PICARD-III: DESCENT CONTINUED

PRAMATHANATH SASTRY

We will use the following convention for referring to results in other lectures of this series. A reference to A.x.y.z is a reference to x.y.z of Picard-A. Thus Theorem II.2.2.4 is a reference to Theorem 2.2.4 of Picard-II. We will sometimes use references of the form II.(2.2.1), if (2.2.1) is a displayed formula (or map, or diagram, or ...) in Picard-II.

In Theorem II.3.2.1, we proved faithful flat descent for quasi-coherent sheaves with respect to faithfully flat quasi-compact maps of schemes, given the truth of the statement for faithfully flat maps of affine schemes (i.e. given Theorem II.2.2.4). We now prove the affine case of faithful flat descent.

Throughout this lecture we fix a ring[1] A, an A-module M, and a faithfully flat A-algebra B. We keep the notations of § 2 of Picard-II. Recall that $B^{\otimes r}$ is the r-fold tensor product of B with itself over A and $\alpha_M \colon M \to B \otimes_A M$ is the map $m \mapsto 1 \otimes m$ (cf. §§ II.2.1).

1. The Čech complex for faithfully flat algebras

1.1. Define a sequence of A-maps

$$\begin{aligned}(1.1.1)\qquad & 0 \to M \xrightarrow{\alpha_M} B \otimes_A M \xrightarrow{d^0} B^{\otimes 2} \otimes_A M \xrightarrow{d^1} \dots \\ & \dots \xrightarrow{d^{r-2}} B^{\otimes r} \otimes_A M \xrightarrow{d^{r-1}} B^{\otimes r+1} \otimes_A M \xrightarrow{d^r} \dots\end{aligned}$$

where $d^r = \sum_i (-1)^i e_i$ and

$$e_i(b_0 \otimes \dots \otimes b_r \otimes m) = b_o \otimes \dots \otimes b_{i-1} \otimes 1 \otimes b_i \otimes \dots b_r \otimes m.$$

For consistency write $d^{-1} = \alpha_M$ and $B^{\otimes 0} = A$. The usual arguments give

$$d^r \circ d^{r-1} = 0, \qquad r \geq 0$$

whence (1.1.1) defines a complex of A-modules which we denote $C^{\bullet}_{B/A}(M)$.

Proposition 1.1.2. *$C^{\bullet}_{B/A}(M)$ is exact.*

Proof. Suppose we have a "retract" of the algebra structure map $\alpha_A \colon A \to B$, i.e. a map of rings $g \colon B \to A$ such that the composite $g \circ \alpha_A$ is the identity. (In other words, suppose $\operatorname{Spec} B \to \operatorname{Spec} A$ has a section.) For $r \geq 1$ define

$$k_r \colon B^{\otimes r+2} \otimes_A M \to B^{\otimes r+1} \otimes_A M$$

by

$$b_0 \otimes \dots \otimes b_{r+1} \otimes m \mapsto g(b_0) b_1 \otimes \dots \otimes b_{r+1} \otimes m.$$

Set $k_{-2} = 0$. One checks that

$$k_r d^r + d^{r-1} k_{r-1} = 1$$

Date: September 12, 2008.

[1] Always commutative with 1.

for $r \geq -1$. Thus $\{k_r\}$ is a contracting homotopy on $C^{\bullet}_{B/A}(M)$, whence, in this case, the assertion follows.

For an A-algebra A', let $B' := B \otimes_A A'$. Then, as is easily checked, for $r \geq 1$ $B' \otimes_{A'} B' \otimes_{A'} \cdots \otimes_{A'} B' = B^{\otimes r} \otimes_A A'$, where the number of tensor factors on the left is r. In other words $B'^{\otimes r} = B^{\otimes r} \otimes_A A'$, and this is true for $r \geq -1$ (and not just $r \geq 1$). It is then obvious that

$$(*) \qquad C^{\bullet}_{B/A}(M) \otimes_A A' = C^{\bullet}_{B'/A'}(M \otimes_A A')$$

Now suppose A' is faithfully flat over A, and $C^{\bullet}_{B'/A'}(M \otimes_A A')$ is exact. Then by $(*)$ and faithful flatness, it follows that $C^{\bullet}_{B/A}(M)$ is also exact. Set $A' = B$ so that $B' = B^{\otimes 2}$, and the structure map $\alpha_{A'} \colon A' \to B'$ is $b \mapsto b \otimes 1$. Clearly the map $g' \colon B' \to A'$ given by $b_1 \otimes b_2 \mapsto b_1 b_2$ is a retract of $\alpha_{A'}$. Thus, as we saw earlier in this proof, $C^{\bullet}_{B'/A'}(M \otimes_A A')$ is exact. But A' is faithfully flat over A, since $A' = B$. Hence we are done. $\square$

Remark 1.1.3. Note that $d^0 \colon B \otimes_A M \to B^{\otimes 2} \otimes_A M$ is given by $b \otimes m \mapsto 1 \otimes b \otimes m - b \otimes 1 \otimes m$. Indeed, by definition, $d^0 = e_0 - e_1$ where $e_0(b \otimes m) = 1 \otimes b \otimes m$ and $e_1(b \otimes m) = b \otimes 1 \otimes m$. It follows that

$$(1.1.3.1) \qquad M = \ker(e_0 - e_1)$$

2. Proof of faithful flat descent for affine schemes

2.1. Now suppose (N, ψ) is a descent datum on B/A. Let α and β be the maps

$$\begin{aligned} \alpha &\colon N \to B \otimes_A N, & n &\mapsto 1 \otimes n; \\ \beta &\colon N \to B \otimes_A N, & n &\mapsto \psi(n \otimes 1). \end{aligned}$$

Define (with (1.1.3.1) in mind)

$$(2.1.1) \qquad M := \ker(\alpha - \beta).$$

We claim that there is an isomorphism

$$\theta (= \theta_{N,\psi}) \colon (B \otimes_A M, \psi_M) \xrightarrow{\sim} (N, \psi)$$

in $\mathrm{Mod}_{A \to B}$, where $\psi_M \colon (B \otimes_A M) \otimes_A B \to B \otimes_A (B \otimes_A M)$ is the map given by $b \otimes m \otimes b' \mapsto b \otimes b' \otimes m$ (cf. Proposition II.2.2.3). Note that the claim implies, in particular, that $N \cong B \otimes_A M$.

Let $\theta \colon B \otimes_A M \to N$ be $b \otimes m \mapsto b f(m)$, where

$$f \colon M \hookrightarrow N$$

is the natural inclusion map of A-modules. It is clear that θ is functorial in (N, ψ) (since α and β are). We leave it to the reader to check that θ is a map of descent data, i.e., to check that the diagram

$$\begin{tikzcd} B \otimes_A M \otimes_A B \arrow[r, "\theta\otimes 1"] \arrow[d, "\psi_M"'] & N \otimes_A B \arrow[d, "\psi"] \\ B \otimes_A B \otimes_A M \arrow[r, "1\otimes\theta"'] & B \otimes_A N \end{tikzcd}$$

commutes using the fact that by definition of M, $1 \otimes f(m) = \psi(f(m) \otimes 1)$.

Next, as in Picard II, let $\iota_M\colon M\otimes_A B\to B\otimes_A M$ be the natural map given by $m\otimes b\mapsto b\otimes m$. Consider the diagram with exact rows

$$\text{(D)}\qquad \begin{tikzcd} 0 \arrow[r] & M\otimes_A B \arrow[r] \arrow[d, "\theta\circ\iota_M"'] & N\otimes_A B \arrow[r, "(\alpha-\beta)\otimes 1"] \arrow[d, "\psi"'] & (B\otimes_A N)\otimes_A B \arrow[d, "\psi_1"] \\ 0 \arrow[r] & N \arrow[r, "d^{-1}=\alpha_N"'] & B\otimes_A N \arrow[r, "d^0=e_0-e_1"'] & B\otimes_A B\otimes_A N \end{tikzcd}$$

where $\psi_1(b\otimes n\otimes b')=b\otimes\psi(n\otimes b')$ (cf. II.(2.2.1)). The rows of (D) are exact for the following reasons. First, by definition of M, $0\to M\to N\xrightarrow{\alpha-\beta}B\otimes_A N$ is exact, and tensoring this with the *flat* A-algebra B gives us the top row of (D). The exactness of the bottom row of (D) follows from the exactness of $C^\bullet_{B/A}(N)$.

We claim that (D) commutes. As, before, it is convenient to denote the M in N by $\iota\colon M\hookrightarrow N$. We leave the commutatvity of the rectangle on the left to the reader. The following two facts are helpful for this. First, the image of $m\otimes b\in M\otimes_A B$ in $B\otimes_A N$ under the "south followed by east" route is $1\otimes b(f(m))\in B\otimes_A N$. To see this is also the image under the "east followed by south" route, use the fact that ψ is a $B^{\otimes 2}$-module map, whence $\psi((1\otimes b)x)=(1\otimes b)\psi(x)$.

The commutativity of the rectangle on the right uses the co-cycle rule namely

$$\psi_1\circ\psi_3=\psi_2 \tag{2.1.2}$$

which is the requirement for ψ to be a descent datum on N. Recall from II.(2.2.1) that $\psi_3(n\otimes b_1\otimes b_2)=\psi(n\otimes b_1)\otimes b_2$ and $\psi_2(n\otimes b'\otimes b)=\sum_\alpha b^*_\alpha\otimes b'\otimes n^*_\alpha$ where $\sum_\alpha b^*_\alpha\otimes n^*_\alpha=\psi(n\otimes b)$. In particular (with $b'=1$ in the above formula for ψ_2) we have

$$\begin{aligned}\psi_2(n\otimes 1\otimes b)&=\sum_\alpha b^*_\alpha\otimes 1\otimes n^*_\alpha\\ &=e_1\Big(\sum_\alpha b^*_\alpha\otimes n^*_\alpha\Big)\\ &=(e_1\circ\psi)(n\otimes b).\end{aligned}\tag{2.1.3}$$

We will show that

$$\psi_1\circ(\alpha\otimes 1)=e_0\circ\psi \tag{i}$$

and

$$\psi_1\circ(\beta\otimes 1)=e_1\circ\psi. \tag{ii}$$

The relation (i) is easy since $\alpha(n)=1\otimes n$ and $e_0(b\otimes n)=1\otimes b\otimes n$. We leave the details to the reader. The relation (ii) is trickier. Here are the details for (ii).

$$\begin{aligned}\psi_1\circ(\beta\otimes 1)(n\otimes b)&=\psi_1(\beta(n)\otimes b)\\ &=\psi_1(\psi(n\otimes 1)\otimes b)\\ &=\psi_1(\psi_3(n\otimes 1\otimes b))\\ &=\psi_1\circ\psi_3(n\otimes 1\otimes b)\\ &=\psi_2(n\otimes 1\otimes b) && \text{(by (2.1.2))}\\ &=e_1\circ\psi(n\otimes b) && \text{(by (2.1.3))}\end{aligned}$$

In view of (i) and (ii) we get $\psi_1 \circ ((\alpha - \beta) \otimes 1) = (e_0 - e_1) \circ \psi$. Thus the rectangle on the right in diagram (D) commutes. Since the rows of (D) are exact and ψ and $\psi_1$are isomorphisms, $\theta \circ \iota_M$, whence θ, is also an isomorphism.

Clearly the assignment $(N, \psi) \mapsto M$ is functorial in $(N, \psi) \in \mathrm{Mod}_{A \to B}$. Moreover, it is evident from the above discussion, as well as (1.1.3.1) and (2.1.1), that it provides a pseudo-inverse to the functor $M \mapsto (B \otimes_A M, \psi_{_M})$ on Mod_A. This completes the proof of Theorem II.3.2.1. □

References

[FGA] A. Grothendieck, *Fondements de la Géométrie Algébrique*, Sém, Bourbaki, exp. no$^\circ$ 149 (1956/57), 182 (1958/59), 190 (1959/60), 195(1959/60), 212 (1960/61), 221 (1960/61), 232 (1961/62), 236 (1961/62), Benjamin, New York, (1966).

[EGA] ——— and J. Dieudonné, *Élements de géométrie algébrique I*, Grundlehren Vol **166**, Springer, New York (1971).

[FGA-ICTP] B. Fantechi, L. Göttsche, L. Illusie, S.L. Kleiman, N. Nitsure, A. Vistoli, *Fundamental Algebraic Geometry, Grothendieck's FGA explained*, Math. Surveys and Monographs, Vol **123**, AMS (2005).

[BLR] S. Bosch, W. Lütkebohmert, M. Raynaud, *Néron Models*, Ergebnisse Vol **21**, Springer-Verlag, New York, 1980.

[M] H. Matsumura, *Commutative Ring Theory*, Cambridge Studies **89**.