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Picard VII

Let $f: X \to S$ be strongly quasi-projective, and let $R \to X \times_S X$ be a scheme theoretic equivalence relation which is proper and flat and finitely presented (Recall, R is proper, flat and finitely presented if any one (and hence both) of the projections $R \rightrightarrows X$ is proper and flat.)

Since R is a proper equivalence relation, by Theorem 2.1.4 of Picard 6, $R \to X \times_S X$ is a closed immersion.

We use the following notations: For any $T \in \mathrm{Sch}_{/S}$, $X_T := X \times_S T$, $f_T: X_T \to T$ the base change of $f: X \to S$, $\pi_1, \pi_2: X \times_S X \rightrightarrows X$ the two projections (so that, $\pi_2 = f_X$) and $p_1, p_2: R \rightrightarrows X$ the two maps given by the composites $R \hookrightarrow X \times_S X \underset{\pi_2}{\overset{\pi_1}{\rightrightarrows}} X$.

We make the following additional assumption. Assume that fibres of $p_2 : R \to X$ have only a finite number of Hilbert polynomials. (This is so, e.g., when S has only finitely many connected components.) The Hilbert polys are computed w.r.t. a fixed embedding $X \hookrightarrow \mathbb{P}(\mathcal{E})$, $\mathcal{E}$ a vector bundle on S.

Let H be the Hilbert scheme of closed subschemes of X which are proper and flat over S. Then $p_2 : R \to X$ defines a flat family of closed subschemes of X, and since p_2 is a proper map, we have a classifying map

$$g : X \to H$$

such that the following diagram is commutative, with each sub-rectangle (and whence the outer rectangle) cartesian. Here $D \to D \times_S H$ is the universal flat family of closed S-subschemes of X and g_R is defined as the base change of g via $p_2 : R \to X$ (The composite of the left column is p_2.)

$$\begin{tikzcd}
R \arrow[r, "g_R"] \arrow[d] & D \arrow[d] \\
X \times_S X \arrow[r, "1 \times g"] \arrow[d, "\pi_2 = f_X"'] & X \times_S H \arrow[d, "f_H"] \\
X \arrow[r, "g"] & H
\end{tikzcd}$$

Our notations regarding maps which involve products are the standard ones. Thus

(a) $\alpha \times \beta : U \times_B V \longrightarrow W \times_B Z$ is the map induced by B-maps $\alpha : U \to W$ and $\beta : V \to Z$. In greater detail, given α, β as above, one has the B-composites $U \times_B V \to U \xrightarrow{\alpha} W$ and $U \times_B V \longrightarrow V \xrightarrow{\beta} W$, whence by the universal property of $W \times_B Z$, one has a unique map $\alpha \times \beta$ as above.

(b) $(\alpha, \beta) : U \longrightarrow V \times_B W$ is the map induced by the B-maps $U \xrightarrow{\alpha} V$ and $U \xrightarrow{\beta} W$.

Note that by our conventions,

$$\alpha \times \beta = (\alpha \circ p_1, \beta \circ p_2)$$

where p_1, p_2 are the two projections on $U \times_B V$.

The graph map: The map $g : X \to H$ gives rise to the graph map $\gamma_g := (1, g) : X \longrightarrow X \times_S H$, and since $f : X \to S$ and $H \to S$ are finitely presented, γ_g is a closed immersion, whence there is a closed subscheme Γ of $X \times_S H$ isomorphic to X such that

γ_g factors as

$$\begin{tikzcd} X \arrow[r, "\sim"] \arrow[rr, bend right, "\gamma_g"'] & \Gamma \subseteq & X \times_S H. \end{tikzcd}$$

T-valued points and their R-equivalence:

Let $x_1, x_2 : T \rightrightarrows X$ be two T-valued points of X over the scheme S. We write:

$$x_1 \sim x_2$$

if $(x_1, x_2) \in R(T)$, where $R(T)$ is thought of as a subset of $(X \times_S X)(T)$.

Let $R_1 \subset X \times_S X_T$ and $R_2 \subset X \times_S X_T$ be the closed subschemes obtained by base changing $R \subset X \times_S X$ via the maps $X_T \rightrightarrows X \times_S X$ induced by x_1, x_2. More precisely R_1, R_2 are defined the cartesian diagrams on the next page:

$$\begin{tikzcd}
R_1 \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & R \arrow[d] \\
X\times_S X\times_X T = X\times_S T \arrow[r, "1\times x_1"] \arrow[d, "f_T"'] \arrow[dr, phantom, "\square"] & X\times_S X \arrow[d, "\pi_2"] \\
T \arrow[r, "x_1"'] & X
\end{tikzcd}
\qquad
\begin{tikzcd}
R_2 \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & R \arrow[d] \\
X\times_S X\times_X T = X\times_S T \arrow[r, "1\times x_2"] \arrow[d, "f_T"'] \arrow[dr, phantom, "\square"] & X\times_S X \arrow[d, "\pi_2"] \\
T \arrow[r, "x_2"'] & X
\end{tikzcd}$$

We wish to examine conditions (necessary & suffic.) under which $x_1 \sim x_2$. To that end, first consider a T'-valued point $t : T' \to T$ of T. Let $(x,t) \in X_T(T')$, i.e., $(x,t) : T' \to X_T = X\times_S T$ is map that lifts $t : T' \to T$. Consider the CD:

(*)

$$\begin{tikzcd}
 & R_i \arrow[r, "r_{x_i}"] \arrow[d] & R \arrow[r, "g_R"] \arrow[d] & D \arrow[d] \\
 & X_T \arrow[r, "1\times x_i"] \arrow[d, "f_T"] & X\times_S X \arrow[r, "1\times g"] \arrow[d, "\pi_2 = f_X"] & X\times_S H \arrow[d, "f_H"] \\
T' \arrow[uur, dashed] \arrow[ur, "{(x,t)}"] \arrow[r, "t"'] & T \arrow[r, "x_i"'] & X \arrow[r, "g"'] & H
\end{tikzcd}$$

Here the R-valued point r_{x_i} is the "restriction" of $1\times x_i$ to the closed subscheme R_i. More precisely it is the base change of $1\times x_i$ under the closed immersion

$R_i \hookrightarrow X_T$. Suppose (x,t) takes values in R_i, i.e. suppose the dotted green arrow can be filled to make the resulting diagram commute (there is at most one way of doing this, since R_i is a closed subscheme of X_T). This can happen if and only if the composite $(1 \times x_i) \circ (x,t) = (x, x_i \circ t)$ takes values in R. Thus

$$(x,t) \in R_i(T') \iff (x, x_i \circ t) \in R(T')$$
$$\iff x \sim x_i \circ t.$$

This means

$$R_i(T') = \{(x,t) \in X_T(T') \mid x \sim x_i \circ t\}.$$

It is evident from the above that

$$x_1 \circ t \sim x_2 \circ t \iff R_1(T') = R_2(T')$$

(as subsets of $X_T(T')$).

Now

$$x_1 \sim x_2 \iff x_1 \circ t \sim x_2 \circ t \quad \forall \, (T' \xrightarrow{t} T) \in Sch_{/T}$$
$$\iff R_1(T') = R_2(T') \quad \forall \; T' \in Sch_{/T}$$
$$\iff R_1 = R_2 \quad \text{(as closed subschemes of } X_T\text{)}.$$

Now, according to the commutative diagram (*), $g \circ x_i$ is the classifying map of the flat and proper family of closed S-subschemes of X given by $R_i \hookrightarrow X_T \xrightarrow{f_T} T$, whence $R_1 = R_2$ if & only if $g \circ x_1 = g \circ x_2$. Thus $x_1 \sim x_2 \iff g x_1 = g x_2$. In fact by considering the commutative diagram with cartesian rectangles

$$
\begin{tikzcd}
& R_2 \arrow[r] \arrow[d] & R \arrow[r, "g_R"] \arrow[d] & D \arrow[d] \\
& X_T \arrow[r, "1 \times x_2"] \arrow[d, "f_T"'] & X \times_S X \arrow[r, "1 \times g"] \arrow[d, "f_X = \pi_2"] & X \times_S H \arrow[d, "f_H"] \\
T \arrow[ur, "(x_1, 1)"] \arrow[r, equal] & T \arrow[r, "x_2"'] & X \arrow[r, "g"'] & H
\end{tikzcd}
$$

one sees that $(x_1, x_2) = (1 \times x_2) \circ (x_1, 1)$ takes values in R if and only if the map

$(x_1, g x_2) = (1 \times g) \circ (1 \times x_2) \circ (x_1, 1)$

takes values in D. We have thus proven most of the following lemma:

Lemma: Let $x_1, x_2 \in X(T)$ for some $T \in Sch/_S$. Then

$$x_1 \sim x_2 \iff g x_1 = g x_2 \iff (x_1, g x_2) \in D(T)$$
$$\iff (x_1, g x_2) \in \Gamma(T)$$

Proof:

We only have to prove the last equivalence. Now $\Gamma(T) = \{(x, h) \in (X \times_S H)(T) \mid g(x) = h\}$. Thus $(x_1, g x_2) \in \Gamma(T) \iff g x_1 = g x_2$, and we're done.

Corollary: Γ is a closed subscheme of D.

Proof: From the Lemma $\Gamma(T) \subseteq D(T)$ as a subset of $(X \times_S H)(T)$. This is true for every T-valued point of $X \times_S H$. Since Γ and D are closed subschemes of $X \times_S H$ we are done by the following Exercise

Exercise: Suppose U and V are closed subschemes of W and we have commutative diagram

$$\begin{tikzcd} U \arrow[r, hook] \arrow[d, "f"'] & W \\ V \arrow[ur, hook] & \end{tikzcd}$$

Then show that f is a closed immersion [Hint: Can assume W is affine. This forces U and V to be affine. The problem then becomes one of rings and surjective maps.]

<u>Remark</u>: Consider $p_1, p_2 : R \rightrightarrows X$. Clearly (p_1, p_2) takes values in R, whence $p_1 \sim p_2$. This means

$$g p_1 = g p_2 .$$

This is a way of saying that the two (flat, proper) families of closed S-subschemes of X given by p_1 and by p_2 are the same. Moreover, by the Lemma on p. 8, since $(p_1, g p_2) \in D(R)$ (via, $g_R : R \to D$), hence $(p_1, g p_2) \in \Gamma(R)$. Thus g_R factors as

$$(**) \qquad \begin{tikzcd} R \arrow[r, "\tilde{g}"] \arrow[rr, bend right, "g_R"'] & \Gamma \arrow[r, hook] & D \end{tikzcd}$$

The map $g_R : R \to D$ and the graph Γ_f:

First note that the closed immersion $R \hookrightarrow X \times_S X$ is the map $(p_1, p_2) : R \longrightarrow X \times_S X$. Let $\theta : R \longrightarrow X \times_S H$ be the composite.

$$R \xrightarrow{g_R} D \subset X \times_S H$$

Then we have a commutative diagram (in fact, the outer rectangle is cartesian)

(***)
$$\begin{tikzcd}
R \arrow[r, "\tilde{g}"] \arrow[d, hook, "{(p_1, p_2)}"'] \arrow[drr, "\theta"] & \Gamma \arrow[r, hook] & D \arrow[d, hook] \\
X \times_S X \arrow[rr, "1 \times g"'] & & X \times_S H
\end{tikzcd}$$
($\tilde{g}$ as in (**))

Moreover, since $p_1 \sim p_2$, $g p_1 = g p_2$, whence

$$\theta = (1 \times g) \circ (p_1, p_2) = (p_1, g p_2)$$
$$= (p_1, g p_1) = (1, g) \circ p_1 = \gamma_g \circ p_1$$

It follows that $\tilde{g} : R \longrightarrow \Gamma$ factors as

(+)
$$\begin{tikzcd}
R \arrow[r, "p_1"] \arrow[dr, "\tilde{g}"'] & X \arrow[d, "\wr"] \\
& \Gamma
\end{tikzcd}$$

Note further that $f_H \circ \theta : R \to H$ is the map $g p_1 = g p_2$.

Now consider the commutative diagram

$$\begin{tikzcd}
R \arrow[r, "\tilde{g}"'] \arrow[d, "p_2"'] & \Gamma \arrow[r, hook] & D \arrow[d] \\
X \arrow[rr, "g"'] & & H
\end{tikzcd}$$

<u>Lemma</u>: $\Gamma \times_H D = D \times_H \Gamma$.

<u>Proof</u>:

Let $T \in Sch/S$. Then

$$(\Gamma \times_H D)(T) = \{((x,h),(x',h)) \mid (x,h) \in \Gamma(T),\ (x',h) \in D(T)\}$$

But $(x,h) \in \Gamma(T) \iff h = g(x)$. Thus

$$(\Gamma \times_H D)(T) = \{((x,g(x)),(x',g(x))) \mid (x',g(x)) \in D(T)\}$$

By the Lemma on p. 8, $(x',g(x)) \in D(T)$

$$\iff x \sim x' \iff (x',g(x)) \in \Gamma(T).$$

Thus

$$(\Gamma \times_H D)(T) = (\Gamma \times_H \Gamma)(T).$$

The same argument gives

$$(D \times_H \Gamma)(T) = (\Gamma \times_H \Gamma)(T). \quad \blacksquare$$

By Corollary 2.2.3 of Picard $\underline{V}$, the above lemma then says that the closed subscheme Γ of D descends to a closed subscheme Z of H giving a cartesian square (with $\Gamma \to Z$ being denoted g').

$$\begin{tikzcd}
\Gamma \arrow[r, hook] \arrow[d, "g'"'] & D \arrow[d] \\
Z \arrow[r, hook] & H
\end{tikzcd}$$

Since $X \xrightarrow{\sim} \Gamma \subset D \longrightarrow H$ composes to g (in fact it is $f_H \circ \gamma_g$) therefore

$$X \xrightarrow{\sim} \Gamma \xrightarrow{g'} Z \subset H$$

also composes to g. Thus g factors through

$Z \subset H$. Let $q : X \to Z$ be the composite $X \xrightarrow{\sim} \Gamma \xrightarrow{q'} Z$.

We then have a diagram

$$\begin{tikzcd} R \arrow[r, "p_1"] \arrow[d, "p_2"'] & X \arrow[r, "\sim"] \arrow[d, "q"'] & \Gamma \arrow[r, hook] \arrow[d, "q'"'] & D \arrow[d] \\ X \arrow[r, "q"'] & Z \arrow[r, equal] & Z \arrow[r, hook] & H \end{tikzcd}$$

The outer rectangle commutes since the top row composes to g_R by (+) and (**).
The bottom row composes to g by the argument we gave just above the diagram.
The two subrectangles on the right commute by definition of q and q' — in fact they're both cartesian. As for the subrectangle on the left, if $i : Z \hookrightarrow H$ denotes the closed immersion, then $i \circ q \circ p_1 = i \circ q \circ p_2$, since

the left side is the "east followed by south" composite of the outer rectangle and the right side is the "south followed by east" composite of the outer rectangle. Now $i : Z \hookrightarrow H$ is a closed immersion, whence a monomorphism (see Definition 2·1·1 of Picard $\underline{\text{VI}}$, and exercise below). Thus $q \circ p_1 = q \circ p_2$, i.e. the subrectangle on the left commutes. It is necessarily cartesian since the remaining subrectangles and the outer rectangle are. Now the universal flat family $D \to H$ is necessarily surjective (at the level of maps of sets) by universality. Therefore it is faithfully flat, whence fppf (locally finitely presented having been included in the definition of the Hilbert functor).

Moreover, the graph $\Gamma \hookrightarrow X \times_S H$ has to be finitely presented since X is finitely presented over S. This forces $Z \subset H$ to be finitely presented, and since this takes values in only finite number of connected components of H (here is where the hypothesis of a finite # of Hilbert polynomials occurring in R comes into play) and each component is strongly quasi-projective over S therefore Z is strongly quasi-projective over S.

Now $q : X \to Z$ being the base change of $D \to H$, is fppf and we have a cartesian square

$$\begin{tikzcd} R \arrow[r, "\varphi_1"] \arrow[d, "\varphi_2"'] \arrow[dr, phantom, "\square"] & X \arrow[d, "q"] \\ X \arrow[r, "q"'] & Z \end{tikzcd}.$$

By Proposition 1.2.4 of Picard $\underline{\text{VI}}$, $X \xrightarrow{q} Z$ is an effective quotient of X by R.

Thus we have proven (modulo the exercise below)

Theorem: Let $f: X \to S$ be strongly quasi-projective and $R \to X$ a proper flat finitely presented equivalence relation with only a finite number of Hilbert polynomials occurring amongst the fibers of $pr_2 : R \to X$. Then the quotient X/R exists as an effective quotient which is strongly quasi-projective. The quotient map $X \xrightarrow{q} X/R$ is fppf.

Exercise: Suppose $U \underset{\beta}{\overset{\alpha}{\rightrightarrows}} V$ are two maps and $i : V \hookrightarrow W$ a closed immersion. Then,

$$i \circ \alpha = i \circ \beta \iff \alpha = \beta.$$

[Hint: Question is local on U and W and hence can assume both are affine. This forces V also to be affine. So we have to show that if $A \underset{\psi}{\overset{\phi}{\rightrightarrows}} B$ are two maps of commutative rings and if $C \overset{\eta}{\twoheadrightarrow} A$ is surjective, then $\phi \circ \eta = \psi \circ \eta \iff \phi = \psi$.]