f/I$. It is clear that $\psi\otimes_{A_f} B$ is an isomorphism. Let $Z_{\mathfrak{p}}$ be the closed subscheme of $U_{\mathfrak{p}}$ defined by B. It is easy to see (since presentations of the form $(**)$ pull-back well) that if $W\xrightarrow{f} U_{\mathfrak{p}}$ is a map of schemes, such that $f^*(\tilde{\psi})$ is an isomorphism, then W factors through $Z_{\mathfrak{p}}$ uniquely. Patching the pairs $(U_{\mathfrak{p}}, Z_{\mathfrak{p}})$ as $\mathfrak{p}$ varies over prime ideals $\mathfrak{p}$ with $e(\mathfrak{p})=n$, we see that there is a locally closed subscheme Z_n of $\operatorname{Spec} A$ such that $\tilde{M}|_{Z_n}$ is locally free of rank n, and if $Y \xrightarrow{g} \operatorname{Spec} A$ is any map of schemes s.t. $g^*\tilde{M}$ is locally free of rank n, then g factors through Z_n. This can be further upgraded to the following situation. Let S be a (noetherian scheme), $\mathcal{F}$ a coherent $\mathcal{O}_X$-module, and n a non-negative integer. Then there exists a locally closed subscheme Z_n of S such that if $g: Y \longrightarrow S$ is a map of schemes such that $g^*\mathcal{F}$ is locally free on Y of rank n, then g factors uniquely throug Z.

$H^{(n)}$ is smooth:

Let $h \in H^{(n)}$ be a point. (since k is not assumed algebraically closed, we cannot assume $k \longrightarrow k(h)$ is an isomorphism even if h is a closed point, and so there is no advantage to assuming h is a closed point.) To lighten notation we write

$$H = H^{(n)}, \quad \mathcal{D} = \mathcal{D}^{(n)}, \quad X_h = X \times_k \operatorname{Spec} k(h).$$

Let $\mathcal{D}_h = \mathcal{D}|_{X_h}$, or, more precisely,

$$\mathcal{D}_h = \mathcal{D} \times_H \operatorname{Spec} k(h).$$

Note that $\mathcal{D}_h$ is an effective Cartier divisor on X_h. Note also that since X is geometrically integral (by hypothesis), X_h is integral.

Now

$$\mathcal{D}_h = D_1 + \dots + D_m \qquad \text{———} (**)$$

where each D_i is supported on a closed point (say $P_i \in X_h$), these points being distinct.

We will show that $\mathcal{O}_{H,h}$ is geometrically regular over k. For this it is enough to show that the completion $A = \hat{\mathcal{O}}_{H,h}$ of $\mathcal{O}_{H,h}$ is geometrically regular over k. We use the "m-smoothness" version of infinitesimal lifting to prove this. We have to show the following (see Theorem 28.7 on p.219 of Matsumura's "Commutative Ring Theory"):

If B is a k-algebra, I an ideal of B such that $I^2 = 0$, and we have a k-algebra map $\theta_0 : A \longrightarrow B_0$ with $\theta_0(\mathfrak{m}_A^{\nu}) = 0$ for $\nu \gg 0$ (i.e. θ_0 is continuous for the $\mathfrak{m}_A$-adic topology on A and the discrete topology on B) then θ_0 lifts to a k-algebra map $\theta : A \longrightarrow B$.

Note the requirement of continuity on θ_0. This is slightly different from the usual infinitesimal lifting – but required in the "formal" situation.

Notations: For any k-algebra C, let $X_C = X_{\operatorname{Spec} C}$. If C is an $\mathcal{O}_{H,h}$-algebra, then we write $\mathcal{D}_C$ for the pull-back of $\mathcal{D}$ to X_C, i.e.

$$\mathcal{D}_C := \mathcal{D} \otimes_{\mathcal{O}_{H,h}} C.$$

Note that $\mathcal{D}_C \longrightarrow \operatorname{Spec} C$ is a flat family of effective degree n divisors on X parameterised by $\operatorname{Spec} C$, and since the map is finite, $\mathcal{D}_C$ is affine.

Since $\mathcal{D}_A$ is finite over $\operatorname{Spec} A$, and A is complete, therefore by Serre's version of the Weierstrass Preparation Theorem, if $\mathcal{D}_A = \operatorname{Spec} R$, then $R = \prod_{i=1}^{m} R_i$, where each R_i is a complete local ring. Moreover, R being flat over A forces each R_i to be flat over A. Note that the integer m occurring in the product

$R = \prod_{i=1}^{m} R_i$ is the same as the integer m occurring in (**), since $\mathcal{D}_h$ is the closed fibre of $\mathcal{D}_A \twoheadrightarrow \operatorname{Spec} A$. Let

$$\mathcal{D}_i = \operatorname{Spec}(R_i) \qquad i = 1, \ldots, m$$

and assume that the subscripts used are compatible with the subscripts in (**). In other words, we choose our subscripts so that

$$\mathcal{D}_i \otimes_A k(h) = D_i \qquad i = 1, \ldots, m.$$

The point $P_i \in X_h$ on which D_i is supported maps to a closed point $\bar{P}_i \in X$ under the natural map $X_h \longrightarrow X$. Pick affine open $U^1, \ldots, U^m$ in X such that $\bar{P}_i \in U^i$, and if $j \neq i$, $\bar{P}_i \notin U^j$. Let $U^1_h, \ldots, U^m_h$ be the inverse images of $U^1, \ldots, U^m$ in X_h. Then U^i_h are affine open in X_h, $P_i \in U^i_h$, and for $j \neq i$, $P_i \notin U^j_h$. As usual, for any k-algebra C, we write $U^i_C = U^i \times_k \operatorname{Spec}(C)$, and these are all affine schemes. In particular, we have A-algebras $S_1, \ldots, S_m$ such that

$$U^i_A = \operatorname{Spec}(S_i) \qquad i = 1, \ldots, m.$$

Moreover $\mathcal{D}_i \subseteq U^i_A$, but $\mathcal{D}_i \cap U^j_A = \phi$ if $i \neq j$, and we have a surjective map of A-algebras

$$S_i \twoheadrightarrow R_i \qquad i = 1, \ldots, m$$

defining the closed immersion $\mathcal{D}_i \hookrightarrow U^i_A$. By shrinking $\bar{U}_i$ around $\bar{P}_i$ if necessary, we may assume that $\ker(S_i \twoheadrightarrow R_i)$ is principal, say generated by $s_i \in S_i$. Note s_i is a non-zero divisor in S_i.

Now let us return to our map of k-algebras $\theta_0 : A \longrightarrow B_0$, where $B_0 = B/I$, B a k-algebra, I a B-ideal such that $I^2 = 0$. Recall that we required $\theta_0(M_A^{\nu}) = 0$ for $\nu \gg 0$. We wish to lift θ_0 to a

k-algebra map $\theta : A \longrightarrow B$.

Write

$$U^i_B = \operatorname{Spec}(S_{i,B}) \qquad i = 1, \ldots, m.$$

Since $U^i_{B_0}$ is a closed subscheme of U^i_B, we have a surjection of B-algebras $S_{i,B} \twoheadrightarrow S_i \otimes_A B_0$. Let $\sigma_i \in S_{i,B}$ be any lift of $s_i \otimes 1 \in S_i \otimes_A B_0$. Clearly σ_i is a non-zero divisor in $S_{i,B}$, (we are using the geometric irreducibility of $U_i \subseteq X$ here), and the local flatness criterion (see e.g., the Corollary above Thm 22.6 on p. 172 of Matsumura's "Comm. Ring. Theory") shows that $S_{i,B}/\sigma_i S_{i,B}$ is flat over B. If $D_{i,B} := \operatorname{Spec}(S_{i,B})$ then we have $D_{i,B}$ is an effective relative Cartier divisor in U^i_B (whence in X_B) over $\operatorname{Spec} B$. Let

$$D_B = D_{1,B} + \cdots + D_{m,B}.$$

where the sum is regarded as the sum of effective relative Cartier divisors on the B-scheme X_B. It is clear that $D_B \times_B \operatorname{Spec}(B_0) = D_{B_0}$ where the right side makes sense since B_0 is an $\mathcal{O}_{H,h}$-algebra, which a-priori B isn't (see notations given above for $\mathcal{O}_{H,h}$-algebras). In fact D_B is a flat family of degree n divisors parametrised by $\operatorname{Spec} B$. We therefore have a unique map $r_B : \operatorname{Spec} B \longrightarrow H$ such that $(1 \times r_B)^{-1}(D) = D_B$. Using, once again, the universal property of (H, D) we have a commutative diagram

$$\begin{tikzcd}
\operatorname{Spec} B \arrow[r, "r_B"] & H \\
\operatorname{Spec} B_0 \arrow[u, hook] \arrow[r] & \operatorname{Spec} A \arrow[u]
\end{tikzcd}$$

where the unlabelled arrows are the obvious ones. Consider the composite $\operatorname{Spec} B_0 \longrightarrow \operatorname{Spec} A \longrightarrow H$. If $y \in \operatorname{Spec} B_0$, then either y maps to h or to a generalisation of h in H, since it factors through $\operatorname{Spec} A$, and whence through $\operatorname{Spec}(\mathcal{O}_{H,h})$. The underlying continuous map for the just considered map $\operatorname{Spec}(B_0) \longrightarrow H$ is the same as that of γ_B, since as topological spaces, $\operatorname{Spec}(B_0)$ and $\operatorname{Spec}(B)$ are identical and the above diagram commutes. It follows that if V is any open subscheme of H containing h, then γ_B factors as $\operatorname{Spec} B \longrightarrow V \subseteq H$. Thus γ_B can be written (scheme theoretically) as a composite $\operatorname{Spec} B \longrightarrow \operatorname{Spec}(\mathcal{O}_{H,h}) \xrightarrow{\text{natural}} H$. Let $\mathcal{O}_{H,h} \xrightarrow{\gamma_h} B$ be the corresponding map of rings. We have then a commutative diagram

$$\begin{array}{ccc} \mathcal{O}_{H,h} & \xrightarrow{\gamma_h} & B \\ \downarrow & & \downarrow \\ A & \xrightarrow[\theta_0]{} & B_0 \end{array}$$

Since $\theta_0(m_A^{\nu}) = 0$ for $\nu >> 0$, and $A/m_A^{\nu} = \mathcal{O}_{H,h}/m_h^{\nu}$, we see that $\gamma_h(m_h^{\nu}) \subseteq I$ for $\nu >> 0$. Since $I^2 = 0$, this further means $\gamma_h(m_h^{\nu}) = 0$ for $\nu >> 0$. In particular γ_h factors as $\mathcal{O}_{H,h} \twoheadrightarrow \mathcal{O}_{H,h}/m_h^{\nu} \xrightarrow{\theta_\nu} B$ for some ν.
Let $\theta : A \longrightarrow B$ be the composite

$$A \twoheadrightarrow A/m_A^{\nu} = \mathcal{O}_{H,h}/m_h^{\nu} \xrightarrow{\theta_\nu} B$$

Then θ lifts θ_0 and we are done.

P. T. O. $\longrightarrow$

$H^{(n)}$ is complete geometrically irreducible and of dimension n:

Since $X \longrightarrow \operatorname{Spec} k$ is geometrically irreducible, so is the m-fold product

$$X^m := \underbrace{X \times_k \cdots \times_k X}_{m\text{-times}}$$

for every $m \geq 1$. Moreover X^m is smooth, complete and of dimension m.

Let $m = n+1$. Write

$$X^m = X \times_k X^n.$$

Let $\pi_0, \pi_1, \ldots, \pi_n$ be the projections $X \times X^n \longrightarrow X$ in an obvious notation, and $\pi = (\pi_1, \ldots, \pi_n) : X \times_k X^n \longrightarrow X^n$ the projection to the last n factors. We have sections

$$s_i : X^n \longrightarrow X \times_k X^n \quad , \; i = 1, \ldots, n$$

of π given by

$$s_i(x_1, \ldots, x_n) = (x_i, x_1, \ldots, x_n)$$

for $x = (x_1, \ldots, x_n) : T \longrightarrow X^n$ a T-valued point of X^n.

Since $X^{n+1} \xrightarrow{\pi} X^n$ is smooth, $D_i^\pi = s_i(X^n)$ are smooth relative Cartier divisors over X^n for the map π.

Let

$$D^\pi = D_1^\pi + \cdots + D_n^\pi.$$

From the universal property of $(\mathcal{H}^{(n)}, \mathcal{D}^{(n)})$ we get a unique classifying map

$$\Psi : X^n \longrightarrow H^{(n)}$$

such that $(1 \times \Psi)^{-1}(\mathcal{D}^{(n)}) = D^\pi$.

It is easy to see Ψ is surjective. Indeed if $h \in H = H^{(n)}$, and if $D_h \subset X_h$ is the divisor corresponding to h $(D_h = \mathcal{D}^{(n)} \times_H \operatorname{Spec} k(h))$, then we can find a finite field extension $k(h) \longrightarrow K$ on which $D_K := \mathcal{D}^{(n)} \times_H \operatorname{Spec} K$ is supported on K rational points, and hence $D_K = P_1 + \cdots + P_n$, where $P_i \in X_K$ are K-rational points, perhaps not distinct. The P_i can be regarded as maps $P_i = \operatorname{Spec} K \longrightarrow X$, whence we have a point $(P_1, \ldots, P_n) \in X^n_K := X^n \times_k K$. It is clear that the composite $X^n_K \longrightarrow X^n \longrightarrow H$ sends $(P_1, \ldots, P_n)$ to h. If $Q \in X^n$ is the image of $(P_1, \ldots, P_n)$ then clearly $\Psi(Q) = h$.

It is immediate that H is complete, connected being the surjective image of X^n. In particular it is irreducible, being smooth. Moreover, if $k \longrightarrow K$ is a field extension then H_K is clearly (by universal properties) the Hilbert scheme on effective degree n divisors on X_K, and since X_K is geometrically irreducible, H_K is irreducible by our just concluded argument. Thus H is geometrically irreducible.

It is easy to see that $X^n \xrightarrow{\Psi} H$ is finite. In fact, we have essentially given the argument when we showed Ψ is surjective, and a little thought shows that the fibres have cardinality $\leq n!$ (the order of the symmetric group S_n), and the generic fibre of Ψ in fact has cardinality $n!$.

Exercises: Let $\mathcal{D}^e \subseteq \mathcal{D}^{(n)}$ be the locus on which the finite surjective map $\mathcal{D}^{(n)} \longrightarrow H^{(n)}$ is étale.

1. Show that there exists $h \in H^{(n)}$ such that $D^n|_{X_h}$ lies entirely in D^e (in particular D^e is non-empty).

[Hint: Make a base change $k \longrightarrow K$ where $K = \bar{k}$, and consider the locus on which $D^{(n)}_K$ consists of divisors supported on n distinct points of X_K.]

2. Let $Z = D^{(n)} \smallsetminus D^{(e)}$, $\bar{Z} = p_2(Z) \subseteq H^{(n)}$, and $E = H^{(n)} \smallsetminus \bar{Z}$. Show that E is non-empty and that $D_E \longrightarrow E$ is étale, where (as always) $D_E = D^{(n)} \times_{H^{(n)}} E$. (Note that E is an open subscheme of H.)

3. For distinct $i, j \in \{1, \dots, n\}$, set $Z_{ij} = D_i^{\pi} \cap D_j^{\pi}$ in X^n (see above for the definition of $D_i^{\pi} \subset X^n$). Let

$$\Gamma = \bigcup_{i \neq j} Z_{ij}$$

and let V be the open subscheme of X^n given by

$$V := X^n \smallsetminus \Gamma.$$

Show that $\Psi^{-1}(E) = V$, where E is as in Problem 2, and $\Psi : X^n \longrightarrow H^{(n)}$ the map we defined earlier.

4. Show that the geometric fibres of $V \xrightarrow{\Psi} E$ have cardinality $n!$

5. Suppose $X(k) = \phi$ (X smooth, complete, geom. irred $/k$). By choosing a line bundle L_0 of degree larger than $2g+n$ (for example $\omega_X^{\otimes m}$ for suitable m, $g > 1$) one can again construct $H^{(n)}$ with the same universal property. Does $H^{(1)}$ have k-rational points?