Characteristic polynomials of homogeneous equations

This is an attempt to unify the treatments of solutions of homogeneous linear ODEs with constant coefficients and of homogeneous linear recurrence relations with constant coefficients.

The abstract theory: Let k be a field and $S \subset k[X]$ a subset which is multiplicatively closed. We are mainly interested in

(a) $S =$ non-zero polynomials (for the DE case)

(b) $S =$ polynomials with non-zero constant term (for the recurrence relations case).

Let V be a k-vector space and

$$T : V \longrightarrow V$$

a linear operator such that

$$\ker(p(T)) = \deg(p(T))$$

for all non-zero polynomials p in S.

Examples

1. Let $D = \frac{d}{dt}$ be the usual differentiation operator on differentiable functions on $\mathbb{R}$. Suppose $p(x) = c_0 x^n + c_1 x^{n-1} + \dots + c_{n-1}x + c_n$ is a polynomial of degree n over $\mathbb{C}$ ($\deg p = n \Leftrightarrow c_0 \neq 0$), and suppose $g : \mathbb{R} \to \mathbb{C}$ is a solution of $P(D)f = 0$. Then

$$D^n g = -\left(c_1 D^{n-1} g + c_2 D^{n-2} g + \dots + c_{n-1} Dg + c_n g\right)$$

and hence it is clear that $D^{n+1} g$ makes sense. In fact, a little thought shows that $D^{n+k} g$ makes sense. Thus g is infinitely differentiable.

In view of the above observation, let V be the $\mathbb{C}$-vector space of infinitely differentiable functions $f : \mathbb{R} \to \mathbb{C}$. Let $p(x) = \sum_{i=0}^{n} c_i x^{n-i} \in \mathbb{C}[x]$ be as above, with $c_0 \neq 0$. It is well-known that given $(a_0, a_1, \dots, a_{n-1}) \in \mathbb{C}^n$, there exists a unique solution of the linear homogeneous

$$p(D)f = 0$$

satisfying the initial conditions

$$f(0) = a_0,\ (Df)(0) = a_1, \dots, (D^{n-1}f)(0) = a_{n-1},$$

and that these solutions depend linearly upon $(a_0, \dots, a_{n-1}) \in \mathbb{C}^n$. Thus

$$\dim_{\mathbb{C}} \ker(p(D)) = \deg p.$$

Thus with V as above, $T = D$, and S the set of non-zero polynomials, we are in the situation we are considering.

Remark: We have implicitly assumed p above is non-zero by talking about its degree.

2. Let V be the $\mathbb{C}$ vector space of maps

$$f : \mathbb{N}_0 \longrightarrow \mathbb{C}$$

where $\mathbb{N}_0 = \{0, 1, 2, \dots\}$, and

$$A : V \longrightarrow V$$

the advancement operator given by

$$(Af)(n) = f(n+1), \qquad f \in V,\ n \in \mathbb{N}_0.$$

a linear homogeneous recurrence relation with constant coefficients $c_0, c_1, \dots, c_d \in \mathbb{C}$, with $c_0 \neq 0$, $c_d \neq 0$ is an equation of the form

$$c_0 a_{n+d} + c_1 a_{n+d-1} + \dots + c_{d-1} a_{n+1} + c_d a_n = 0, \quad n \in \mathbb{N}_0.$$

This can obviously be re-written as

$$p(A) f = 0$$

where $p(x) = c_0 x^d + c_1 x^{d-1} + \dots + c_{d-1} x + c_d \in \mathbb{C}[x]$,

and $f \in V$, where complex sequence $(a_0, a_1, \dots, a_n, \dots)$ are identified with maps $f : \mathbb{N}_0 \longrightarrow \mathbb{C}$ in the obvious way.

Now, clearly, given $(\alpha_0, \dots, \alpha_{d-1}) \in \mathbb{C}^d$, there is unique solution to the equation $\sum_{i=0}^{d} c_i a_{n+d-i} = 0$, $n \in \mathbb{N}_0$, satisfying the initial conditions $a_0 = \alpha_0$, $a_1 = \alpha_1, \dots, a_{d-1} = \alpha_{d-1}$. Moreover, these solutions obviously depend linearly on $(\alpha_0, \dots, \alpha_{d-1}) \in \mathbb{C}^d$. In other words,

$$\dim_k(\ker p(A)) = \deg p.$$

If we set S equal to the subset of $\mathbb{C}[x]$ consisting of polynomials with non-zero constant terms, and $T = A$, we find ourselves in the situation we are considering.

We now state and prove our main result.

Theorem: Let $\Phi, \Psi \in S$ be s.t. $\gcd(\Phi, \Psi) = 1$. Then

$$\ker((\Phi\Psi)(T)) = \ker(\Phi(T)) \oplus \ker(\Psi(T)).$$

Proof:

Let $W_1 = \ker(\Phi(T))$, $W_2 = \ker(\Psi(T))$ and $W = \ker(\Phi\Psi(T))$.

Then $W_i \subset W$, $i = 1, 2$, whence $W_1 + W_2 \subset W$. We claim that

$$W_1 + W_2 = W_1 \oplus W_2.$$

Since $\gcd(\Phi, \Psi) = 1$, there exist $p, q \in k[T]$ s.t.

$$p\Phi + q\Psi = 1.$$

Let $w \in W_1 \cap W_2$. Then

$$\begin{aligned} w &= (p(T)\Phi(T) + q(T)\Psi(T))w \\ &= p(T)\Phi(T)w + q(T)\Psi(T)w \\ &= 0. \end{aligned}$$

Thus $W_1 \cap W_2 = 0$, proving the claim.

The theorem follows from the observation that

$$\begin{aligned} \dim_k(W_1 \oplus W_2) &= \dim_k(W_1) + \dim_k(W_2) \\ &= \deg(\Phi) + \deg(\Psi) \\ &= \deg(\Phi\Psi) \\ &= \dim_k(W) \end{aligned}$$

Since $W_1 \oplus W_2$ $(= W_1 + W_2)$ is a subspace of W, we get

$$W = W_1 \oplus W_2$$

as required. q.e.d.

The following two corollaries are useful.

Corollary 1: Let $\varphi_1, \varphi_2, \ldots, \varphi_k$ be a collection of pairwise co-prime polynomials in S. Let $\varphi = \varphi_1 \varphi_2 \cdots \varphi_k$. Then
$$\ker(\varphi(T)) = \bigoplus_{i=1}^{k} \ker(\varphi_i(T)).$$
Proof:

This is straightforward. We only need to observe that if $k \geq 2$, then $\varphi_1 \cdots \varphi_{k-1}$ and φ_k are coprime, and then apply induction. q.e.d.

Corollary 2: Let $\varphi_1, \ldots, \varphi_k$ and φ be as in Corollary 1. For each $i \in \{1, \ldots, k\}$, let $(g_{ij} \mid 1 \leq j \leq \deg(\varphi_i))$ be a basis of $\ker(\varphi_i(T))$. Then $(g_{ij} \mid 1 \leq j \leq \deg(\varphi_i),\ 1 \leq i \leq k)$ is a basis of $\ker(\varphi(T))$.

Proof:

Obvious (from Corollary 1)! q.e.d.

Solutions of $(D-\lambda)^n f = 0$

As above, let $D = \frac{d}{dt}$ be the differentiation operator.

Theorem: Let $\lambda \in \mathbb{C}$, $n \in \mathbb{N}$, $j \in \{0, 1, \ldots, n-1\}$, and $g : \mathbb{R} \to \mathbb{C}$ the map
$$g(t) = t^j e^{\lambda t}, \quad t \in \mathbb{R}.$$
Then g is a solution of the DE
$$(D-\lambda)^n f = 0.$$

Proof:

It is clearly enough to prove that
$$(D-\lambda)^n (t^{n-1} e^{\lambda t}) = 0, \quad n \in \mathbb{N}. \qquad (*)$$

We prove $(*)$ by induction on n. The case $n=1$ is easy and standard.

Suppose that for some $m \in \mathbb{N}$ we have

$$(D-\lambda)^m (t^{m-1} e^{\lambda t}) = 0.$$

Then

$$(D-\lambda)^{m+1}(t^m e^{\lambda t}) = \sum_{p=0}^{m+1} \binom{m+1}{p} (-\lambda)^{m+1-p} D^p (t^m e^{\lambda t})$$

$$= \binom{m+1}{0} (-\lambda)^{m+1} t^m e^{\lambda t} + \sum_{p=1}^{m+1} \binom{m+1}{p} (-\lambda)^{m+1-p} \left\{ D^0(t) \cdot D^p (t^{m-1} e^{\lambda t}) + \binom{p}{1} D(t) D^{p-1} (t^{m-1} e^{\lambda t}) \right\}$$

(Leibnitz rule)

$$= t \sum_{p=0}^{m+1} \binom{m+1}{p} (-\lambda)^{m+1-p} D^p (t^{m-1} e^{\lambda t}) + \sum_{p=1}^{m+1} \binom{m+1}{p} \binom{p}{1} (-\lambda)^{m-(p-1)} D^{p-1} (t^{m-1} e^{\lambda t})$$

$$= t (D-\lambda)^{m+1} (t^{m-1} e^{\lambda t}) + (m+1) \sum_{p=1}^{m+1} \binom{m}{p-1} (-\lambda)^{m-(p-1)} D^{p-1} (t^{m-1} e^{\lambda t})$$

$$= 0 + (m+1) \sum_{k=0}^{m} \binom{m}{k} (-\lambda)^{m-k} D^k (t^{m-1} e^{\lambda t})$$ (Induction hypothesis)

$$= (m+1) (D-\lambda)^m (t^{m-1} e^{\lambda t})$$

$$= 0.$$ (Induction hypothesis)

Remark 1: We have used the identity

$$\binom{m+1}{p} \cdot p = (m+1) \binom{m}{p-1}.$$

The algebraic proof is obvious. The combinatorial proof is as follows. Pick a team of p players from $m+1$ players and then pick a captain from the p players. There are $\binom{m+1}{p} \cdot p$ ways of doing this. On the other hand, one could pick the captain first and the remaining $p-1$ players later. There are

$(m+1)\binom{m}{p-1}$ ways of doing this.

Remark 2: Since $e^{\lambda t}, te^{\lambda t}, \ldots, t^{n-1}e^{\lambda t}$ are linearly independent (indeed $\sum_{i=1}^{n} c_i t^{i-1} e^{\lambda t} = 0 \;\; \forall t \in \mathbb{R} \Rightarrow \sum_{i=1}^{n} c_i t^{i-1} = 0 \;\; \forall t \in \mathbb{R} \Rightarrow c_i = 0 \;\; \forall i$), and since $\dim_{\mathbb{C}}(\ker(D-\lambda)^n) = n$, therefore they form a basis for $\ker(D-\lambda)^n$.

Solutions of $(A-\lambda)^n f = 0$

Let $V = \{f \mid f : \mathbb{N}_0 \longrightarrow \mathbb{C}\}$ and let $A : V \longrightarrow V$ be the usual advancement operator, i.e. $(Af)(n) = f(n+1)$ for $f \in V$, $n \in \mathbb{N}_0$.

Theorem: Let $\lambda \in \mathbb{C} \setminus \{0\}$, $m \in \mathbb{N}$, $j \in \{0, 1, \ldots, m-1\}$, and $g : \mathbb{N}_0 \longrightarrow \mathbb{C}$ the map

$$g(n) = n^j \lambda^n, \quad n \in \mathbb{N}_0.$$

Then g is a solution of the recurrence relation

$$(A-\lambda)^m f = 0.$$

Note: The constant term of $(x-\lambda)^m$ is λ^m which is non-zero.

Proof:

If $j = 0$, then $(A-\lambda) g = 0$. Indeed since $g(n) = \lambda^n$ in this case, $(A-\lambda)g(n) = Ag(n) - \lambda g(n) = g(n+1) - \lambda g(n) = \lambda^{n+1} - \lambda^{n+1} = 0$

This means $(A-\lambda)^m g = 0$.

Let us assume $1 \leq j \leq m-1$. Let $n \in \mathbb{N}_0$. We have

$$(A-\lambda)^m g(n) = \sum_{k=0}^{m} \binom{m}{k} (-1)^k \lambda^k A^{m-k} g(n)$$

$$= \sum_{k=0}^{m} \binom{m}{k} (-1)^k \lambda^k g(n+m-k)$$

$$= \sum_{k=0}^{m} \binom{m}{k} (-1)^k \lambda^k (n+m-k)^j \lambda^{n+m-k}$$

$$= \lambda^{n+m} \sum_{k=0}^{m} \binom{m}{k} (-1)^k \lambda^k (n+m-k)^j$$

$$= \lambda^{n+m} \; |X|$$

where X is the set of maps

$$\varphi : \{1,2,\dots,j\} \longrightarrow \{1,2,\dots,m+n\}$$

such that the image of φ contains the set $\{1,2,\dots,m\}$. Since $1 \le j \le m-1$, X is clearly empty, whence $|X| = 0$.

Thus

$$(A-\lambda)^m g = 0.$$

q.e.d.

Remark 3: The equality $|X| = \sum_{k=0}^{m} \binom{m}{k} (-1)^k (m+n-k)^j$ can be seen using the inclusion-exclusion principle. Another way to show that $\sum_{k=0}^{m} \binom{m}{k} (-1)^k (m+n-k)$ is zero is to consider the power series expansion of

$$\theta(x) = (e^x - 1)^m e^{nx}.$$

It is clear (since x "divides" $e^x - 1$ in the power series ring over $\mathbb{R}$) that the coefficient of x^j is zero. On the other hand, this coefficient is $\frac{1}{j!} \sum_{k=0}^{m} \binom{m}{k} (-1)^k (m+n-k)^j$.

Remark 4: For $j = \{1,\dots,m\}$, let $g_j \in V$ be given by

$$g_j(n) = n^{j-1} \lambda^n, \quad n \in \mathbb{N}_0.$$

We have seen that $g_1,\dots,g_j \in \ker\left((A-\lambda)^m\right)$. Moreover, they are linearly independent, for if $c_1,\dots,c_m \in \mathbb{C}$ are s.t.

$$c_1 g_1 + \dots + c_m g_m = 0$$

then

$$\left(c_1 + c_2 n + \dots + c_m n^{m-1}\right) \lambda^n = 0, \quad n \in \mathbb{N}_0,$$

i.e.

$$c_1 + c_2 n + \dots + c_m n^{m-1} = 0, \quad n \in \mathbb{N}_0.$$

This means that $c_1 = c_2 = \dots = c_m = 0$, proving that $g_1,\dots,g_m$ are linearly independent. Since

$$\dim_{\mathbb{C}} (\ker (A-\lambda)^m) = \deg (x-\lambda)^m = m$$

this means that $g_1,\dots,g_m$ is a basis for $\ker (A-\lambda)^m$.

Corollary 2, together with Remarks 2 and 4 above give us the following results.

Theorem:

(a) Let $\lambda_1, \dots, \lambda_k$ be distinct complex numbers, $m_1, \dots, m_k$ positive integers and W the complex vector space of solutions to the homogeneous linear ODE

$$(D-\lambda_1)^{m_1} \cdots (D-\lambda_k)^{m_k} f = 0.$$

Then the functions g_{ij}, $1 \le j \le m_i$, $1 \le i \le k$ given by

$$g_{ij}(t) = t^{j-1} e^{\lambda_i t} \quad , \quad t \in \mathbb{R}$$

form a basis for W.

(b) Let $\lambda_1, \dots, \lambda_k$ be distinct non-zero complex numbers, $m_1, \dots, m_k$ positive integers and W the complex vector space of solutions of the homogeneous linear recurrence relation

$$(A-\lambda_1)^{m_1} (A-\lambda_2)^{m_2} \cdots (A-\lambda_k)^{m_k} f = 0.$$

Then the functions $g_{ij} : \mathbb{N}_0 \to \mathbb{C}$, $1 \le j \le m_i$, $1 \le i \le k$, given by

$$g_{ij}(n) = n^{j-1} \lambda_i^n \quad , \quad n \in \mathbb{N}_0$$

form a basis for W.