PICARD-VI: RELATIVE REPRESENTABILITY

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1. Equivalence relations

Let S be a scheme. We saw that every S-scheme X is an fpqc-sheaf on $\mathbb{S}\mathrm{ch}_{/S}$. Recall that this means the following: Suppose $p\colon T' \to T$ is an fpqc-map and as usual we set $T'' := T' \times_S T'$, and let $p_1, p_2 \colon T'' \rightrightarrows T'$ denote the two projections. Suppose we have a map $f'\colon T' \to X$ in $\mathbb{S}\mathrm{ch}_{/S}$ such that $f' \circ p_1 = f' \circ p_2$. Then there is a unique map $f\colon T \to X$ such that $f' = f \circ p$. In other words if we have a commutative diagram below of solid arrows in $\mathbb{S}\mathrm{ch}_{/S}$ (with the square being cartesian) then the dotted arrow can be filled in a unique way to make the whole diagram commutative.

(1.1)
$$\begin{tikzcd}
T'' \arrow[r, "p_2"] \arrow[d, "p_1"'] & T' \arrow[d, "p"] \arrow[ddr, bend left, "f'"] & \\
T' \arrow[r, "p"] \arrow[drr, bend right, "f'"'] & T \arrow[dr, dotted, "f"] & \\
& & X
\end{tikzcd}$$

Here our attention is on X and the cartesian diagram of T's is allowed to vary. If we transfer our attention to the commutative square (fixing it) and allow X to vary in $\mathbb{S}\mathrm{ch}_{/S}$ then we get the scheme-theoretic notion of quotients by equivalence relations, or more generally of co-equalizers, notions that we now discuss.

1.2. **Equivalence relations and co-equalizers.** The notion of an equivalence relation on a set has the following natural generalization in the category $\mathbb{S}\mathrm{ch}_{/S}$.

Definition 1.2.1. Let $X \in \mathbb{S}\mathrm{ch}_{/S}$. A *schematic equivalence relation* on X over S is an object $R \in \mathbb{S}\mathrm{ch}_{/S}$ together with a morphism $f\colon R \to X \times_S X$ such that for every $T \in \mathbb{S}\mathrm{ch}_{/S}$ the map of sets
$$f(T)\colon R(T) \to X(T) \times X(T)$$
is *injective* and its image is (the graph of) an equivalence relation on the set $X(T)$. Here, for any $Z \in \mathbb{S}\mathrm{ch}_{/S}$, in keeping with our identification of Z with the functor h_Z, the set $Z(T)$ denotes the set $h_Z(T) := \mathrm{Hom}_{\mathbb{S}\mathrm{ch}_{/S}}(T, Z)$ for any $T \in \mathbb{S}\mathrm{ch}_{/S}$.

For example, the scheme T'' in (1.1) is a schematic equivalence relation on T' over S, or more precisely, the natural map $T'' \to T \times_S T$, is a schematic equivalence relation on T' over S. We will see—from the definition we give below of quotients by equivalence relations—that $p\colon T' \to T$ is the scheme theoretic quotient of T' with respect to this equivalence relation.

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Definition 1.2.2. Let $f\colon R\colon X$ be an equivalence relation on $X \in \mathbb{S}\mathrm{ch}_{/S}$ and $f_1, f_2\colon R \rightrightarrows X$ the natural maps arising from f and the projections $X \times_S X \rightrightarrows X$. A morphism $q\colon X \to Q$ in $\mathbb{S}\mathrm{ch}_{/S}$ s a *quotient* for $R \to X$ (or simply of X by R) if $q \circ f_1 = q \circ f_2$ and given any map $g\colon X \to Z$ in $\mathbb{S}\mathrm{ch}_{/S}$ satisfying $g \circ f_1 = g \circ f_2$ there is a unique map $h\colon Q \to Z$ such that $g = h \circ q$, in other words, as in (1.1), if—in the diagram below—the solid arrows form a commutative diagram, then the dotted arrow can be filled in a unique way to make the whole diagram commute:

(1.2.2.1)
$$\begin{tikzcd} R \arrow[r, "f_2"] \arrow[d, "f_1"'] & X \arrow[d, "q"] \arrow[ddr, bend left, "g"] & \\ X \arrow[r, "q"] \arrow[drr, bend right, "g"'] & Q \arrow[dr, dotted, "h"] & \\ & & Z \end{tikzcd}$$

If the quotient $q\colon X \to Q$ of X by R exists, then we say it is an *effective quotient* if the natural map $(f_1, f_2)\colon R \to X \times_Q R$ is an isomorphism, i.e., if the square in Diagram (1.2.2.1) is cartesian. We often denote the quotient Q, if it exists, by X/R.

Remark 1.2.3. Clearly, from the universal property of quotients by (schematic) equivalence relations, if such a quotient $q\colon X \to Q$ exists, it is unique up to unique isomorphism. In category theory terms, the universal property of $q\colon X \to Q$ makes it a *co-equalizer* for the maps f_1 and f_2. Co-equalizers are clearly unique up to unique isomorphisms.

Consider the situation in Diagram (1.1). The fpqc-map $p\colon T' \to T$ is an effective quotient of T' with respect to the equivalence relation T''. One can say more, namely:

Proposition 1.2.4. *Let* $\mathbf{top} \in \{\mathrm{Zar},\ \text{ét},\ \mathrm{fppf},\ \mathrm{fpqc}\}$[1]*. Let* $p\colon T' \to T$ *be a map in* $\mathfrak{M}_{\mathbf{top}}$*,* $T'' = T' \times_T T'$ *and*

$$\begin{tikzcd} T'' \arrow[r, "p_2"] \arrow[d, "p_1"'] & T' \arrow[d, "p"] \\ T' \arrow[r, "p"'] & T \end{tikzcd}$$

the resulting cartesian diagram. Then the map $p\colon T' \to T$ *is a co-equalizer for* p_1 *and* p_2 *in the category of* $\mathbf{top}$*-sheaves (where, by a* $\mathbf{top}$*-sheaf we mean an* $\mathfrak{M}_{\mathbf{top}}$*-sheaf) on* $\mathbb{S}ch_{/S}$*. In greater detail, if*

$$F\colon (\mathbb{S}ch_{/S})^{\circ} \to (\mathrm{Sets})$$

is an $\mathbf{top}$*-sheaf on* $\mathbb{S}ch_{/S}$ *and we have a map* $f'\colon T' \to F$ *of* $\mathbf{top}$*-sheaves on* $\mathbb{S}ch_{/S}$ *such that* $f' \circ p_1 = f' \circ p_2$*, then there is a unique map* $f\colon T \to F$ *of* $\mathbf{top}$*-sheaves on* $\mathbb{S}ch_{/S}$ *such that* $f \circ p = f'$*. In particular* $T' \to T$ *is a co-equalizer for* p_1 *and* p_2 *in* $\mathbb{S}ch_{/S}$*, whence it is the quotient of* T' *by the equivalence relation* T''*.*

[1] Where these topologies are over $\mathbb{S}\mathrm{ch}_{/S}$. One should perhaps write $\mathrm{Zar}_{/S}$, $\text{ét}_{/S}$, etc. Perhaps one day, in a revised form of these notes . . . .

Proof. Recall from previous notes that a map $T' \to F$ is the same as a section of F over T. Thus $f' \in F(T')$ and $p_1^*(f') = p_2^*(f') \in F(T'')$. Since F is an **top**-sheaf and $p\colon T' \to T$ is a map in $\mathfrak{M}_{\mathbf{top}}$, we have a unique element $f \in F(T)$ such that $f' = p^*(f)$. Now re-interpret $f \in F(T)$ as a map $f\colon T \to F$ and we are done. □

In order to represent $\mathscr{P} = (\mathscr{P}ic_{X/S})_{(\text{fppf})}$ it becomes important to realize $\mathscr{P}$—or at least, some "open and closed" subfunctors of $\mathscr{P}$ which "cover" $\mathscr{P}$—as a co-equalizer of an fppf-map from a scheme to $\mathscr{P}$. The next subsection gives a necessary and sufficient condition under which a map of sheaves can be regarded as a co-equalizer of two maps.

1.3. **Fiber product of functors.** Recall that if $A \to C$ and $B \to C$ are maps of sets, then the fiber-product $A \times_C B$ exists in the category (Sets). In fact an explicit description of this fiber-product is

$$A \times_C B = \{(a,\, b) \in A \times B \,|\, \text{the image of } a \text{ in } C \text{ equals the image of } b \text{ in } C\}.$$

More precisely, $A \times_C B$ is the above set together with the natural projections to A and B. This data has the required universal property for a fiber product, as is readily verified.

One can use the existence of fiber-products in (Sets) to deduce their existence in $\widehat{\mathscr{C}}$, where, as in Lecture 5, $\mathscr{C} = \mathbb{S}\mathrm{ch}_{/S}$, and $\widehat{\mathscr{C}}$ is the category of contravariant (Sets)-valued functors on $\mathbb{S}\mathrm{ch}_{/S}$. To that end, suppose $F, G \rightrightarrows H$ are a pair of maps in $\widehat{\mathscr{C}}$. Set

$$(F \times_H G)(T) = F(T) \times_{H(T)} G(T) \qquad (T \in \mathbb{S}\mathrm{ch}_{/S})$$

(an assignment which is clearly functorial in T). It is easy to see that $F \times_H G$ is indeed a fiber products in $\widehat{\mathscr{C}}$.

Definition 1.3.1. Fix a topology **top** $\in$ {Zar, ét, fppf, fpqc} on $\mathbb{S}\mathrm{ch}_{/S}$. A map $F \to G$ of **top**-sheaves is said to be *surjective* or an *epimorphism* if, given $T \in \mathbb{S}\mathrm{ch}_{/S}$ and an element $\theta \in G(T)$, there exists $T' \to T$ in $\mathfrak{M}_{\mathbf{top}}$ and an element $\eta \in F(T')$ such that the image of η in $G(T')$ under $F(T') \to G(T')$ equals the image of θ in $G(T')$ under $G(T) \to G(T')$. (In other words the diagram

(1.3.1.1)
$$\begin{tikzcd} T' \arrow[r, "\eta"] \arrow[d] & F \arrow[d] \\ T \arrow[r, "\theta"'] & G \end{tikzcd}$$

commutes.)

Proposition 1.3.2. *Let* **top** $\in$ {Zar, ét, fppf, fpqc} *be a topology on* $\mathbb{S}\mathrm{ch}_{/S}$. *A map of* **top**-*sheaves* $F \to G$ *is surjective if and only if it is the co-equalizer of* $F \times_G F \rightrightarrows F$ *in the category of* **top**-*sheaves.*

First suppose $F \to G$ is surjective. Let $\varphi\colon F \to H$ be a map of **top**-sheaves such that the two maps from $F \times_G F$ to H are equal. We have to show that there is a unique map $\varphi\colon G \to H$ such φ' is the composite $F \to G \xrightarrow{\varphi} H$. Suppose $\theta \in F(T)$, for some $T \in \mathbb{S}\mathrm{ch}_{/S}$. Since $F \to G$ is surjective, there is a map $T' \to T$ in **top** and element $\eta \in F(T')$ such that the diagram in (1.3.1.1) commutes. Let $T'' = T \times_T T'$. Then η gives rise to a natural element $\eta^* \in (F \times_G F)(T'')$ via the two images of η in $F(T'')$. Let $\xi' \in H(T')$ be the image of $\eta \in F(T')$ under φ'. The two images of ξ' in $H(T'')$ (under the maps $H(T') \rightrightarrows H(T''))$ coincide, since both agree with the image of $\eta^* \in (F \times_G F)(T'')$ under either composite inherent in the double composite:

$$(F \times_G)(T'') \rightrightarrows F(T'') \xrightarrow{\varphi'(T'')} H(T'').$$

Since H is a **top**-sheaf, ξ' gives rise to a unique element $\xi \in H(T)$.

Let us check that $\xi \in H(T)$ is independent of the data $(T' \to T, \eta)$ making Diagram (1.3.1.1) commute. Suppose we have (T'_1, η_1) and (T'_2, η_2) such that $T'_i \to T$ is in $\mathfrak{M}_{\mathbf{top}}$ and the image of θ in $G(T'_i)$ is the image of η_i in $G(T'_i)$ for $i = 1, 2$. Let $\xi_1 \in H(T)$ and $\xi_2 \in H(T)$ be the elements obtained by the process in the previous paragraph. Let $T^* = T'_1 \times_T T'_2$. Let $\zeta^* = (\eta_1, \eta_2)\colon T^* \to F \times_G F$, and let $\xi^* \in H(T^*)$ be the image of $\zeta^* \in (F \times_G F)(T^*)$ in $H(T^*)$. For $i = 1, 2$, let $\xi'_i = \varphi'(T'_i)(\eta_i) \in H(T'_i)$. Then $\xi^* \in H(T^*)$ is the pull back of both $\xi'_1 \in H(T'_1)$ as well as of $\xi'_2 \in H(T'_2)$ under $T'_1 \to T$ and $T'_2 \to T$ respectively. Since H is a **top**-sheaf and $T'_i \to T$, and $T^* \to T'_i$ are maps in $\mathfrak{M}_{\mathbf{top}}$, the arrows $H(T) \to H(T'_i)$ and $H(T'_i) \to H(T^*)$ are injective for $i = 1, 2$. We have just argued that ξ_1 and ξ_2 have the same image—namely ξ^*—under the injective map $H(T) \to H(T^*)$. This means $\xi_1 = \xi_2$.

Thus we have a well defined map of sets $\varphi(T)\colon G(T) \to H(T)$ given by $\theta \mapsto \xi$. This is clearly functorial in $T \in \mathbb{S}\mathrm{ch}_{/S}$, whence we get a map of sheaves $\varphi\colon G \to H$, and the composite $F \to G \xrightarrow{\varphi} H$ is indeed φ'. Uniqueness of φ satisfying this property follows from the uniqueness of $\xi \in H(T)$ arising from $\theta \in G(T)$.

We do not need the converse for the remaining lectures, therefore we only sketch the proof. The converse needs the notion of a sheafication of a pre-sheaf, which we will construct in a later lecture. Assuming such a process exists, let G' be the sheafification of the presheaf

$$T \mapsto \operatorname{im}(F(T) \to G(T)) \qquad (T \in \mathbb{S}\mathrm{ch}_{/S}).$$

We have a natural maps $G' \to G$ and $F \to G'$. The map $F \to G$ clearly factors as $F \to G' \to G$. One checks that $F \times_{G'} F = F \times_G F$. The map $F \to G'$ is surjective, by construction. Hence, by what we proved above, $F \to G'$ is a co-equalizer of $F \times_G F \rightrightarrows F$. Thus, if $F \to G$ is a co-equalizer of $F \times_G F \rightrightarrows F$, then the natural map $G' \to G$ is an isomorphism, whence $F \to G$ is surjective. □

2. Monomorphisms

2.1. **Closed immersions.** If R is a schematic equivalence relation on $X \in \mathbb{S}\mathrm{ch}_{/S}$, then it is not clear that the map $R \to X \times_S X$ is an immersion (i.e., a closed subscheme of an open subscheme. We will show in Theorem 2.1.4 (1) that if X is separated over S and R is a proper equivalence relation, then in fact $R \to X \times_S X$ is a closed immersion.

Definition 2.1.1. A map $f\colon V \to W$ in $\mathbb{S}\mathrm{ch}_{/S}$ is said to be a *monomorphism* if for every $T \in \mathbb{S}\mathrm{ch}_{/S}$, the induced map $f(T)\colon V(T) \to W(T)$ is injective.

Remarks 2.1.2. Let $f\colon V \to W$ be a map in $\mathbb{S}\mathrm{ch}_{/S}$.

(1) To say f is a monomorphism is the same as saying that given two maps $\alpha, \beta\colon T \rightrightarrows V$ in $\mathbb{S}\mathrm{ch}_{/S}$, such that $f \circ \alpha = f \circ \beta$, we have $\alpha = \beta$. In other words f can be "cancelled from the left". This agrees with usual notion of a monomorphism in a category.

(2) If f is a monomorphism, it is necessarily a *universal monomorphism* in the following sense: Let $W' \to W$ be a map in $\mathbb{S}\mathrm{ch}_{/S}$, and let $f_{(W')}: V' \to W'$ be the resultingbase change of f. Then $f_{(W')}$ is also a monomorphism. To see this observe that if $A \to C$ is an injective map of sets, then for any map of sets $B \to C$, the map $A \times_C B \to B$ is injective.

Proposition 2.1.3. *Let $f: V \to W$ be a proper map in $\mathbb{S}ch_{/S}$ which is a monomorphism. Then f is a closed immersion.*

Proof. Since f is proper, $f(V)$ is a closed subset of W. Let W' be the closed subscheme structure on $f(V)$ given by the ideal $\mathscr{I} := \ker(\mathscr{O}_W \to f_*\mathscr{O}_V)$, and let $i: W' \to W$ be the resulting closed immersion of schemes. Then f factors as

$$V \xrightarrow{g} W' \xrightarrow{i} W.$$

Notice that $V = V \times_W W'$, and g equals $f_{(W')}$, the base change of f by $i: W' \to W$. Replacing W by W' and f by g, if necessary, we see it is enough to prove the seemingly less general result:

P: *Let $f: V \to W$ be a proper map such that W is the scheme theoretic image of f (i.e., $\ker(\mathscr{O}_W \to f_*\mathscr{O}_V) = 0$) and such that it is a monomorphism. Then f is an isomorphism.*

We proceed to prove **P**. We first show that if f is a monomorphism, then it is (set theoretically) injective. To that end suppose v_1 and v_2 are two points in V with the same image $w \in W$ under f. Let $A = k(v_1) \otimes_{k(w)} k(v_2)$, and $T = \operatorname{Spec} A$. We have maps $T \to \operatorname{Spec} k(v_i)$, $i = 1, 2$, inducing maps $\varphi_i: T \to V$, via the natural maps $\operatorname{Spec} k(v_i) \to V$. Clearly $f \circ \varphi_1 = f \circ \varphi_2$. Canceling f from the left, we get $\varphi_1 = \varphi_2$, whence $v_1 = v_2$.

Now assume f satisfies the hypothesis of **P**. Since f is proper and its fibers are finite (singletons!), it is a finite map. The standard proof of this (see Hartshorne's *Algebraic Geometry* for instance) requires Stein factorization, which needs W to be locally noetherian. Deligne, however, has extended the result—that quasi-finite and proper maps are finite maps—to W an arbitrary scheme [EGA IV$_4$, p. 182, Corollaire (18.12.4)]. We are thus reduced to the case where V and W in **P** are affine, say $V = \operatorname{Spec} B$ and $W = \operatorname{Spec} A$. Note that B is a finite A-algebra (i.e., as an A-module B is finitely generated). By Nakayama, it is enough to prove that for any maximal ideal $\mathfrak{m}$ of A, the natural map $A/\mathfrak{m} \to B/\mathfrak{m}B$ is an isomorphism. Indeed, Nakayama would show that $A \to B$ is surjective, and since by the hypothesis of **P**, $\ker(A \to B) = 0$ we would be done. In other words, by making the base change $W' = \operatorname{Spec} A/\mathfrak{m} \to \operatorname{Spec} A = W$, and using the hypothesis that f is a monomorphism (whence a universal monomorphism by Remark 2.1.2(2)), we are reduced to the case where $W = \operatorname{Spec} k$ in **P** (and $V = \operatorname{Spec} B$). Since the map f is set-theoretically one-to-one, B must be an Artin local ring. Let $\mathfrak{m}_B$ denote the maximal ideal of B. By making the faithfully flat base change to the algebraic closure of k if necessary, we may assume that k is algebraically closed. It follows that the finite field extension extension $k \to B/\mathfrak{m}$ (given by $k \to B \twoheadrightarrow B/\mathfrak{m}_B$) is an isomorphism. Consider the k-algebra endomorphism of B given by the composite;

$$B \twoheadrightarrow B/\mathfrak{m}_B \xrightarrow{\sim} k \to B$$

where the last arrow is the k-algebra structure map on B. By the functorial injectivity of f, the above map must be the identity map on B. This forces $\mathfrak{m}_B = 0$

and hence $k \to B$ is an isomorphism, thus proving $\mathbf{P}$. (Cf. [EGA IV$_4$, p. 182, Corollaire (18.12.6)].) $\square$

Theorem 2.1.4. *Let $X \in \mathbb{S}ch_{/S}$ be separated over S and $R \to X \times_S X$ a schematic equivalence relation on X.*

(1) *If $R \to X \times_S X$ is a schematic equivalence relation such that R is a proper scheme over X (via either and hence both projections), then the natural map $R \to X \times_S X$ is a closed immersion.*
(2) *Let $\mathbf{top} \in \{\text{Zar, ét, fppf, fpqc}\}$ be topology on $\mathbb{S}ch_{/S}$. Let $X \xrightarrow{\pi} \mathscr{P}$ be a map of $\mathbf{top}$-sheaves. If $T \to \mathscr{P}$ is a map of $\mathbf{top}$-sheaves, with $T \in \mathbb{S}ch_{/S}$ and $X_T := X \times_{\mathscr{P}} T$ is (representable by) a proper scheme over T, then the natural map $X_T \to X \times_S T$ is a closed immersion.*

Proof. For part (1), note that $R \to X \times_S X$ is proper since both projection $X \times_S X \to X$ are separated (recall X is separated over S) and $R \to X$ is proper. We are using [EGA II, p. 101, Corollaire (5.4.3)] which states that if we have a composite of maps $u \circ v$ which is proper, and u is separated, then v is proper. By definition of an equivalence relation $R \to X \times_S X$ is a monomorphism. The result follows from Proposition 2.1.3. The proof of (2) is similar to that of (1). Since $X \to S$ is separated, so is $X \times_S T \to T$. Now $X_T \to T$ is proper, and $X \times_S T \to T$ is separated, we conclude (again) by [EGA II, p. 101, Corollaire (5.4.3)] that $X_T \to X \times_S T$ is proper. By definition of a fiber product of functors, the map $X_T \to X \times_S T$ is a monomorphism. Proposition 2.1.3 again gives the result. $\square$

3. Representing functors

Our major goal is to represent the functor $(\mathscr{P}ic_{X/S})_{(\text{fppf})}$ for a suitable S scheme X, where $(\mathscr{P}ic_{X/S})_{(\text{fppf})}$ is the sheafification of $\mathscr{P}ic_{X/S}$ in the fppf-topology. We will therefore focus on a general technique which allows one to represent functors.

3.1. **Relative representability.** As in $\mathbb{S}\mathrm{ch}_{/S}$, we often regard fiber products of functors (contravariant, (Sets)-valued, ...) as base changes. This is especially so if one of the factors in the fiber product is a scheme over S. This leads to the notion of relative representability.

Definition 3.1.1. A map $f\colon F \to G$ in $\widehat{\mathscr{C}}$ is said to be *relatively representable* if $F \times_G T\colon (\mathbb{S}\mathrm{ch}_{/T})^\circ \to (\text{Sets})$ is representable for every map $T \to G$ in $\widehat{\mathscr{C}}$ with $T \in \mathbb{S}\mathrm{ch}_{/S}$. If $\mathbf{P}$ is a property of maps of schemes (e.g., $\mathbf{P}$ = flat, smooth, finite type , ...) then we say that $f\colon F \to G$ has property $\mathbf{P}$ (or f is $\mathbf{P}$) if f is relatively representable and for each $T \in \mathbb{S}\mathrm{ch}_{/S}$ the map of schemes $F \times_G T \to T$ has property $\mathbf{P}$.

We often write F_T for $F \times_G T$ ($T \in \mathbb{S}\mathrm{ch}_{/S}$) emphasizing the nature of $F \times_G T$ as the base change of F via $T \to G$. Thus we have a "cartesian" diagram in $\widehat{\mathscr{C}}$:

$$\begin{tikzcd} F_T \arrow[r] \arrow[d, "f_T"'] & F \arrow[d, "f"] \\ T \arrow[r] & G \end{tikzcd}$$

Theorem 3.1.2. *Let $\mathbf{top} \in \{\text{Zar, ét, fppf, fpqc}\}$. Suppose $X \xrightarrow{\pi} \mathscr{P}$ is a relatively representable map of $\mathbf{top}$-sheaves on $\mathbb{S}ch_{/S}$, with X a scheme, π a map in $\mathbf{top}$*

(i.e., every base change of π by a scheme is in **top***). Let $R := X \times_{\mathscr{P}} X$ (note that R is a scheme by the relative representability of π), and let $p_1, p_2\colon R \rightrightarrows X$ be the two projections (necessarily in* **top***), so that we have a cartesian diagram of* **top***-sheaves on $\mathbb{S}ch_{/S}$:*

$$\begin{tikzcd} R \arrow[r, "p_2"] \arrow[d, "p_1"'] & X \arrow[d, "\pi"] \\ X \arrow[r, "\pi"'] & \mathscr{P} \end{tikzcd} \tag{3.1.3}$$

Then $X \xrightarrow{\pi} \mathscr{P}$ is a co-equalizer for $p_1, p_2\colon R \rightrightarrows X$ in the category of **top***-sheaves over S.*

Proof. Suppose $T \in \mathbb{S}\mathrm{ch}_{/S}$ and $\theta \in \mathscr{P}(T)$. Then $\theta\colon T \to \mathscr{P}$ is a map and $X_T := X \times_{\mathscr{P}} T$ is a scheme over T and over X. Let $R_T := R \times_X X_T$. We have a cartesian diagram:

$$\begin{tikzcd} R_T \arrow[r, "p_2'"] \arrow[d, "p_1'"'] & X_T \arrow[d, "\pi_T"] \\ X_T \arrow[r, "\pi_T"'] & T \end{tikzcd} \tag{3.1.4}$$

induced by (3.1.3) and $\theta\colon T \to \mathscr{P}$. Now, by our hypothesis on π, π_T is a map in **top**. Suppose $\varphi\colon X \to \mathscr{G}$ is a map **top**-sheaves on $\mathbb{S}\mathrm{ch}_{/S}$ such that $\varphi \circ p_1 = \varphi \circ p_2$. Let $\varphi_{{}_T}\colon X_T \to \mathscr{G}$ be the composite $X_T \to X \xrightarrow{\varphi} \mathscr{G}$ where the first map is induced by $\theta\colon T \to \mathscr{P}$. Clearly $\varphi_{{}_T} \circ p_1' = \varphi_{{}_T} \circ p_2'\colon R_T \to \mathscr{G}$. Since $\mathscr{G}$ is a **top**-sheaf, the sequence of sets

$$\mathscr{G}(T) \to \mathscr{G}(X_T) \rightrightarrows \mathscr{G}(R_T)$$

is exact. Therefore the element $\varphi_{{}_T} \in \mathscr{G}(T)$ arises from a unique element $\theta^* \in \mathscr{G}(T)$. The association $\theta \mapsto \theta^*$ gives a map of sets $\mathscr{P}(T) \to \mathscr{G}(T)$ which is clearly functorial in $T \in \mathbb{S}\mathrm{ch}_{/S}$, i.e., we have a map of sheaves $\psi\colon \mathscr{P} \to \mathscr{G}$. Using the above exact sequence of sets, it is clear that $\psi \circ \pi = \varphi$. □

Remark 3.1.5. (Added on July 6, 2019) Rahul Hirwani pointed out to me that the proof of Theorem 3.1.2 is trivial once one observes that π is clearly a surjective map of sheaves by definition of relative representability.

3.2. **Representing functors.** Let **top** $\in \{\mathrm{Zar}, \text{ét}, \mathrm{fppf}, \mathrm{fpqc}\}$ be a topology on $\mathbb{S}\mathrm{ch}_{/S}$. Suppose $X \xrightarrow{\pi} \mathscr{P}$ is a map of **top**-sheaves, with $X \in \mathbb{S}\mathrm{ch}_{/S}$. Let $R = X \times_{\mathscr{P}} X$, and suppose R is representable. We have a cartesian diagram:

$$\begin{tikzcd} R \arrow[r, "p_2"] \arrow[d, "p_1"'] & X \arrow[d, "\pi"] \\ X \arrow[r, "\pi"'] & \mathscr{P} \end{tikzcd} \tag{3.2.1}$$

Clearly R is an equivalence relation on X. Let us make the following two assumptions:

(1) The maps p_1 and p_2 are in **top**. (A little thought shows that if one of p_1 or p_2 is in **top**, so is the other, since each of them can be regarded as the "base change of π by π".)
(2) The quotient X/R exists and is an effective quotient and the quotient map $q\colon X \to Q := X/R$ is in **top**.

By Proposition 1.2.4, $q\colon X \to Q$ is a co-equalizer of p_1 and p_2 in the category of **top**-sheaves. Therefore, we have a unique map of **top**-sheaves

$$X/R \to \mathscr{P} \tag{3.2.2}$$

such that $(3.2.2)\circ q = \pi$. Since co-equalizers are unique up to unique isomorphism, for (3.2.2) to be an isomorphism it is necessary and sufficient that $\pi\colon X \to \mathscr{P}$ be a co-equalizer for p_1 and p_2 in the category of **top**-sheaves. Proposition 1.3.2 and Theorem 3.1.2 give us different sufficient conditions for $\pi\colon X \to \mathscr{P}$ to be a co-equalizer for p_1 and p_2. If either condition is verified, $\mathscr{P}$, is representable by $Q = X/R$. (Of course, *a-fortiori*, it follows that π is relatively representable.) We summarize our discussion in the form of the following theorem.

Theorem 3.2.3. *Let $X \xrightarrow{\pi} \mathscr{P}$, be a map of* **top***-sheaves, with $X \in \mathbb{S}ch_{/S}$. Let $R = X \times_{\mathscr{P}} X$, and suppose R is representable and the projections p_i, $i = 1, 2$ as in the cartesian diagram* (3.2.1) *are in* **top**. *Assume that the quotient X/R exists, is an effective quotient, and $X \to X/R$ is in* **top**. *If $\pi\colon X \to \mathscr{P}$ is surjective then $\mathscr{P}$ is representable, in fact by X/R. In particualr if π is* relatively representable*, then $\mathscr{P}$ is representable by X/R.*

3.3. **Strategy for constructing the Picard scheme.** There are two main ingredients, both of which require Hilbert schemes. The first is a general statement regarding the existence of quotient schemes X/R under certain, often occurring, situations, and the second is special to the Picard functor.

1) If $R \to X \times_S X$ is an equivalence relation such that both projections $R \to X$ are flat proper finitely presented and the structural map $X \to S$ is strongly quasi-projective, then X/R exists, is an effective quotient, and $X \to X/R$ is faithfully flat and finitely presented. The proof—which needs the existence of Hilbert schemes—was given by Grothendieck in slightly less general form (projective rather than strongly quasi-projective). The form in which we have stated the result is due to Altmann and Kleiman, and we will give their proof later.

2) Let Z be strongly projective over S. Let $\mathscr{P}' := (\mathscr{P}ic_{Z/S})_{(\mathrm{fppf})}$. One can show that for each $T \in \mathbb{S}\mathrm{ch}_{/S}$, $\mathscr{P}'(T)$ is a disjoint union, $\mathscr{P}'(T) = \coprod_{\mathbf{\Phi}} (\mathscr{P}')^{\mathbf{\Phi}}(T)$ where $(\mathscr{P}')^{\mathbf{\Phi}}(T)$ is the component of $(\mathscr{P}ic_{Z/S})_{(\mathrm{fppf})}$ parameterizing line bundles with Hilbert polynomial $\mathbf{\Phi}$. Note that $(\mathscr{P}')^{\mathbf{\Phi}} \xrightarrow{\sim} (\mathscr{P}')^{\mathbf{\Phi_n}}$, where $\mathbf{\Phi_n}(t) = \mathbf{\Phi}(t{+}n)$. Fix $\mathbf{\Phi}$—large enough so that the higher cohomolgies of the participating line bundles vanish and they are generated by their global sections—and let $\mathscr{P} := (\mathscr{P}')^{\mathbf{\Phi}}$. Now $\mathscr{P}$ is an open and closed subfunctor of $\mathscr{P}'$. Since Z is strongly quasi-projective over S, there exists an fppf-map $\pi\colon X \to \mathscr{P}$, with X a strongly projective S-scheme, namely, the Hilbert scheme of effective relative divisors of $Z \to S$, such that the corresponding line bundles have Hilbert polynomial $\mathbf{\Phi}$. In other words, X will be the scheme which parameterizes effective Cartier divisors $D \hookrightarrow Z$ such that $D \to S$ is flat., and $\mathscr{O}_Z(D)$ has Hilbert polynomial $\mathbf{\Phi}$ over S. The strategy for representing $\mathscr{P}$ is to show:

(a) The equivalence relation $R = X \times_{\mathscr{P}} X$ is proper flat and finitely presented over X (by either projection).
(b) The map $\pi \colon X \to \mathscr{P}$ is a surjective map of fppf-sheaves.

The condition (b) can be replaced by the condition

(b') The $\pi \colon X \to \mathscr{P}$ is relatively representable.

If either (a)+(b) or (a)+(b') hold, then by Theorem 3.2.3, $\mathscr{P}$ (whence $(\mathscr{P}ic_{Z/S})_{(\mathrm{fppf})}$) is representable (by X/R in fact). My original strategy was to prove (a) and (b') since I was not aware of Proposition 1.3.2. It is much more efficient to prove (b) rather than (b') in this particular case, and that is the strategy I follow these coming lectures. The appendix below is for those interested in seeing a proof of (b') (without assuming representabiity of $\mathscr{P}$) for the curious reader. It may well be of independent interest.

Appendix A

A.1. Let Z be strongly projective over S and let $\mathscr{P} := (\mathscr{P}ic_{Z/S})^{\mathbf{\Phi}}_{(\mathrm{fppf})}$, and $\pi \colon X \to \mathscr{P}$ be as in Subsection 3.3. (Recall that $\mathbf{\Phi}$ is "large" and X is the Hilbert scheme of effective relative divisors on $Z \to S$ such that $\mathscr{O}_Z(D)$ has Hilbert polynomial $\mathbf{\Phi}$ over S. If $\theta \colon T \to \mathscr{P}$ is a map in $\widehat{\mathscr{C}}$ with $T \in \mathbb{S}\mathrm{ch}_{/S}$, then it is not a priori clear if the base change $X_T = X \times_{\mathscr{P}} T$ of π is a scheme (i.e., is representable). However this is so, if the element $\theta \in \mathscr{P}(T)$ is given by a line bundle on $Z \times_S T$ as we will see later in these lectures. Since this situation can be arranged by a base change $T' \to T$ of T by an fppf-map, it is eventually possible to show that X_T is representable. This was our original strategy for representing $\mathscr{P}$. However, there is a more elegant way outlined in Subsection 3.3. We have kept this subsection, since the result may be of later interest.

The problem is best addressed in the abstract, without the added clutter of special properties of $(\mathscr{P}ic_{Z/S})_{(\mathrm{fppf})}$ intervening. To that end we prove the following Lemma, and for simplicity, we agree to call a sheaf on a topology **top** over $\mathbb{S}\mathrm{ch}_{/S}$ as simply a **top**-sheaf.

Proposition A.1.1. *Let* **top** $\in \{\mathrm{Zar}, \text{ét}, \mathrm{fppf}, \mathrm{fpqc}\}$ *where these topologies are on* $\mathbb{S}ch_{/S}$. *Let* $X \to \mathscr{P}$ *and* $T \to \mathscr{P}$ *be maps of* **top**-*sheaves with* X *and* T *schemes over* S, *and* X *separated over* S. *Suppose* $T' \to T$ *is a map in* **top** *such that* $X \times_{\mathscr{P}} T'$ *is representable (as a contravariant functor on* $(\mathbb{S}ch_{/T})$*) by (say)* $X_{T'}$ *and* $X_{T'}$ *is proper over* T'. *Then* $X \times_{\mathscr{P}} T$ *is representable by a scheme* X_T *which is proper over* T. *Moreover, if* $X_{T'}$ *is faithfully flat over* T', *then* X_T *is faithfully flat over* T.

Proof. Let T'' and $p_1, p_2 \colon T'' \rightrightarrows T'$ have their usual meanings. Let $\pi_1, \pi_2 \colon X \times_S T'' \rightrightarrows X \times_S T'$ and $p'_1, p'_2 \colon X \times_{\mathscr{P}} T'' \rightrightarrows X \times_{\mathscr{P}} T' (= X_{T'})$ be the maps induced by

p_1 and p_2. We have a commutative cube with cartesian faces:

$$
\begin{tikzcd}
 & X \times_S T'' \arrow[rr, "\pi_2"] \arrow[dd, "\pi_1"] & & X \times_S T' \arrow[dd] \\
X \times_{\mathscr{P}} T'' \arrow[ur] \arrow[rr, "p_2'"] \arrow[dd, "p_1'"'] & & X \times_{\mathscr{P}} T' \arrow[ur] \arrow[dd] & \\
 & X \times_S T' \arrow[rr] & & X \times_S T \\
X \times_{\mathscr{P}} T' \arrow[ur] \arrow[rr] & & X \times_{\mathscr{P}} T \arrow[ur] &
\end{tikzcd}
$$

Examining the (cartesian) top square, we see that $X_{T''} := X \times_{\mathscr{P}} T''$ is representable since the square shows that $X_{T''}$ is the base change of $X \times_S T''$ by $X_{T'} \to X \times_S T'$ and $X_{T'}$ is a scheme. Consider the map $X_{T'} = X \times_{\mathscr{P}} T' \to X \times_S T'$ (see either the edge shared by the top face and the right face, or the edge shared by the bottom face and the left face). The map is proper, by [EGA II, p. 101, Corollaire (5.4.3)], for $X \times_S T' \to T'$ is separated (since X is separated over S) and $X_{T'} \to T'$ is proper. Using the definition of fiber product of functors, one sees that $X_{T'} \to X \times_S T'$ is a monomorphism (see Definition 2.1.1). Therefore Proposition 2.1.3 applies, whence the map $X_{T'} \to X \times_S T'$ is a closed immersion. The diagram shows that $\pi_1^{-1}(X_{T'}) = \pi_2^{-1}(X_{T'})$ (both equal $X_{T''}$), whence by Corollary 2.2.3 in Lecture 5 (see also Remark 2.2.4 of *ibid*) we get a closed subsecheme X_T of $X \times_S T$ and the diagram

$$
\begin{tikzcd}
X_{T'} \arrow[r] \arrow[d] & X_T \arrow[d] \\
T' \arrow[r] & T
\end{tikzcd}
$$

is cartesian. We leave it to the reader to show that X_T represents the functor of T-schemes $X \times_{\mathscr{P}} T$. Using standard facts about faithful flatness (see [EGA IV$_2$, p. 29, Proposition (2.7.1)]) and the above cartesian diagram, we see that $X_T \to T$ is proper, and if $X_{T'}$ is faithfully flat over T', then X_T is faithfully flat over T. $\square$

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