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Picard VIII

Sheafifications: Let $\mathcal{M}$ be a class of morphisms of $Sch/_S$ which are closed under composition, fiber products, and contains every isomorphism. In other words $\mathcal{M}$ is a topology.

Let

$$P : Sch/_S^{\circ} \longrightarrow Sets$$

be a presheaf. A map of presheaves

$$\theta : P \longrightarrow P^+$$

is called a **sheafification** of P if

(a) P^+ is an $\mathcal{M}$-sheaf, and

(b) whenever $\psi : P \to G$ is a map of presheaves with G an $\mathcal{M}$-sheaf, $\exists!$ map $\psi^+ : P^+ \to G$ of $\mathcal{M}$-sheaves such that the diagram

$$\begin{tikzcd}
P \arrow[dd, "\theta"'] \arrow[dr, "\psi"] & \\
& G \\
P^+ \arrow[ur, "\psi^+"'] &
\end{tikzcd}$$

commutes.

"Construction" of P^+ from P: Let $T \in Sch/S$. If $T' \to T$ is a map in $\mathcal{M}$, define $H^0(T'/T, P)$ to be the subset of $P(T')$ consisting of elements elements $\xi \in P(T')$ satisfying the condition that there exists a map $\tilde{T} \to T'' := T' \times_T T'$ in $\mathcal{M}$ with the property that the two pullbacks of ξ under the two composites $\tilde{T} \longrightarrow T'' \rightrightarrows T'$ are equal. Here, the double arrow indicates the two projections from T'' to T'.

Fix a universe for the class $\mathcal{M}$ and set

$$P^+(T) = \varinjlim_{T'} H^0(T'/T, P).$$

This direct limit depends — a priori — on the choice of universe, but does not if $\mathcal{M}$ is one of Zariski, étale, or fppf topologies. The assignment $T \mapsto P^+(T)$ is easily checked to functorial (in a contravariant fashion), and one has a natural map $\theta : P \longrightarrow P^+$. One checks (quite easily) that θ is a sheafification of P. We now restrict $\mathcal{M}$ to one of $\mathcal{M}_{Zar}$, $\mathcal{M}_{ét}$, or $\mathcal{M}_{fppf}$.
From its universal property, clearly (P^+, θ) is unique

upto unique isomorphism, and therefore it make sense to talk of "the" sheafification of P.

The Picard sheaf: Recall that if $X \in Sch/_S$ then
$\mathrm{Pic}_{X/S}(T) := \mathrm{Pic}(X \times_S T)/\mathrm{Pic}(T)$ for $T \in Sch/_S$.
We will represent — under certain hypotheses on X/S — the fppf sheafification

$$\mathcal{P} := \left(\mathrm{Pic}_{X/S}\right)_{(fppf)}$$

of $\mathrm{Pic}_{X/S}$. Actually, our proof is such that we can represent (just as well) $(\mathrm{Pic}_{X/S})_{(ét)}$ and we will try to point this out as we proceed with the proof.

Facts about cohomology and base change. For simplicity, from now on, our schemes are Noetherian. The following facts can be found in Hartshorne's Algebraic Geometry.

1. Suppose $f: X \to S$ is proper and $\mathcal{F}$ is a coherent $\mathcal{O}_X$-module which is flat over S, and $R^i f_* \mathcal{F} = 0$ for $i \geq i_0 + 1$. Then
(a) $R^{i_0} f_* \mathcal{F}$ is a locally free $\mathcal{O}_S$-module.
(b) For any quasi-coherent $\mathcal{O}_S$-module $\mathcal{M}$, the natural

map

$(BC\text{-}\mathcal{M})_i$ $$R^i f_* \mathcal{F} \otimes_{\mathcal{O}_S} \mathcal{M} \longrightarrow R^i f_* (\mathcal{F} \otimes_{\mathcal{O}_X} f^* \mathcal{M})$$

is an isomorphism for $i \geq i_0$.

(c) If we have a cartesian diagram

$$\begin{tikzcd} X' \arrow[r, "v"] \arrow[d, "f'"'] \arrow[dr, phantom, "\square"] & X \arrow[d, "f"] \\ S' \arrow[r, "u"'] & S \end{tikzcd}$$

then the natural map

$$u^* R^i f_* \mathcal{F} \longrightarrow R^i f'_* v^* \mathcal{F}$$

is an isomorphism for every $i \geq i_0$

2. If $f: X \to S$ and $\mathcal{F}$ are as above, then

$$\boxed{R^i f_* \mathcal{F} = 0 \text{ for } i > i_0 \iff H^i(X_s, \mathcal{F}_s) = 0 \ \forall s \in S \text{ and } i > i_0}$$

The above facts follow from descending induction on i and the last result from Hartshorne's "Algebraic Geometry" in the chapter on "semi continuity", namely that $(BC\text{-}\mathcal{M})_{i+1}$ is an isomorphism, then:

$$(BC\text{-}\mathcal{M})_i \text{ is an isomorphism} \iff R^{i+1} f_* \mathcal{F} \text{ is locally free.}$$

Note: The proof of this is simpler than that given in Hartshorne or EGA both of which use difficult "formal function thm." Nakayama is enough.

Theorem: Let $f: X \to S$ be proper and $\mathcal{P}$ the Picard sheaf defined earlier (i.e. $\mathcal{P} = (\mathrm{Pic}_{X/S})_{(fppf)}$). Suppose $\xi \in \mathcal{P}(T)$ is represented by a line bundle L on $X_T := X \times_S T$. Then ξ is trivial if and only if there is a Zariski open cover $\{T_i\}$ of T such that L is trivial on each X_{T_i}.

Proof: Without loss of generality we may assume that $T = S$. We are assuming S is noetherian (otherwise we have to assume f is finitely presented, in addition to being proper, and use the fact that such maps are necessarily inverse limits of proper maps between noetherian schemes). Since S is noetherian, the map f has a (unique) Stein factorization

(SF)
$$\begin{tikzcd} X \arrow[r, "g"] \arrow[rr, bend right, "f"'] & Y \arrow[r, "h"] & S \end{tikzcd}$$

where g has connected fibers (so that $g_*(\mathcal{O}_X = \mathcal{O}_Y)$) and h is a finite (whence affine) map.

Claim: The natural map $g^* g_* L \to L$ is an isomorphism.

Proof of claim: The claim can be tested after making a faithful flat base change on X. We know that $\exists$ an fppf map $S' \to S$ such that the pull back of L to $X_{S'}$ is trivial. Now $X_{S'} \to X$ is fppf. Hence without loss of generality, in order to prove the claim, we may assume that $L \simeq \mathcal{O}_X$. Since g has connected fibers the claim follows (for $g_*\mathcal{O}_X = \mathcal{O}_S$ and $g^*\mathcal{O}_S = \mathcal{O}_X$).

From the claim it follows that $L = g^*M$ for a line bundle M on Y. Now $Y \xrightarrow{h} S$ is a finite map, whence affine. It is not hard to show that in this case S can be covered by open sets $\{S_i\}$ such that $M|_{h^{-1}(S_i)}$ is trivial. The problem reduces to the following problem (since h is finite). Let $A \to B$ be a map of rings such that B is a finitely generated A-module. Let P be a finitely generated projective module on B. Then for every prime ideal $\mathfrak{p}$ of A, $P_{\mathfrak{p}}$ is free over $B_{\mathfrak{p}}$. In order to prove this, it is enough to assume A is local and $\mathfrak{p}$ its unique maximal ideal. Let $k = A/\mathfrak{p}$. Then $P \otimes_A k$ is free over $B \otimes_A k$ since the latter is an Artin ring. Nakayama's lemma (and the fact that P is projective) gives the result, whence the theorem.

<u>Note</u>: The proof goes through even if the map $A \to B$ is only integral. The non-noetherian case makes use of this as the direct limit of finite maps of rings need not be finite, but is certainly integral.

Strictly speaking this is a cor. to the proof & not to the thm.

Corollary: Suppose in addition that $f_*\mathcal{O}_X = \mathcal{O}_S$ and this holds universally (e.g. if the geometric fibers are connected, then this is true, and conversely). Then $L = f_T^* M$ for an invertible sheaf M on T.

Proof: With this assumption, the map h in the Stein factorization is the identity and $f = g$. The result follows from the claim in the proof of the theorem.

Remarks:

1. The Corollary says that the natural map

$$\mathrm{Pic}_{X/S}(T) \longrightarrow \mathcal{P}(T)$$

is injective for every T. It need not be surjective, i.e., it is possible to have elements $\xi \in \mathcal{P}(T)$ which are not represented by line bundles on X_T.

2. If in addition to the existing hypotheses if $f: X \to S$ has a section, then

$$\mathrm{Pic}_{X/S}(T) \xrightarrow{\sim} \mathcal{P}(T) \qquad \forall T \in \mathrm{Sch}/S.$$

Thus $\mathrm{Pic}_{X/S}$ is already a sheaf in this case.

Representing the functor $\mathcal{P} = (\underline{\mathrm{Pic}}_{X/S})_{(fppf)}$:

From now on we suppose that $f: X \to S$ is flat, strongly projective, with integral geometric fibres.

Notation(s): If $T \in Sch/_S$ and $t \in T$ (i.e. t is a scheme point of T) then we write X_t for the fiber of $f_T : X_T \to T$ over T. Note that $X_t = X_{\mathrm{Spec}(k(t))}$ where $k(t) := \mathcal{O}_{X,t}/\underline{m}_{X,t}$ is the residue field at t. If k is a field and $\mathrm{Spec}\, k \to S$ a map, then we often write X_k for $X_{\mathrm{Spec}\, k} := X_{X_S}(\mathrm{Spec}\, k)$. Note that $X_t = X_{k(t)}$ by our notations and $X_k = X_\theta$ where $\theta \in \mathrm{Spec}\, k$ is the only scheme point of the scheme $\mathrm{Spec}\, k$.

Definition: Let k, k' be fields, $\mathrm{Spec}\, k \to S$, $\mathrm{Spec}\, k' \to S$ two maps of schemes, L and L' line bundles on X_k and $X_{k'}$ respectively. We say that L and L' are representative of the same class if there is a common field extension k'' of k and k' such that the pullbacks of L and L' on $X_{k''}$ are isomorphic.

Since field extensions are faithfully flat, and since cohomology commutes with flat base change, if L and L' represent the same class then their Hilbert polynomials are the

the same, if both are computed with respect to the pull back of $\mathcal{O}_{X/S}(1)$ on X (Recall $X \to S$ is strongly projective).

Definition: Let $\xi \in P(T)$. We say that the class of a line bundle L on X_k is a class of ξ, if there is a field k' and a map $\operatorname{Spec} k' \to T$ such that the pull back of ξ to $X_{k'}$ is represented by a line bundle L' on $X_{k'}$ which is in the same class as L.

Note that the classes of ξ can be obtained in the following way: Let $T' \to T$ be an fppf map such that the pull back of ξ to T' is represented by a line bundle $\mathcal{L}$ on T'. Then the classes of ξ are exactly the classes of the various line bundles $\mathcal{L}_{t'}$ on $X_{t'}$ as t' varies over the scheme points of T'. Here $\mathcal{L}_{t'}$ is the restriction of $\mathcal{L}$ to the fiber $X_{t'}$.

Note also that since $X_{T'} \xrightarrow{f_{T'}} T'$ is flat, $\mathcal{L}$ is flat over T', where the Hilbert polynomials of the various $\mathcal{L}_{t'}$, $t' \in T$ is constant on connected components of T'. We say φ is the Hilbert polynomial of ξ, if every class associated with ξ has φ as its Hilb. poly.

<u>Definition</u>: Let $X \xrightarrow{f} S$ be as above (i.e. flat, strongly projective...). Let $\varphi \in \mathbb{Q}[z]$. For $T \in \mathrm{Sch}/S$ define $\mathcal{P}^\varphi(T)$ to be the subset of $\mathcal{P}(T)$ consisting of elements $\xi \in \mathcal{P}(T)$ such that the Hilbert polynomial of L^{-1} is φ for every L representing a class of ξ.

Clearly $\mathcal{P}^\varphi$ is a subsheaf of $\mathcal{P}$. Further, if $\xi \in \mathcal{P}(T)$ is represented by $\mathcal{L}$ on X_T, then $\mathcal{L}^{-1}$ forms a flat family of line bundles parametrized by T, whence it has constant Hilbert polynomials on connected components of T. Let T^φ be the component where $\mathcal{L}_t^{-1}$ has Hilbert polynomial φ. Clearly T^φ is open and closed in T, and

$$T^\varphi \xrightarrow{\sim} \mathcal{P}^\varphi \times_{\mathcal{P}} T.$$

Thus $\mathcal{P}^\varphi \to \mathcal{P}$ is relatively representable and is an open & closed subfunctor of $\mathcal{P}$. Moreover $\{\mathcal{P}^\varphi\}$ cover $\mathcal{P}$ (since $\{T^\varphi\}$ cover T).
Thus it is enough to represent $\mathcal{P}^\varphi$.

(In the event $\xi \in \mathcal{P}(T)$ is not represented by $\mathcal{L}$ on X_T, one can still define T^{Φ} using classes in an obvious way and if $T' \to T$ is fppf with the pull back of ξ represented by $\mathcal{L}$, then T'^{Φ} is the inverse image of T^{Φ}, thus T^{Φ} is open and closed in T. Moreover $T^{\Phi} = \mathcal{P}^{\Phi} \times_{\mathcal{P}} T$. We are using the fact that since $T' \to T$ is fppf, T has the quotient (Zariski) topology induced by T'.)

<u>Definition</u>: For $T \in Sch/S$, and $m \in \mathbb{Z}_{\geq 0}$, define $\mathcal{P}^{\Phi}_m(T)$ to be the subset of $\mathcal{P}^{\Phi}(T)$ consisting of elements $\xi \in \mathcal{P}^{\Phi}(T)$ such that if L on X_k is a class of ξ, then

$$H^i(X_k, L(n)) = 0 \qquad \forall\, i \geq 1 \text{ and } n \geq m.$$

By the semi-continuity theorem, vanishing of the above cohomology is an open condition. Thus if $\xi \in \mathcal{P}^{\Phi}(T)$, and say for simplicity ξ is represented by $\mathcal{L}$, then there is an open subscheme T_m of T such that $R^i f_{T_m *}(h^*\mathcal{L}(n)) = 0$ for $i \geq 1$, $n \geq m$, where $h: X_{T_m} \to X_T$ is the inclusion (open) map. Clearly $T_m = \mathcal{P}^{\Phi}_m \times_{\mathcal{P}} T$. Thus $\mathcal{P}^{\Phi}_m$ is an open subfunctor of $\mathcal{P}^{\Phi}$. By Serre's theorem $\bigcup_m T_m = T$ (this is an increasing union, for $T_m \subseteq T_{m+1}$). Thus $\{\mathcal{P}_m^{\varphi}\}$ is an open cover of $\mathcal{P}$.

It is clearly enough to represent each $\mathcal{P}_m^{\varphi}$.

<u>Notation</u>: Let $\varphi \in \mathbb{Q}[t]$ and $n \in \mathbb{Z}$. Define $\tau_n \varphi \in \mathbb{Q}[t]$ by

$$(\tau_n \varphi)(t) = \varphi(t+n).$$

Clearly $\mathcal{P}_m^{\varphi} \xrightarrow{\sim} \mathcal{P}_0^{\tau_m \varphi}$, under $L \mapsto L(m)$.

Thus it is enough to represent $\mathcal{P}_0^{\varphi}$, which is what we intend to do.