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Line bundles, linear systems, & maps to $\mathbb{P}^n$

This note is to help you think about some aspects of line bundles and divisors. There are no (or very few) proofs in this note.

Let k be an alg. closed field and V a finite dimensional vector space. The projective space of lines through the origin in V is denoted $\mathbb{P}(V)$. Scheme theoretically there are various ways of defining $\mathbb{P}(V)$. Here are some.

1. $\mathbb{P}(V) := \mathrm{Proj}(k[V])$, where $k[V]$ is the polynomial ring associated to V. More precisely, for $n \geq 0$, let $T^n(V^*) = V^* \otimes_k \cdots \otimes_k V^*$, the n-fold tensor product of V^* with itself (with $T^0(V^*) = k$) and consider the tensor algebra (V^* = dual of V)

$$T(V^*) = \bigoplus_{n \geq 0} T^n(V^*)$$

with obvious product. Then

$$k[V] := T(V^*)/I$$

where I is the two sided ideal generated by elements of the form $f \otimes g - g \otimes f$, $f, g \in V^*$. The alg $k[V]$ has the following universal property: if A is a k-algebra and we have a k-linear map $\phi: V^* \longrightarrow A$, then ϕ extends to a k-algebra map $k[V] \longrightarrow A$. The elements of $k[V]$ are regarded a "functions on V" (details left to you). If we pick a basis $v_1^*, \ldots, v_m^*$ of V^* (equivalently, a basis $v_1, \ldots, v_m$ of V) then we have an isomorphism $k[V] \xrightarrow{\sim} k[x_1, \ldots, x_m]$ with

$v_i^* \longmapsto x_i$.

2. One can also define $\mathbb{P}(V)$ as the Grassmannian of one-dim'l subspaces of V, or, what is the same thing, one-dim'l quotients of V^*. In greater detail, consider the functor

$$\mathcal{P}(V) : \mathrm{Sch}_{/k}^{\circ} \longrightarrow (\mathrm{Sets}) \qquad (\mathrm{Sch}_{/k}^{\circ} = \text{opp. cat of } \mathrm{Sch}_{/k})$$

given by

$\mathcal{P}(V)(T) =$ equivalence class of quotients of the form $V^* \otimes_k \mathcal{O}_T \twoheadrightarrow \mathcal{L}$, where $\mathcal{L}$ is a line bundle on T.

Here, two quotients $V^* \otimes_k \mathcal{O}_T \twoheadrightarrow \mathcal{L}$ and $V^* \otimes_k \mathcal{O}_T \twoheadrightarrow \mathcal{M}$ are considered equivalent if there is an isomorphism $\mathcal{L} \xrightarrow{\sim} \mathcal{M}$ s.t. the following diagram commutes:-

$$\begin{tikzcd} & \mathcal{L} \arrow[d, "\wr"] \\ V^* \otimes_k \mathcal{O}_T \arrow[ur, two heads] \arrow[r, two heads] & \mathcal{M} \end{tikzcd}$$

It is not hard to show $\mathcal{P}(V)$ is representable, and one can do it without the Proj construction. The representing scheme is then the definition of $\mathbb{P}(V)$ in this approach.

The result in [H, p. 150, Thm 7.1] (where [H] = Hartshorne's Alg. Geom) is a way of saying these two constructions yield the same thing (take $A = k$ in loc. cit.)

Remark: Since $\mathbb{P}(V)$ represents $\mathcal{P}(V)$, by definition, for every $T \in Sch/_k$,

$$\mathcal{P}(V)(T) \cong \mathrm{Hom}_{Sch/_k}(T, \mathbb{P}(V))$$

Taking $T = \mathbb{P}(V)$, we have a universal element $\xi_u \in \mathcal{P}(V)(\mathbb{P}(V))$ corresponding to the identity map in $\mathrm{Hom}_{Sch/_k}(\mathbb{P}(V), \mathbb{P}(V))$.

ξ_u is an equiv. class of a quotient $V^* \otimes_k \mathcal{O}_{\mathbb{P}(V)} \longrightarrow \mathcal{L}_u$. (The subscript u in ξ_u and $\mathcal{L}_u$ is for "universal") and $\mathcal{L}_u$ is denoted commonly as $\mathcal{O}_{\mathbb{P}(V)}(1)$, or simply as $\mathcal{O}(1)$.

Now suppose $\xi \in \mathcal{P}(V)(T)$, say ξ is represented by

$$V^* \otimes_k \mathcal{O}_T \twoheadrightarrow \mathcal{L}. \qquad (\mathcal{L} \text{ a line bdle on } T)$$

Then, dualising, we get

$$\mathcal{L}^{-1} \hookrightarrow V \otimes_k \mathcal{O}_T$$

and this is sub-bundle, the quotient being the vector bundle dual to the kernel of $(V^* \otimes_k \mathcal{O}_T \longrightarrow \mathcal{L})$.

This means that for each k-rational point t on T, we get an inclusion $\mathcal{L}^{-1}|_{\{t\}} \hookrightarrow V$, and hence a one dim'l subspace of V. Using the naive (classical) defn of $\mathbb{P}(V)$, this gives a point $\phi(t)$ in $\mathbb{P}(V)$. So one should get a map $\phi: T \longrightarrow \mathbb{P}(V)$. Representability of $\mathcal{P}(V)$ shows exactly this.

We regard the quotient $V^* \otimes_k \mathcal{O}_T \twoheadrightarrow \mathcal{L}$ as a family of one dim'l subspace of V parametrised by T.

One further remark. In EGA, Grothendieck preferred to call $\mathbb{P}(V)$ the space of 1-dim'l quotients of V. This has some advantages, and the literature, post Grothendieck, has both conventions. So Grothendieck's $\mathbb{P}(V)$ is our $\mathbb{P}(V^*)$.

Facts: 1. $H^0(\mathbb{P}(V), \mathcal{O}_{\mathbb{P}(V)}) = k$, $H^0(\mathbb{P}(V), \mathcal{O}(1)) = V^*$ and the natural map

$$\begin{array}{ccc} H^0(\mathbb{P}(V), V^* \otimes_k \mathcal{O}_{\mathbb{P}(V)}) & \longrightarrow & H^0(\mathbb{P}(V), \mathcal{O}(1)) \\ \| & & \| \\ V^* \otimes_k H^0(\mathbb{P}(V), \mathcal{O}_{\mathbb{P}(V)}) & & V^* \\ \| & & \\ V^* & & \end{array}$$

is the identity map.

2. Recall $V^* \subseteq k[V]$. We regard elements of V^* as degree 1 polynomials on V. V^* is also regarded, from the previous section, as the space of sections of the universal line bundle $\mathcal{O}(1)$. Combining the two views, if $0 \neq f \in V^*$, then regarding f as a section of $\mathcal{O}(1)$, the locus on $\mathbb{P}(V)$ that f vanishes on is a hyperplane H. Conversely every hyperplane H in $\mathbb{P}(V)$ is the zero locus of a non-zero section f of $\mathcal{O}(1)$.

3. Hyperplanes in $\mathbb{P}(V)$ are effective Cartier divisors. Any two are linearly equivalent, and if D is an effective divisor on $\mathbb{P}(V)$ then D is a hyperplane.

4. Since $\mathbb{P}(V)$ is smooth (non-singular), every Weil divisor on $\mathbb{P}(V)$ is locally principal and hence is a Cartier divisor.

(Look up definitions if the terminology is unfamiliar)

Line bundles and divisors:

The correspondence between line bundles and divisors is something you should look up. Not in great detail. Just the fact that divisors (always considered as locally principal, i.e., as Cartier divisors) give rise to line bundles, and for the kind of schemes we are interested in, one can go the other way too (see [H, pp. 144-145, Rmk. 6.14.1 & Prop. 6.15]). What follows can be read even if you haven't looked up the correspondence.

Given a Cartier divisor D on an integral scheme X, one can define an $\mathcal{O}_X$-module, denoted $\mathcal{O}(D)$ (unfortunately denoted $\mathcal{L}(D)$ in Hartshorne as follows) as follows:

For $U \overset{\text{open}}{\subset} X$, $U \neq \emptyset$

$$\Gamma(U, \mathcal{O}_X(D)) := \left\{ f \in k(X) \,\middle|\, f \neq 0 \text{ and } (f)|_U + D|_U \text{ is effective} \right\} \cup \{0\}.$$

Here $k(X) = k(U)$ is the function field of X, namely $k(X) = \mathcal{O}_{X,\xi}$ where ξ is the generic point of X.

One checks that $\mathcal{O}_X(D)$ is a locally free $\mathcal{O}_X$-module of rank one, in other words, an invertible $\mathcal{O}_X$-module, i.e. a line bundle on X. Thus we have a map

(*) $\quad \mathrm{Div}(X) \longrightarrow \mathrm{Pic}(X)$ (= iso. classes of line bdles on X)

It turns out, two divisors D_1 and D_2 give isomorphic line bundles on X if and only if they are linearly equivalent, i.e. $\exists$ a non-zero element $f \in k(X)$ s.t. $D_1 = (f) + D_2$. The map (*) is surjective (since X is integral). The idea is, suppose $\mathcal{L}$ is a line bundle. For simplicity first assume $H^0(X,\mathcal{L}) \neq 0$. Let s be a non-zero section of $\mathcal{L}$. Let D be the scheme given by the vanishing of s. Then D is an effective Cartier divisor and $\mathcal{L} \simeq \mathcal{O}(D)$. (Incidentally, if D is an effective Cartier divisor on X, then $H^0(X,\mathcal{O}(D)) \neq 0$, and D is indeed the zero scheme of a non-zero section of $\mathcal{O}(D)$. Which one?) More generally, one looks for "meromorphic sections" of $\mathcal{L}$. If s is a meromorphic section of X, then the divisor $D = (s)$, where (s) is the divisor of zeros and poles of X, is a divisor such that $\mathcal{L} = \mathcal{O}(D)$. Hence the map $(*)$ is surjective. The kernel of $(*)$ is the subgroup of principal divisors.

Something similar can be done when X is not integral. One replaces $k(X)$ by the total quotient ring. There are technical points to be addressed. The map $(*)$ can always be defined. It need not be surjective, but has all the other properties mentioned. If X is a projective scheme over k, then $(*)$ is surjective.

Remark: If D is divisor on X, (Now assumed to be a k-variety) then $|D|$ denotes the collection of effective divisors linearly equivalent to X. $|D|$ is non-empty if and only if $H^0(X,\mathcal{O}(D)) \neq 0$, and in this case every member of $|D|$ is obtained as the zero locus of a non-zero section of $\mathcal{O}(D)$. Note that two non-zero sections s and σ give rise to the same effective divisor if $\exists$ a non-zero scalar $\alpha \in k$ s.t. $\sigma = \alpha s$. In fact this is an 'if and only if statement. It follows that $|D|$ can be identified with $\mathbb{P}(H^0(X, \mathcal{O}(D)))$.

X a projective scheme $/k$

Base point free linear systems: Fix $\mathcal{L} = \mathcal{O}(D)$, and set $V = \Gamma(X, \mathcal{L})$.

1. The collection $|D|$ is called a complete linear system.
2. If W is a non-zero vector subspace of V, then the collection of effective divisors D' consisting of zero schemes of sections of $\mathcal{L}$ which are in W is called a linear system. It is a sub linear system of the complete linear system $|D|$.
3. A non-empty linear system $|D|$ is base point free if

$$B = \bigcap_{D' \in |D|} D'$$

is empty. The set B is called the base locus of $|D|$, and it has the natural structure of a closed subscheme of X.

Fact: Let $V^* = \Gamma(X, \mathcal{O}(D))$. There is a natural map $X \xrightarrow{\varphi} \mathbb{P}(V)$ such that $|D|$ is the pull back under φ of the complete linear system $|H|$ of hyperplanes in $\mathbb{P}(V)$ if and only if $|D|$ is base point free.

This is best seen through line bundles and their sections, and below is a sketch of the ideas. For simplicity X is a complete k-variety.

First note that there is canonical map

(**) —— $\boxed{H^0(X, \mathcal{L}) \otimes_k \mathcal{O}_X \longrightarrow \mathcal{L}}$

for any line bundle $\mathcal{L}$. There are many ways to see this. The slickest way is note that by abstract nonsense and adjointness of the pair (π_*, π^*) where $\pi : X \longrightarrow \operatorname{Spec} k$ is the structure map, we have a natural map $\pi^* \pi_* \mathcal{L} \longrightarrow \mathcal{L}$, corresponding to identity map on $\pi_* \mathcal{L}$. $\left(\operatorname{Hom}_k(\pi_*\mathcal{L}, \pi_*\mathcal{L}) \xrightarrow{\sim} \operatorname{Hom}_{\mathcal{O}_X}(\pi^*\pi_*\mathcal{L}, \mathcal{L}).\right)$

This gives (**) since $\pi_*\mathcal{L} = \Gamma(X, \mathcal{L})$.

A more naive but perhaps more illuminating way is this: Let $U = \operatorname{Spec} A$ be an affine open subscheme of X, and $P = \Gamma(U, \mathcal{L})$. The map (**) restricted to U corresponds to a map $H^0(X, \mathcal{L}) \otimes_k A \longrightarrow P$, and this map is

$$s \otimes a \longmapsto a(s|_U) \qquad s \in H^0(X, \mathcal{L}),\ a \in A.$$

It is easy to check that these maps "patch" as we vary U amongst affine open subschemes of X giving (**).

Let $x \in X$ be a point (k-rational as always). Then tensoring (**) with $k(x)$ (over $\mathcal{O}_X$) we get

$$(**)_x \qquad \begin{array}{c} H^0(X, \mathcal{L}) \otimes_k k(x) \\ \| \\ H^0(X, \mathcal{L}) \end{array} \longrightarrow \mathcal{L} \otimes_{\mathcal{O}_X} k(x) = \mathcal{L}|_{\{x\}}$$

and this corresponds to $s \longmapsto s(x)$, s a section of $\mathcal{L}$.

It is, by Nakayama, clear that (**) is

surjective if and only if $(**)_x$ is surjective $\forall\, x \in X$. The last condition is the same as saying that given a point $x \in X$, there exists a global section s of $\mathcal{L}$ on X such that s does not vanish at x. This amounts to saying that there is an effective divisor D in the complete linear system represented by $\mathcal{L}$ which does not pass through x. Let $|D|$ be the linear system given by $\mathcal{L}$. We have just shown that $(**)$ is surjective if and only if $|D|$ is base point free.

Definition: $\mathcal{L}$ is said to be *generated by global sections* if $(**)$ is surjective.

We often, in a loose way, say that "$\mathcal{L}$ is base point free" instead of "$\mathcal{L}$ is gen'd by global sections".

So suppose $\mathcal{L}$ is generated by global sections. Recall we set $V^* = \Gamma(X, \mathcal{L})$, so that $V = \Gamma(X, \mathcal{L})^*$. By $(**)$ we have a surjective map

$$V^* \otimes_k \mathcal{O}_X \longrightarrow\!\!\!\!\rightarrow \mathcal{L}.$$

By our earlier discussion (regarding the above quotient a family of lines through the origin in V, parametrised by X) we get a natural map

$$\phi : X \longrightarrow \mathbb{P}(V).$$

Here, the view of $\mathbb{P}(V)$ as the Grassmannian of 1-dim'l subspaces of V is more useful than the Proj definition of $\mathbb{P}(V)$. On other occasions, the Proj construction is useful. It is important to know that the two constructions give the same object and are equivalent constructions.

Ample line bundles: $\mathcal{L}$ is said to be very ample if it is generated by global sections and the resulting map

$X \longrightarrow \mathbb{P}(\Gamma(X, \mathcal{L})^*)$ is an embedding. $\mathcal{L}$ is said to be ample if $\mathcal{L}^{\otimes n}$ is very ample for some $n \geq 1$.

Lemma: Let X be a complete variety and $\mathcal{L}$ a line bundle on X generated by global sections. Let $f: X \longrightarrow \mathbb{P}(V)$ be the resulting map where $V = \Gamma(X, \mathcal{L})^*$. A connected reduced subscheme Y contracts to a point in $\mathbb{P}(V)$ under f if and only if $\mathcal{L}|_Y$ is trivial.

Proof:

The "only if" part is obvious since $\mathcal{L} \cong \phi^* \mathcal{O}(1)$, and $\mathcal{L}|_Y$ is therefore the pull-back of $\mathcal{O}(1)|_{\{p\}}$ where $\{p\} = \phi(Y)$. All line bundles on a point are trivial.

For the converse it is enough to assume Y is irreducible by restricting to each irreducible component of the reduced connected scheme Y. Thus Y is a subvariety of X. By assumption $\mathcal{L}|_Y \simeq \mathcal{O}_Y$ whence $H^0(Y, \mathcal{L}|_Y) \cong H^0(Y, \mathcal{O}_Y) \cong k$. Thus $H^0(Y, \mathcal{L}|_Y)$ is one dimensional.

Since $\mathcal{L}$ is generated by global section

$$H^0(X, \mathcal{L}) \otimes_k \mathcal{O}_Y \longrightarrow \mathcal{L}|_Y$$

is also surjective by Nakayama's lemma (indeed from the surjectivity of $H^0(X,\mathcal{L}) \otimes_k \mathcal{O}_X \longrightarrow \mathcal{L}$, we get $H^0(X,\mathcal{L}) \otimes_k k(y) \longrightarrow \mathcal{L} \otimes_{\mathcal{O}_X} k(y) = (\mathcal{L}|_Y) \otimes_{\mathcal{O}_Y} k(y) \quad \forall\, y \in Y$).

Now consider the commutative diagram

$$\begin{tikzcd}
H^0(X,\mathcal{L}) \otimes_k \mathcal{O}_Y \arrow[d] \arrow[dr] & \\
& \mathcal{L}|_Y \\
H^0(Y,\mathcal{L}|_Y) \otimes_k \mathcal{O}_Y \arrow[ur] &
\end{tikzcd}$$

From its commutativity, it is clear that $H^0(Y,\mathcal{L}|_Y) \otimes \mathcal{O}_Y \longrightarrow \mathcal{L}|_Y$ is surjective. Since both are rank 1 line bundles, it follows that $H^0(Y,\mathcal{L}|_Y) \otimes \mathcal{O}_Y \longrightarrow \mathcal{L}|_Y$ is an isomorphism.

It follows immediately from the above commutative diagram that $H^0(X,\mathcal{L}) \otimes_k \mathcal{O}_Y \longrightarrow H^0(Y,\mathcal{L}|_Y) \otimes_k \mathcal{O}_Y$ is surjective. Tensoring with $k(y)$ for a point $y \in Y$, we see that the natural map $H^0(X,\mathcal{L}) \longrightarrow H^0(Y,\mathcal{L}|_Y)$ is surjective. Since $H^0(Y,\mathcal{L}|_Y)$ is 1 dim'l, by the universal property of $\mathbb{P}(V)$ this corresponds to a point $p \in \mathbb{P}(V)$. (Indeed, the surjection $H^0(X,\mathcal{L}) \twoheadrightarrow H^0(Y,\mathcal{L}|_Y)$ can be regarded as a surjection of vector bundles on Spec k, namely $H^0(X,\mathcal{L}) \otimes_k \mathcal{O}_{\mathrm{Spec}\,k} \longrightarrow M$, with $M = H^0(Y,\mathcal{L}|_Y)$ a line bundle on Spec k, giving a map $\mathrm{Spec}\,k \xrightarrow{p} \mathbb{P}(V)$, i.e. a point $p \in \mathbb{P}(V)$.) It is easy to see that Y contracts to this point on $\mathbb{P}(V)$ under ϕ. In greater detail, let $\psi : Y \longrightarrow \mathbb{P}(V)$ be the composite $Y \longrightarrow \mathrm{Spec}\,k \xrightarrow{p} \mathbb{P}(V)$ where the first arrow is the structure map. Then as p is the arrow corresponding to $H^0(X,\mathcal{L}) \twoheadrightarrow H^0(Y,\mathcal{L}|_Y)$, the pull back $\psi^*(H^0(X,\mathcal{L}) \otimes_k \mathcal{O}_{\mathbb{P}(V)} \twoheadrightarrow \mathcal{O}(1))$ is the surjection $H^0(X,\mathcal{L}) \otimes_k \mathcal{O}_Y \twoheadrightarrow H^0(Y,\mathcal{L}|_Y) \otimes \mathcal{O}_Y$. This is the same as the pull back of the universal surjection on $\mathbb{P}(V)$ via $\phi|_Y$. Thus $\phi|_Y = \psi$, and so $\phi|_Y$ is a constant map.