Mar 16, 2010.

Throughout this lecture, divisors will mean ~~effective~~ Cartier divisors. An effective divisor (i.e. an effective Cartier divisor) D of a scheme X is therefore a closed subscheme D of X such that the ideal sheaf $\mathcal{I}$ of D is invertible. As is standard in this situation, if $\mathcal{F}$ is any quasi-coherent $\mathcal{O}_X$-module, we set

$$\mathcal{F}(nD) := \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{I}^{-n} \qquad n \in \mathbb{Z}.$$

In particular

$$\mathcal{O}_X(-D) = \mathcal{O}_X \otimes_{\mathcal{O}_X} \mathcal{I} = \mathcal{I},$$

and we have an exact sequence

$$0 \longrightarrow \mathcal{O}_X(-D) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_D \longrightarrow 0$$

whence an exact sequence

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_X(D) \longrightarrow \mathcal{O}_D(D) \longrightarrow 0.$$

The map $\mathcal{O}_X \longrightarrow \mathcal{O}_X(D)$ gives rise to a section. This is not just any section, but one that induces an injective map, and as such, is called a regular section. This motivates:

Definition: If L is any line bundle on X, a section $s \in \Gamma(X, L)$ is said to be regular if the map $\mathcal{O}_X \xrightarrow{s} L$ is injective, or equivalently, $L^{-1} \xrightarrow{s} \mathcal{O}_X$ is injective.

The image of L under s in the above Defn. is clearly an invertible ideal sheaf, and hence gives rise to an effective divisor D. Note $L \simeq \mathcal{O}_X(D)$. If $\Gamma(X, L)_{reg}$ denotes the set of sections in $\Gamma(X, L)$ which are regular, and $|L|$ the set of effective divisors D s.t. $\mathcal{O}_X(D) \simeq L$ (i.e. the complete linear system associated to L) then from our discussions, clearly

$$|L| \simeq \Gamma(X, L)_{reg} / \Gamma(X, \mathcal{O}_X^*)$$

where, of course, $\mathcal{O}_X^*$ is the sheaf of units of the sheaf of rings $\mathcal{O}_X$.

Defn: Let X be an S-scheme. A relative effective (Cartier) divisor on X/S is an effective divisor D of X which is flat over S.

$$\begin{tikzcd} D \arrow[r, hook, "\text{closed}"] \arrow[dr, "\text{flat}"'] & X \arrow[d] \\ & S. \end{tikzcd}$$
(ideal $\mathcal{I}$ of $D \subset X$ invertible).

(*) Theorem: Let $X \in Sch_{/S}$ and D a closed subscheme of X. Let $x \in D$, and let its image in S be denoted s. TFAE:

(a) D is an effective relative divisor at x (i.e. in an open nbhd of x).

(b) X and D are flat over S at x, and the fibre D_s is an effective divisor in the fibre X_s.

(c) X is flat at x, and D is cut out in a nbhd. of x by a single element which gives an effective divisor of X_s (i.e. D_s is an effective divisor of X_s).

Proof: Can be found in Matsumura, in the language of commutative algebra.

Examples

1. Let A be a ring and $a \in A$ an element. The map
$$A \longrightarrow A \qquad \text{“multiplication by } a\text{”}$$
is injective if and only if a is a nonzero divisor in A. Such elements are called regular elements.
In this case, xA is a free rank one module, and defines an effective divisor D of $\operatorname{Spec} A$. Moreover,

if A is integral, then by what we've said above, every non zero element of A is necessarily a regular element.

(b) From the local example above, if X is an integral scheme, L a line bundle on X, and $s \in \Gamma(X, L)$ a non-zero element, then s is a regular section. Thus for an integral scheme X

$$\Gamma(X,L)_{reg} = \Gamma(X,L) \setminus \{0\}.$$

<u>Definition</u>: Let $X \in Sch_{/S}$. Define, for any $T \in Sch_{/S}$,

$$\mathrm{Div}_{X/S}(T) = \left\{ D \;\middle|\; \begin{array}{l} D \text{ is a relative effective divisor} \\ \text{for } X_T \longrightarrow T \end{array} \right\}$$

<u>Theorem</u>: Let $f : X \longrightarrow S$ be strongly projective and flat. Then $\mathrm{Div}_{X/S}$ is representable.

<u>Proof</u>: Let $H = \mathrm{Hilb}_{X/S}$ and $W \subset X \times_S H = X_H$ the universal flat family over H. Let $V \subset W$ be the open set of points w such that W is a relative effective divisor at w over H.

$$\begin{array}{ccccc} V & \subset & W & \subset & X_H \\ & & \downarrow & \swarrow_{f_H} & \\ & & H & & \end{array}$$

Let $Z = W \setminus V$. Z is closed, whence so is $f_H(Z)$. Define

$$U = H \setminus f_H(Z).$$

Note that if $h \in H$ is such that W_h is an effective divisor of X_h, then $h \in U$, for, if not, then $h \in f_H(Z)$, which means $h = f_H(z)$ for some $z \in W \setminus V$. Now $z \in f_H^{-1}(h) = X_h$, whence $z \in (W \setminus V) \cap X_h = W_h \setminus V_h$. This means $z \in W_h$, and W_h is not an effective divisor at z. This contradicts the fact that W_h is an effective divisor ~~at~~ in X_h.

We now check that U represents $\mathrm{Div}_{X/S}$. Indeed if

$D \in \mathrm{Div}_{X/S}$, then D is flat over T, whence $\exists!$ map

$$g : T \longrightarrow H$$

such that $(g \times 1)^{-1}(W) = D$. We have to show g takes values in $\mathcal{U}$. Let $t \in T$. We have a cartesian diagram

$$\begin{tikzcd} D_t \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & W_{g(t)} \arrow[d] \\ \mathrm{Spec}(k(t)) \arrow[r] & \mathrm{Spec}(k(g(t))) \end{tikzcd}$$

Since D_t is an effective divisor and the above base change is faithfully flat, $W_{g(t)}$ is an effective divisor of $X_{g(t)}$. It follows that $g(t) \in \mathcal{U}$ ∎

<u>Proposition</u> : Suppose $f : X \longrightarrow S$ is strongly projective, flat, with <u>integral</u> <u>fibres</u>. Suppose L is a line bundle on X such that $R^i f_* L = 0$ for $i \geq 1$. Let $\sigma \in \Gamma(X, L) = \Gamma(S, f_* L)$. The following are equivalent.

(a) The section σ of L is a regular section of L ~~and the~~ such that the corresponding effective divisor D_σ of X is a relative effective divisor for $f : X \longrightarrow S$.

(b) Regarding σ as a section of $f_* L$, for every $s \in S$, $\sigma(s) \neq 0$. Here, with $\underline{m}_s$ the maximal ideal of $\mathcal{O}_{S,s}$, and $(f_* L)_s$ the stalk of the coherent sheaf $f_* L$ at $s \in S$, $\sigma(s)$ is element $\sigma \otimes k(s) \in (f_* L)_s / \underline{m}_s \quad (= f_* L \otimes_{\mathcal{O}_S} k(s))$ induced by σ.

<u>Proof</u> :

~~Since~~ By semi-continuity we have (a) for arbitrary base changes $u : T \longrightarrow S$, the natural map $u^* R^i f_* L \longrightarrow R^i f_{T*} L_T$

is an isomorphism for $i \geq 0$ (note that $i=0$ is included) and (b) $f_* L$ is a locally free $\mathcal{O}_S$-module.

Suppose ~~$D_\sigma \in$~~ σ is a regular section of L, and the associated effective divisor D^σ in X is flat over S. Let $s \in S$ be any point. Then $D^\sigma_s \subset X_s$ is an effective divisor of X_s. On the other hand, identifying (via the base change $\operatorname{Spec} k(s) \to S$) $f_* L \otimes_{\mathcal{O}_S} k(s)$ with $\Gamma(X_s, L|_{X_s})$, we see that D^σ_s is the divisor given by ~~the vanishing of $\sigma(s)$~~). ~~Thus $\sigma(s$~~ the section $\sigma(s) \in \Gamma(X_s, L|_{X_s})$. Thus $\sigma(s) \neq 0$, for D^σ_s is an effective divisor.

Conversely, suppose $\sigma(s) \neq 0$ for any $s \in S$. ~~Then~~ Let $D^\sigma \subset X$ be the closed subscheme defined locally by the locus where σ (thought of as a section of L) vanishes, i.e. the closed subscheme whose ideal is the image of L^{-1} under the map $\sigma: L^{-1} \to \mathcal{O}_X$. Then D^σ is locally cut out by a single equation, and $D^\sigma_s = D^\sigma \cap X_s$ for any $s \in S$ is given by the vanishing of $\sigma(s) \in \Gamma(X_s, L|X_s)$. Now $\sigma(s) \neq 0$ by hypothesis, and X_s is integral. Hence D^σ_s is an ~~___~~ effective divisor of X_s. By our earlier theorem, it follows that D^σ is a relative effective divisor, whence σ is a regular section of L. ~~by~~

∎.

<u>Remark</u>: The last requires slight elaboration. Recall that if A is a ring and F is a flat A-module, ~~the~~ and that $0 \to M \to F_k \to F_{k-1} \to \cdots \to F_1 \to F \to 0$ is an exact sequence of A-modules with F_i, $i=1,\ldots,k$ flat, then M must be flat. This easily seen by breaking up

the exact sequence into short exact sequences, hence reducing to the case $k=1$. Now we have

$$0 \to M \to F_1 \to F \to 0$$

is exact. Let N be any module. ~~Tensoring the above sequence~~ Considering the long exact sequence involving $\mathrm{Tor}_i^A(-,N)$, and using the fact that $\mathrm{Tor}_i^A(F,N) = \mathrm{Tor}_i^A(F_1,N)$ for $i \geq 1$, we see that $\mathrm{Tor}_i^A(M,N) = 0$ for all $i \geq 1$, i.e. M is flat.

Now suppose $K = \ker(\mathcal{O}_X \xrightarrow{\sigma} L)$. We have an exact sequence

$$(E): \quad 0 \to K \to \mathcal{O}_X \xrightarrow{\sigma} L \to \mathcal{O}_{D^\sigma}(D^\sigma) \to 0$$

and we have proven that D^σ is flat over S. Now (E) is an exact sequence of $\mathcal{O}_X$-modules which are flat over S, except possibly K. By arguing with stalks and using the result on A-modules just proved, K is also flat over S. It follows that $(E) \otimes_{\mathcal{O}_{S,s}} k(s)$ is also exact for every $s \in S$, or more precisely, if $x \in X$ and $s \in S$ is its image, then $(E)_x \otimes_{\mathcal{O}_{S,s}} k(s)$ is exact. In particular, since $\sigma(s) \neq 0$, $K_x \otimes_{\mathcal{O}_{S,s}} k(s) = 0$. Thus $K_x = 0$ by Nakayama, whence $K=0$, and $\sigma: \mathcal{O}_X \to L$ is injective. This proves σ is regular. //

Perhaps better to say $\sigma(s) \neq 0$ & X_s integral $\Rightarrow \mathcal{O}_{X_s} \xrightarrow{\sigma(s)} L|_{X_s}$ is injective $\Rightarrow K|_{X_s} = 0$ for every $s \Rightarrow K = 0$.

<u>Definition</u>: Let $f: X \to S$ be an S-scheme and L a line bundle on X. Define a (contravariant) set-valued functor $\mathrm{Div}^L_{X/S}$ on $\mathrm{Sch}_{/S}$ by

$$\mathrm{Div}^L_{X/S}(T) = \left\{ D \in \mathrm{Div}_{X/S} \;\middle|\; \begin{array}{l} \mathcal{O}_{X_T}(D) \simeq L_T \otimes f_T^* N \text{ for} \\ \text{some line bdle } N \text{ on } T \end{array} \right\}$$

<u>Theorem</u>: Let $f: X \to S$ be strongly projective, flat, with fibres which are geometrically integral. Let L be a line bundle on X such that $R^i f_* L = 0$ for $i \geq 1$. Then $\mathrm{Div}^L_{X/S}$ is representable by $\mathbb{P}(f_* L)$. In particular $\mathrm{Div}^L_{X/S}$ is smooth and strongly projective over S.

<u>Proof</u>: By semi-continuity, $f_* L$ is locally free and commutes with arbitrary base changes $T \to S$, as do all higher direct images (we are using the hypothesis that $R^i f_* L = 0$ for $i \geq 1$ for ~~both~~ all assertions). In particular our hypotheses are stable under arbitrary base changes. Since $f_* L$ is locally free, $\mathbb{P}(f_* L)$ is smooth and strongly projective over S.

Let $V = f_* L$, $\mathbb{P} = \mathbb{P}(V)$, and $\pi: \mathbb{P} \to S$ the resulting map. Let $V' = f^* V$, $\mathbb{P}' = \mathbb{P}(V')$, and $\pi': \mathbb{P}' \to X$ the resulting map. One checks easily that

$$\mathbb{P}' = X \times_S \mathbb{P}. \quad \text{— (1)}$$

Moreover, $\pi'^* V' = f_{\mathbb{P}}^* \pi^* V$. — (2)

We have a cartesian sq.

$$\begin{tikzcd} X \times_S \mathbb{P} = \mathbb{P}' \arrow[r, "\pi'"] \arrow[d, "f_{\mathbb{P}}"'] \arrow[dr, phantom, "\square"] & X \arrow[d, "f"] \\ \mathbb{P} \arrow[r, "\pi"'] & S \end{tikzcd}$$

Now if

$$E: \quad 0 \to \mathcal{O}_{\mathbb{P}}(-1) \to \pi^* V \to Q \to 0$$

is the universal exact sequence on $\mathbb{P} = \mathbb{P}(V)$ and

$$E': \quad 0 \to \mathcal{O}_{\mathbb{P}'}(-1) \to \pi'^* V' \to Q' \to 0$$

the universal exact sequence on $\mathbb{P}' = \mathbb{P}(V')$, then one checks easily that

$$E' = f_{\mathbb{P}}^{*} E,$$

under the standard identification $f_{\mathbb{P}}^{*}(\mathcal{O}_{\mathbb{P}}(-1)) = \mathcal{O}_{\mathbb{P}'}(-1)$. In fact the identification ① in the last page is through the map π' and the map $\mathbb{P}' \longrightarrow \mathbb{P}$ obtained from E' and the ~~exact sequence~~ universal property of $\mathbb{P}$, and identification ②.

On $\mathbb{P}$ we have (by looking at the exact sequence $E(1)$): a section

$$(**) \qquad \sigma : \mathcal{O}_{\mathbb{P}} \longrightarrow \pi^{*} V \otimes \mathcal{O}_{\mathbb{P}}(1).$$

Now, as $f_{*} L$ commutes with arbitrary base change, therefore

$$\pi^{*} V = f_{\mathbb{P}*} L_{\mathbb{P}}.$$

Thus

$$\pi^{*} V \otimes \mathcal{O}_{\mathbb{P}}(1) = f_{\mathbb{P}*}(L_{\mathbb{P}} \otimes f_{\mathbb{P}}^{*} \mathcal{O}_{\mathbb{P}}(1))$$

$$= f_{\mathbb{P}*}(L_{\mathbb{P}} \otimes \mathcal{O}_{\mathbb{P}'}(1)).$$

Now $\sigma \in \Gamma(\mathbb{P}, \pi^{*} V \otimes \mathcal{O}_{\mathbb{P}}(1)) = \Gamma(\mathbb{P}, f_{\mathbb{P}*}(L_{\mathbb{P}} \otimes \mathcal{O}_{\mathbb{P}}(1)))$

$$= \Gamma(\mathbb{P}', L_{\mathbb{P}} \otimes \mathcal{O}_{\mathbb{P}'}(1)).$$

Thus σ is a section of $L_{\mathbb{P}} \otimes \mathcal{O}_{\mathbb{P}'}(1)$, and thought as a section of $f_{\mathbb{P}*}(L_{\mathbb{P}} \otimes \mathcal{O}_{\mathbb{P}'}(1))$ on $\mathbb{P}$, it is nowhere vanishing, since in its avatar as an ~~map~~ injective map. $\mathcal{O}_{\mathbb{P}} \longrightarrow \pi^{*} V \otimes \mathcal{O}_{\mathbb{P}}(1)$ (see $(**)$) it has a vector bundle as the cokernel, namely $Q(1)$. By the previous Proposition, the associated divisor $D^{\sigma} \subset \mathbb{P}' = X \times_S \mathbb{P}$ is

a relative effective divisor for $f_{\mathbb{P}} : X \times_S \mathbb{P} \to \mathbb{P}$.
We claim $(\mathbb{P}, D^o)$ represents $\mathrm{Div}^L_{X/S}$.
To that end, suppose $D \in \mathrm{Div}_{X/S}(T)$, for $u : T \to S$. Then there exists a line bundle N ~~on T~~ on T such that

$$\mathcal{O}_{X_T}(D) \simeq L_T \otimes f_T^* N.$$

Consider $f_{T*}(L_T \otimes f_T^* N)$

$$= f_{T*}(L_T) \otimes N$$

$$= u^* V \otimes N.$$

since $f_* L = V$ commutes with formation of arbitrary base changes. ~~The~~ We have an exact sequence

$$0 \to \mathcal{O}_{X_T} \xrightarrow{s} \mathcal{O}_{X_T}(D) \to \mathcal{O}_D(D) \to 0$$

where s is the regular section arising from the ~~relative~~ effective divisor D. Since D is a relative effective divisor, by our previous result, s when regarded as a section of $f_* \mathcal{O}_{X_T}(D)$, is nowhere vanishing on T. (we are using the fact that the fibres of $X_T \to T$ are integral, since those of $X \to S$ are geometrically integral). Thus s gives an injective map $\mathcal{O}_T \xrightarrow{s} f_* \mathcal{O}_{X_T}(D)$ whose cokernel is locally free, since $s \otimes k(t) \neq 0$ for any t, and $f_* \mathcal{O}_{X_T}(D)$ is locally free. We thus have an exact sequence

$$0 \to \mathcal{O}_T \to \underset{\substack{\shortparallel \\ u^* V \otimes N.}}{f_* \mathcal{O}_{X_T}(D)} \to W_T \to 0$$

This gives an exact sequence of vector ~~bundles~~ bundles

(+) $\quad 0 \longrightarrow N^{-1} \longrightarrow u^*V \longrightarrow W_T \longrightarrow 0,$

whence, by the universal property of $\mathbb{P} = \mathbb{P}(V)$, an S-map $u' : T \longrightarrow \mathbb{P} = \mathbb{P}(V)$, and u' is the unique map such that (+) is equivalent to the exact sequence $u'^*(E)$, i.e. we have a commutative diagram of exact sequences with vertical arrows as isomorphisms:

$$\begin{tikzcd}
u'^*(E): & 0 \arrow[r] & u'^*\mathcal{O}_{\mathbb{P}}(-1) \arrow[r] \arrow[d, "\wr"] & u'^*\pi^*V \arrow[r] \arrow[d, equal] & u'^*Q \arrow[r] \arrow[d, "\wr"] & 0 \\
(+): & 0 \arrow[r] & N^{-1} \arrow[r] & u^*V \arrow[r] & W_T \arrow[r] & 0
\end{tikzcd}$$

~~In particular~~ Thus the isomorphism $\mathcal{O}_{X_T}(D) \simeq L_T \otimes f_T^*N$ yields, via $L_T \otimes f_T^*N \simeq (1\times u')^*(L_{\mathbb{P}} \otimes f_{\mathbb{P}}^*\mathcal{O}(1))$, an isomorphism $\mathcal{O}_{X_T}(D) = (1\times u')^*\mathcal{O}_{\mathbb{P}'}(D^\sigma)$, since $L_{\mathbb{P}} \otimes f_{\mathbb{P}}^*\mathcal{O}(1) \simeq \mathcal{O}_{\mathbb{P}'}(D^\sigma)$. It is not difficult to then see that $(1\times u')^{-1}(D^\sigma) = D$. Thus $(\mathbb{P}, D^\sigma)$ represents $\mathrm{Div}^L_{X/S}$. ∎

Finally, we ~~give a~~ end with a proposition we could have proven a few lectures ago.

<u>Proposition</u>: Let $f : X \longrightarrow S$ be proper and finitely presented, such that the natural map $\mathcal{O}_S \longrightarrow f_*\mathcal{O}_X$ is an isomorphism, ~~for~~ and this holds universally (i.e. $\mathcal{O}_T \xrightarrow{\sim} f_{T*}\mathcal{O}_{X_T}$ for every S-scheme T). If L is a line bundle on X such that L represents the trivial element in $(\mathrm{Pic}_{X/S})_{(fppf)}$ (resp. $(\mathrm{Pic}_{X/S})_{(\acute{e}t)}$), then $L \simeq f^*N$ for some line bundle N on S.

P. T. O.

Basically,

$$\begin{tikzcd}
0 \arrow[r] & (1\times u')^*(L_{\mathbb{P}}^{-1} \otimes \mathcal{O}_{\mathbb{P}'}(-1)) \arrow[r] \arrow[d, "\wr"] & \mathcal{O}_{X_T} = (1\times u')^*\mathcal{O}_{X\times\mathbb{P}} \arrow[r] \arrow[d, equal] & (1\times u')^*\mathcal{O}_{D^\sigma} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & L_T^{-1} \otimes N^{-1} \arrow[r] & \mathcal{O}_{X_T} \arrow[r] & \mathcal{O}_D \arrow[r] & 0
\end{tikzcd}$$

commutes.

<u>Remarks</u> (a) Recall, from the lecture on sheafifications, L represents the trivial element in $(\mathrm{Pic}_{X/S})_{(\mathrm{top})}$ ($\mathrm{top} \in \{\mathrm{fppf}, \mathrm{\acute{E}t}\}$) if and only if for some member $T \to S$ of $\mathcal{M}_{\mathrm{top}}$, the pull back L_T on X_T is the trivial line bundle.

(b) The Proposition asserts that if $\mathcal{O}_S \xrightarrow{\sim} f_* \mathcal{O}_X$ holds universally, then the natural map (with $\mathrm{top} \in \{\mathrm{\acute{e}t}, \mathrm{fppf}\}$)

$$(*) \qquad \mathrm{Pic}_{X/S}(T) \longrightarrow (\mathrm{Pic}_{X/S})_{(\mathrm{top})}(T)$$

is injective for $T \in \mathrm{Sch}/S$.

(c) If in addition to the hyp. of the Prop^n, if $f: X \to S$ has a section, then $(*)$ above is an isomorphism.

(d) When f is proper and finitely presented, the hypothesis that $\mathcal{O}_S \to f_* \mathcal{O}_X$ is an isomorphism holds universally if f has geometrically connected fibres.

<u>Proof</u>: Let $T \to S$ be an fppf-map (resp. étale surjective map) such that $L_T \simeq \mathcal{O}_{X_T}$. By our hypothesis $\mathcal{O}_T \xrightarrow{\sim} f_{T*} \mathcal{O}_{X_T}$, whence the natural map

$$f_T^* f_{T*} \mathcal{O}_{X_T} \longrightarrow \mathcal{O}_{X_T}$$

is an isomorphism. Now the map $f_T^* f_{T*} \to$ identity is functorial, whence the natural map

$$f_T^* f_{T*} L_T \longrightarrow L_T$$

is an isomorphism. Since $X_T \to X$ is faithfully flat, it follows that the natural map

$$f^* f_* L \longrightarrow L$$

is an isomorphism. Set $N = f_* L$, to get the result. ∎