PICARD-V: SCHEMES ARE SHEAVES

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1. The Yoneda Embedding

1.1. **Schemes over S as functors.** For any category $\mathscr{C}$, let $\widehat{\mathscr{C}}$ be the category of contravariant (Sets)-valued functors on $\mathscr{C}$. Recall that this means that an object F of $\widehat{\mathscr{C}}$ is a functor

$$F\colon \mathscr{C}^{\circ} \to \text{(Sets)}$$

and given two such functors F and G, a morphism from F to G is a natural transformation (or, what is the same thing, a functorial map)

$$F \to G.$$

For the rest of these notes, fix a scheme S, and set $\mathscr{C} := \mathbb{S}\text{ch}_{/S}$. Let X be scheme over S. Define the "functor of points" on X to be the functor on $\mathbb{S}\text{ch}_{/S}$

$$h_X\colon (\mathbb{S}\text{ch}_{/S})^{\circ} \to \text{(Sets)}$$

given by

$$T \mapsto \text{Hom}_{\mathbb{S}\text{ch}_{/S}}(T,\, X) \qquad (T \in \mathbb{S}\text{ch}_{/S}),$$

with an obvious effect on morphisms $\varphi\colon T' \to T$ in $\mathbb{S}\text{ch}_{/S}$, namely,

$$q \mapsto q \circ \varphi$$

for $q \in h_X(T) = \text{Hom}_{\mathbb{S}\text{ch}_{/S}}(T,\, X))$. Note that $h_X \in \widehat{\mathscr{C}}$ for every $X \in \mathbb{S}\text{ch}_{/S}$.

Next, if $f\colon X \to Y$ is a map in $\mathbb{S}\text{ch}_{/S}$ then

$$f \circ (\cdot)\colon \text{Hom}_{\mathbb{S}\text{ch}_{/S}}(T,\, X) \to \text{Hom}_{\mathbb{S}\text{ch}_{/S}}(T,\, Y)$$

defined by composing (on the left) with f, is functorial in T. Hence we get a map in $\widehat{\mathscr{C}}$

$$h_f\colon h_X \to h_Y.$$

It is trivial to check that for a pair of maps in $\mathbb{S}\text{ch}_{/S}$

$$X \xrightarrow{f} Y \xrightarrow{g} Z,$$

the diagram

$$\begin{array}{ccc} h_X & \xrightarrow{\;h_f\;} & h_Y \\ & {\scriptstyle h_{(g\,\circ\, f)}}\searrow \quad \swarrow{\scriptstyle h_g} & \\ & h_Z & \end{array}$$

commutes. In other words the association

$$h_{(\cdot)}\colon \mathbb{S}\text{ch}_{/S} \to \widehat{\mathscr{C}} \tag{1.1.1}$$

defines a functor.

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The process $f \mapsto h_f$ (for $f\colon X \to Y$ a map of S-schemes) can be "reversed". More precisely, given a map $\psi\colon h_X \to h_Y$ in $\widehat{\mathscr{C}}$ (X and Y in $\mathbb{S}\mathrm{ch}_{/S}$), we can find a unique map $f = f_\psi\colon X \to Y$ such that $\psi = h_f$. Indeed, we have a map of sets $\psi(X)\colon h_X(X) \to h(Y)$, and hence we have an element $f_\psi \in h_X(Y) = \mathrm{Hom}_{\mathbb{S}\mathrm{ch}_{/S}}(X, Y)$ defined by the image of $\mathbf{1}_X \in h_X(X) = \mathrm{Hom}_{\mathbb{S}\mathrm{ch}_{/S}}(X, X)$ under $\psi(X)$. It is easy to see that $h_{f_\psi} = \psi$. It is equally easy to see—from the definitions—that if $f\colon X \to Y$ is a map in $\mathbb{S}\mathrm{ch}_{/S}$ and $\psi\colon h_X \to h_Y$ is defined by $\psi = h_f$, then $f_\psi = f$ (i.e., $f_{h_f} = f$). Thus $f \mapsto h_f$ and $\psi \mapsto f_\psi$ are inverse processes. This can be restated in the following compact form:

$$h_X(T) = \mathrm{Hom}_{\mathbb{S}\mathrm{ch}_{/S}}(T, X) \xrightarrow{\sim} \mathrm{Hom}_{\widehat{\mathscr{C}}}(h_T, h_X). \tag{1.1.2}$$

Another way of saying this is that $\mathbb{S}\mathrm{ch}_{/S}$ can be regarded as a full subcategory of $\widehat{\mathscr{C}}$ via the functor $h_{(\cdot)}$ (see Theorem 1.1.4 below).

The isomorphism in (1.1.2) can be extended—as we will see below—to give an isomorphism of sets:

$$F(T) \xrightarrow{\sim} \mathrm{Hom}_{\widehat{\mathscr{C}}}(h_T, F). \tag{1.1.3}$$

Indeed, given $\xi \in F(T)$, and $W \in \mathbb{S}\mathrm{ch}_{/S}$, we can define $\theta_\xi(W)\colon h_T(W) \to F(W)$ as follows: Let $f\colon W \to T$ be an element of $h_T(W)$. Writing $f^* = F(f)$, we have $f^*\colon F(T) \to F(W)$. The map $\theta_\xi(W)$ is defined by $f \mapsto f^*(\xi)$. It is easy to see that $\theta_\xi(W)$ is functorial in $W \in \mathbb{S}\mathrm{ch}_{/S}$, whence we have a natural transformation $\theta_\xi\colon h_T \to F$. The association $\xi \mapsto \theta_\xi$ gives us a map $F(T) \to \mathrm{Hom}_{\widehat{\mathscr{C}}}(h_T, F)$. Conversely, given a map $\theta\colon h_T \to F$ in $\widehat{\mathscr{C}}$, we get an element $\xi_\theta \in F(T)$ defined as the image of $\mathbf{1}_T \in h_T(T) = \mathrm{Hom}_{\mathbb{S}\mathrm{ch}_{/S}}(T, T)$ in $F(T)$ under $\theta(T)\colon h_T(T) \to F(T)$. One checks, in the usual way, that $\theta_{\xi_\theta} = \theta$ and $\xi_{\theta_\xi} = \xi$, whence we get the isomorphism (1.1.3).

The isomorphisms (1.1.2) and (1.1.3) are often referred to as the Yoneda lemmas. They are best summarized as a statement, namely:

Theorem 1.1.4. (Yoneda)

(a) *The functor* $h_{(\cdot)}\colon \mathbb{S}ch_{/S} \to \widehat{\mathscr{C}}$ *of* (1.1.1) *is a fully faithful embedding of* $\mathbb{S}ch_{/S}$ *into* $\widehat{\mathscr{C}}$.

(b) *Given* $T \in \mathbb{S}ch_{/S}$ *and* $F\colon (\mathbb{S}ch_{/S})^\circ \to$ (Sets) *a functor, and identifying* T *with* $h_T \in \widehat{\mathscr{C}}$ *via part (a), we have a one-to-one correspondence between* $F(T)$ *and maps* $T \to F$ *in* $\widehat{\mathscr{C}}$.

From now on we will identify $T \in \mathbb{S}\mathrm{ch}_{/S}$ with h_T, and we will treat the isomorphism (1.1.3) as an identity. Thus, with these identifications, we have

$$\mathrm{Hom}_{\widehat{\mathscr{C}}}(T, F) = F(T) \qquad (T \in \mathbb{S}\mathrm{ch}_{/S}, F \in \widehat{\mathscr{C}}). \tag{1.1.5}$$

This should be compared with the special case $\mathrm{Hom}_{\mathbb{S}\mathrm{ch}_{/S}}(T, X) = X(T)$.

Remark 1.1.6. The alert reader would have recognized that in the proof of Theorem 1.1.4, the category $\mathbb{S}\mathrm{ch}_{/S}$ played no essential role, and could have been replaced by an arbitray category $\mathscr{C}$.

1.2. **The structural morphism for objects in $\widehat{\mathscr{C}}$.** Recall that we are working with schemes over a fixed ambient scheme S. When we write $X \in \mathbb{S}\mathrm{ch}_{/S}$ we are really using a shorthand for $(X \to S) \in \mathbb{S}\mathrm{ch}_{/S}$. The map $X \to S$ is often called the *structural map* or sometimes just the *structure map*. If $S = \mathrm{Spec}\, A$ is affine, we call

$X \in \mathbb{S}\mathrm{ch}_{/S}$ an A-scheme rather than an S-scheme and often write $\mathbb{S}\mathrm{ch}_{/A}$ instead of $\mathbb{S}\mathrm{ch}_{/S}$.

Given an S-scheme X, note that $h_S(X)$ is a singleton set whose only element is the structural map $X \to S$. For $F \in \widehat{\mathscr{C}}$, we have a natural map $F \to h_S$ namely the map such that for $X \in \mathbb{S}\mathrm{ch}_{/S}$, the induced map $F(X) \to h_S(X)$ is the map sending all elements of $F(X)$ to the only element of $h_S(X)$. It is clear that this (as X varies in $\mathbb{S}\mathrm{ch}_{/S}$) is functorial in X. Identifying (as we have agreed to) h_S with S, we thus have a map

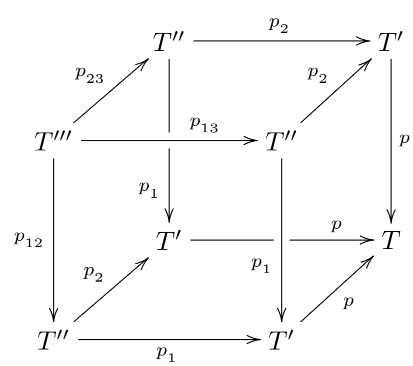
$$F \to S \tag{1.2.1}$$

which we call the structural map for F. In the event the object F of $\widehat{\mathscr{C}}$ lies in the smaller category $\mathbb{S}\mathrm{ch}_{/S}$, clearly the above notion of the structural map coincides with the notion defined for schemes over S.

2. Descent for closed subschemes

2.1. **Decent for fpqc maps.** As usual, for any map $T' \to T$ in $\mathbb{S}\mathrm{ch}_{/S}$, T'' and T''' will be given by $T'' := T' \times_T T'$ and $T''' := T' \times_T T' \times_T T'$. The maps p_1, $p_2 \colon T'' \rightrightarrows T'$ denote the two projections and p_{12}, $p_{13}\, p_{23}$ the three projections from T''' to T''.

Suppose $p \colon T' \to T$ is fpqc. We have a commutative diagram (with all six faces cartesian):



Since descent (obviously) works for Zariski covers, and we have proved that it works for faithfully flat and quasi-compact maps, therefore it works for fpqc maps (see the October 17 notes for the definition of fpqc maps). We're using the fact that the fpqc topology on $\mathbb{S}\mathrm{ch}_{/S}$ is generated by the Zariski topology and the topology given by faithfully flat and quasi-compact maps. In other words if $\mathscr{F}'$ is a quasi-coherent sheaf on T' and we have an isomorphism $\varphi \colon p_2^*\mathscr{F}' \xrightarrow{\ \sim\ } p_1^*\mathscr{F}'$ such that $p_{12}^*(\varphi) \circ p_{23}{}^*(\varphi) = p_{13}^*(\varphi)$, then up to isomorphism, there is a unique quasi-coherent sheaf $\mathscr{F}$ on T satisfying $p^*\mathscr{F} = \mathscr{F}'$.

Exercise: Using the various characterizations of fpqc maps given in the October 17 notes, show directly that descent for fpqc maps follows from descent for faithfully flat and quasi-compact maps. [Hint: First reduce to $T = \operatorname{Spec} A$. Next, pick a quasi-compact open subscheme V' of T' such that $p(V') = T$. Then $V' \to T$ is a quasi-compact faithfully flat map. Descent works for this, and we have obtain a quasi-coherent sheaf $\mathscr{F}$ on T. To show that the end-product (i.e. $\mathscr{F}$) is independent of the process, consider $V = V' \cup V''$ where V'' is another quasi-compact open subscheme of T' which maps surjectively onto T, and use the fact that V is also quasi-compact, so that descent works for $V \to T$. The uniqueness of the descended sheaf should give you the required result.]

Lemma 2.1.1. *Let $p\colon T' \to T$ be a map in $\mathbb{S}ch_{/S}$. With notations as above, suppose we have a map $f'\colon T' \to X$ in $\mathbb{S}ch_{/S}$ such that $f' \circ p_1 = f' \circ p_2$. Then f' is constant on the fibers of p, i.e., if $z_1, z_2 \in p^{-1}(t)$ for some $t \in T$, then $f'(z_1) = f'(z_2)$.*

Proof. As usual, write $k(z_1)$, $k(z_2)$ and $k(t)$ for the residue fields at z_1, z_2 and t respectively. Let $R = k(z_1) \otimes_{k(x)} k(z_2)$. We have two maps $g_1, g_2\colon \operatorname{Spec} R \rightrightarrows T'$ given by the composites $\operatorname{Spec} R \to \operatorname{Spec} k(z_i) \to T'$, $i = 1, 2$. By definition of a fiber product we have a unique map $(g_1, g_2)\colon \operatorname{Spec} R \to T''$ such that $p_i \circ (g_1, g_2) = g_i$ for $i = 1, 2$. Since $f' \circ p_1 = f' \circ p_2$, it follows that $f' \circ g_1 = f' \circ g_2$. Now, g_i factors through $\{z_i\} = \operatorname{Spec} k(z_i)$, for $i = 1, 2$, whence $f'(z_1) = f'(z_2)$. □

2.2. **Descent for quotient sheaves.** The following two results are found in [SGA 1]. For $Z \in \mathbb{S}\mathrm{ch}_{/S}$ for $\mathscr{F}$ a quasi-coherent $\mathscr{O}_Z$-module, let $\operatorname{Quot}(\mathscr{F})$ denote the set of equivalence classes of quasi-coherent quotients of $\mathscr{F}$ in the category of quasi-coherent $\mathscr{O}_Z$-modules. Here, two quotients $\theta_1\colon \mathscr{F} \twoheadrightarrow \mathscr{Q}_1$ and $\theta_2\colon \mathscr{F} \twoheadrightarrow \mathscr{Q}_2$ are considered equivalent if there is an isomorphism $\varphi\colon \mathscr{Q}_2 \xrightarrow{\sim} \mathscr{Q}_1$ such that $\varphi \circ \theta_2 = \theta_1$.

Proposition 2.2.1. [SGA 1, Exposé VIII, Corollaire 1.8] *Let $p\colon T' \to T$ be an* fpqc*-map and $\mathscr{G}$ a quasi-coherent $\mathscr{O}_T$-module. Then, the sequence of set-theoretic maps*

$$\operatorname{Quot}(\mathscr{G}) \to \operatorname{Quot}(\mathscr{G}') \rightrightarrows \operatorname{Quot}(\mathscr{G}'')$$

is exact, where, with the standard notations, $\mathscr{G}' = p^*\mathscr{G}$, $\mathscr{G}'' = q^*\mathscr{G}$, and $q = p \circ p_1 = p \circ p_2\colon T'' \to T$.

We should point out that in [SGA 1] the above result is proved for faithfully flat and quasi-compact maps, but the only ingredient used is descent for faithfully flat and quasi-compact maps, and therefore one can give a proof in the above situation also, as we now proceed to do. I am giving a slightly more elaborate proof than I meant to, making a distinction (for part of the proof) between g^*f^* and $(fg)^*$ for a pair of composable maps f and g. I believe making this distinction gives greater clarity to the proof.

Proof. There is a natural descent datum on $\mathscr{G}'$ since $\mathscr{G}' = p^*\mathscr{G}$. Let $\psi\colon p_2^*\mathscr{G}' \xrightarrow{\sim} p_1^*\mathscr{G}'$ be this descent data. In other words, if $\alpha_1\colon p_1^*p^*\mathscr{G} \xrightarrow{\sim} q^*\mathscr{G}$ and $\alpha_2\colon p_2^*p^*\mathscr{G} \xrightarrow{\sim} q^*\mathscr{G}$ are the natural isomorphisms then $\psi = \alpha_1^{-1} \circ \alpha_2$. Next suppose $\theta'\colon \mathscr{G}' \twoheadrightarrow \mathscr{F}'$ is a quotient such that $p_1^*\theta'$ is equivalent to $p_2^*\theta'$. More precisely, suppose the two composites

$$\mathscr{G}'' = q^*\mathscr{G} \xrightarrow{\sim} p_1^*\mathscr{G}' \xrightarrow{p_1^*\theta'} p_1^*\mathscr{F}' \tag{1}$$

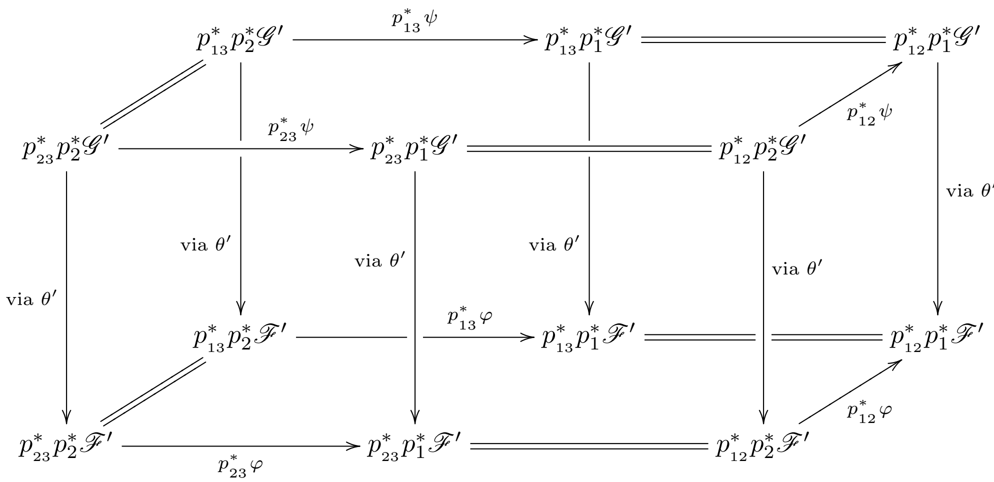
and

$$\mathscr{G}'' = q^*\mathscr{G} \xrightarrow{\sim} p_2^*\mathscr{G}' \xrightarrow{p_2^*\theta'} p_2^*\mathscr{F}' \tag{2}$$

are equivalent. We have to show that there is—up to isomorphism—a unique quasi-coherent $\mathscr{O}_T$-module $\mathscr{F}$ together with a quotient $\theta\colon \mathscr{G} \twoheadrightarrow \mathscr{F}$ such that the quotient $p^*\theta$ is equivalent to θ'. Let $\varphi\colon p_2^*\mathscr{F}' \xrightarrow{\sim} p_1^*\mathscr{F}'$ be the isomorphism resulting from the equivalence of the quotients (1) and (2) above. Note that we then have a commutative diagram:

(2.2.2)
$$\begin{array}{ccc} p_2^*\mathscr{G}' & \xrightarrow[\psi]{\sim} & p_1^*\mathscr{G}' \\ {\scriptstyle p_2^*\theta'}\downarrow & & \downarrow{\scriptstyle p_1^*\theta'} \\ p_2^*\mathscr{F}' & \xrightarrow[\varphi]{\sim} & p_1^*\mathscr{F}' \end{array}$$

This means that if φ is a descent datum on $\mathscr{F}'$ then θ' gives a map of descent data $(\mathscr{G}', \psi) \to (\mathscr{F}', \varphi)$. We will now show that $(\mathscr{F}', \varphi)$ is a descent datum. We will from now on identify $(f \circ g)^*$ with $g^* f^*$ in the usual way for composites of the form $p_1 \circ p_{12}$ etc. Consider the diagram:



By Diagram (2.2.2) all the vertical rectangles commute. The top rectangle commutes, since $(\mathscr{G}', \psi)$ is a descent datum. Since the vertical arrows are surjective (being quotient maps), the bottom rectangle also commutes. The commutativity of the bottom rectangle is the same saying $(\mathscr{F}', \varphi)$ is a descent datum. Now, $p\colon T' \to T$ is fpqc, whence—by faithful flat decsent—the map $\theta'\colon \mathscr{G}' \to \mathscr{F}'$ descends to a unique (up to isomorphism) map of quasi-coherent $\mathscr{O}_T$-modules $\theta\colon \mathscr{G} \to \mathscr{F}$ and the map θ is necessarily surjective by faithful flatness of p. □

We have as an immediate corollary:

Corollary 2.2.3. [SGA 1, Exposé VIII, Corollaire 1.9] *For any scheme X, let $H(X)$ denote the set of closed subschemes of X. With this notation, and the conditions of Proposition* 2.2.1, *the sequence of sets*

$$H(T) \to H(T') \rightrightarrows H(T'')$$

is exact.

Proof. Set $\mathscr{G} = \mathscr{O}_T$ in Proposition 2.2.1. Note that if $(\mathscr{G}', \psi)$ a descent datum such that $\mathscr{G}'$ is a sheaf of rings (resp. algebras) and ψ is a map of sheaves of rings (resp. algebras), then $\mathscr{G}'$ descends to a sheaf of rings (resp.algebras) $\mathscr{G}$. Moreover, if a map of descent datum $\theta'\colon (\mathscr{G}', \psi) \to (\mathscr{F}', \varphi)$ is such that $\mathscr{G}'$, $\mathscr{F}'$ are sheaves of rings (resp. algebras) and ψ, φ, θ' are maps of sheaves of rings (resp. algebras), then the descended map $\theta\colon \mathscr{G} \to \mathscr{F}$ is a map of sheaves of rings (resp. algebras). This follows easily from the proof of faithful flat descent. □

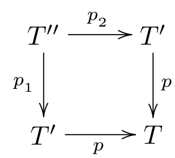
Remark 2.2.4. Corollary 2.2.3 is a compact way of saying the following: Let $T' \xrightarrow{p} T$ be fpqc and let $Z' \hookrightarrow T'$ be a closed subscheme of T' such that $p_1^{-1}(Z') = p_2^{-1}(Z')$. Then there is a unique closed subscheme $Z \hookrightarrow T$ such that $p^{-1}(Z) = Z'$. Here, as is standard, the scheme structures on the closed subspaces $p_1^{-1}(Z') \hookrightarrow T''$, $p_2^{-1}(Z') \hookrightarrow T''$, and $p^{-1}(Z) \hookrightarrow T'$ are given respectively by $p_1^{-1}(Z') = Z' \times_{T'} T''$, $p_2^{-1}(Z') = T'' \times_{T'} Z'$, and $p^{-1}(Z) = Z \times_T T'$.

3. Schemes are fpqc-sheaves

Fix a scheme S. In this section we will prove that a scheme X over S is necessarily an fpqc-sheaf on $\mathbb{S}\mathrm{ch}_{/S}$. More precisely, we will prove that h_X is an fpqc-sheaf over $\mathbb{S}\mathrm{ch}_{/S}$. Note that since the fpqc topology is finer than the Zariski, étale, and fppf topologies on $\mathbb{S}\mathrm{ch}_{/S}$, it follows that X is a Zariski, étale, and an fppf-sheaf.

3.1. The problem restated.

Fix $X \in \mathbb{S}\mathrm{ch}_{/S}$. Suppose



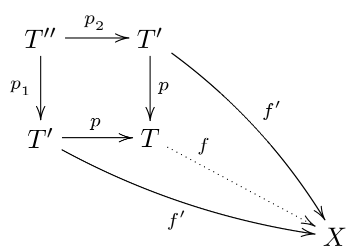
is a cartesian diagram with p (and hence p_1 and p_2) fpqc. In order to show that h_X is an fpqc-sheaf we have to show that the sequence of sets

$$h_X(T) \to h_X(T') \rightrightarrows h_X(T'')$$

is exact, where the first arrow is p^* and the double arrow arises from p_1^* and p_2^*.

The problem can be rephrased as follows. Suppose $f' \colon T' \to X$ is a map in $\mathbb{S}\mathrm{ch}_{/S}$ such that $f' \circ p_1 = f' \circ p_2$. Then there is a unique map $f \colon T \to X$ in $\mathbb{S}\mathrm{ch}_{/S}$ such that $f' = f \circ p$. In other words, if the diagram of solid arrows below commutes, then the dotted arrow can be filled in a unique way to make the whole diagram commute.

(3.1.1)



We will first argue that the problem is local on X. Indeed, if U is an open subscheme of X, then by Lemma 2.1.1 ${f'}^{-1}(U) = p^{-1}(V)$ for a unique subset V of T, for f' is constant on the fibers of p. In Proposition 2.1.1 of the cheat-sheet for faithful flatness[1], we stated that if a map is faithfully flat and quasi-compact then a subset of the target is open if and only if its inverse image (in the source) is open (see [EGA IV$_2$, Corollaire 2.3.12] for a proof). It is not hard to show that this property carries over to fpqc maps (see [FGA-ICTP, p. 28, Proposition 2.35(vi)]

[1] This reference number may unfortunately change since I plan to constantly upgrade the cheat-sheet; flesh it out, supply more proofs . . .

for a proof which assumes the earlier result from [EGA IV$_2$]). It follows that V is open in T. Replacing T' by $V' = p^{-1}(V)$ etc, we see that solving the problem for each member of an open cover of X will solve the problem for X (by the required uniqueness of the map f). Therefore, without loss of generality, we may assume that $X = \operatorname{Spec} A$. The advantage of this is that X is then separated over S, whence the graph of f' is closed in $T' \times_S X$ by [EGA I, p. 135, Corollaire (5.4.3)] or by [EGA, p. 278, Corollaire (5.2.4)]. The reader needs to take note of the fact that in [EGA I], a scheme is by definition separated, and what we today call a scheme is called a prescheme. The terminlogy was changed in [EGA] to reflect the current usage.

Now suppose $Z' \hookrightarrow T' \times_S X$ is the graph of the S-map $f' \colon T' \to X$. Let $q \colon T' \times_S X \to T \times X$, and $q_i \colon T'' \times_S X \to T' \times_S X$, $i = 1, 2$ be the maps induces by p and p_i $(i = 1, 2)$ respectively. Since $f' \circ p_1 = f' \circ p_2$, one checks, $q_1^{-1}(Z") = q_2 - 1(Z')$, both sides being graphs of the common map $f' \circ p_1 = f' \circ p_2$. By Corollary 2.2.3 (see also Remark 2.2.4), there is a unique closed subscheme $Z \hookrightarrow T \times_S X$ such that $q^{-1}(Z) = Z'$. We have a cartesian square

$$\begin{array}{ccc} Z' & \longrightarrow & Z \\ \downarrow & & \downarrow \\ T' & \xrightarrow[p]{} & T \end{array}$$

and since Z' is the graph of f', the downward arrow on the left is an isomorphism. Now, p is faithfully flat, whence the downward arrow on the right is also an isomorphism, as we shall show shortly. Such a $Z \hookrightarrow T \times_S X$ must necessarily be the graph of an S-map $f \colon T \to X$. In fact f is the composite

$$T \xrightarrow{\sim} Z \hookrightarrow T \times_S X \xrightarrow{\text{projection}} X.$$

It remains to show that the downward arrow on the right is an isomorphism. First suppose $A \to A'$ is a flat map of rings, and B is an A-algebra, and $B' = B \otimes_A A'$. Now, by the flatness of $A \to A'$, we have $\ker(A' \to B') = \ker(A \to B) \otimes_A A'$ and $\operatorname{coker}(A' \to B') = \operatorname{coker}(A \to B) \otimes_A A'$. If $A \to A'$ is faithfully flat and $A' \to B'$ is an isomorphism, then from the above it follows that $\ker(A \to B) = 0 = \operatorname{coker}(A \to B)$. Thus $A \to B$ is an isomorphism. The general problem can be reduced to this. First note that the map $Z \to T$ is affine, since the base change of its higher direct images is zero ($Z' \to T'$ being an isomorphism) and the base change map $p \colon T' \to T$ is faithfully flat. Therefore, since the question is local on T, we may assume T (and hence Z) is affine. Next, since p is fpqc, we can find a quasi-compact open subset of T' which maps surjectively on to T. Replacing T' by this open subscheme, we are reduced to the case where T' is quasi-compact. Finally, in the usual way (by taking a finite disjoint union of affine open subschemes) we may replace T' by an affine scheme, and now we are in the affine situation which we handled earlier.

We summarize the above in the form of the following theorem:

Theorem 3.1.2. *Let X be an S-scheme. Then X is a sheaf on the* fpqc *site (whence on the* Zariski, étale*, and* fppf *sites) on $\mathbb{S}ch_{/S}$.*

We point out that the hierarchy of topologies on $\mathbb{S}\mathrm{ch}_{/S}$, with the arrows pointing toward finer topologies, is:

$$\text{Zariski} \to \text{étale} \to \text{fppf} \to \text{fpqc}.$$

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