PICARD-II: DESCENT

PRAMATHANATH SASTRY

All rings are commutative with a multiplicative identity, and all ring maps (i.e., ring homomorphisms) are unital (i.e., 1 maps to 1). If A is a ring, then by an A-algebra we always mean a commutative A-algebra. Some of the results below are in the cheat sheet for faithful flatness. But repetition never hurts.

1. **Faithfully flat algebras**

1.1. Recall the following basic result from commutative algebra (see, for example, [M, Thms 7.2 & 7.3]).

Proposition 1.1.1. [M, Thms. 7.2 and 7.3] *Let $A \to B$ be a map of rings. The following are equivalent:*

1. *B is flat over A and* $\operatorname{Spec} B \to \operatorname{Spec} A$ *is surjective.*[1]
2. *A sequence of A-modules*

(**E**) $$M' \to M \to M''$$

   is exact if and only if (**E**) $\otimes_A B$ *is exact.*

3. *A homomorphism of A-modules $M \to M'$ is injective if and only if the associated homomorphism $M' \otimes_A B \to M \otimes_A B$ is injective.*
4. *B is flat over A, and an A-module M is zero if and only if $M \otimes_A B = 0$.*
5. *B is flat over A, and $\mathfrak{m}B \neq B$ for all maximal ideals $\mathfrak{m}$ of A.*

Definition 1.1.2. A map of rings $A \to B$ is said to be *faithfully flat* if it satisfies any of the equivalent conditions of Proposition 1.1.1. A map of schemes $X \to Y$ is said to be faithfully flat if it is flat and surjective (as a map of sets).

Remark 1.1.3. One can show (along the lines of Proposition 1.1.1) that a map of schemes $f\colon X \to Y$ being faithfully flat is equivalent to any of the following:

1. A sequence **E** of quasi-coherent $\mathscr{O}_Y$-modules

(**E**) $$\mathscr{F}' \to \mathscr{F} \to \mathscr{F}''$$

   is exact if and only if $f^*(\mathbf{E})$ is exact.

2. A map $\theta\colon \mathscr{F} \to \mathscr{F}'$ of quasi-coherent $\mathscr{O}_Y$-modules is injective if and only if $f^*\theta$ is injective.
3. The map f is flat and $f^*\mathscr{F} = 0$ if and only if $\mathscr{F} = 0$ for $\mathscr{F}$ a quasi-coherent $\mathscr{O}_Y$-module.

If $f\colon X \to Y$ is faithfully flat, one can use (3) to prove: *if $\theta\colon \mathscr{F} \to \mathscr{G}$ is a map of quasi-coherent sheaves and $f^*\theta = 0$, then $\theta = 0$.* (In other words the map $\operatorname{Hom}_{\mathscr{O}_Y}(\mathscr{F}, \mathscr{G}) \to \operatorname{Hom}_{\mathscr{O}_X}(f^*\mathscr{F}, f^*\mathscr{G})$ given by $\varphi \mapsto f^*\varphi$ is injective.) Indeed, let $\mathscr{K} = \ker(\theta)$, $\mathscr{I} = \operatorname{im}(\theta)$ and $\mathscr{C} = \operatorname{coker}(\theta)$. Applying he exact functor f^*

Date: September 5, 2008.

[1]As a map of sets.

to the exact sequences $0 \to \mathscr{K} \to \mathscr{F} \to \mathscr{I} \to 0$ and $0 \to \mathscr{I} \to \mathscr{G} \to \mathscr{C} \to 0$, we see that $f^*\mathscr{I}$ is the image of $f^*\theta$. By our hypothesis, $f^*\mathscr{I} = 0$, whence $\mathscr{I} = 0$ by (3) above. Note that this means that two maps $\theta_1, \theta_2 \colon \mathscr{F} \rightrightarrows \mathscr{G}$ of quasi-coherent sheaves are equal if and only if $f^*\theta_1 = f^*\theta_2$. This has the far reaching implication that a diagram (Δ) of arrows of quasi-coherent sheaves on X is commutative if (and clearly—only if) $f^*(\Delta)$ is commutative. Informally one can say: the commutativity of a diagram of quasi-cohrent sheaves on X can be detected by checking its commutativity over X'.

2. Descent for Modules

2.1. Notations. For any ring A the category of A-modules will be denoted Mod_A. Now suppose B is an A-algebra and M is an A-module. Then the map $M \otimes_A B \to B \otimes_A M$ of B-modules given by $m \otimes b \mapsto b \otimes m$ will be denoted ι_M. Note that we have:

$$\iota_M \colon M \otimes_B A \xrightarrow{\ \sim\ } B \otimes_A M.$$

With A and B as above, and $M \in \mathrm{Mod}_A$, set

(i) $B^{\otimes r} := \underbrace{B \otimes_A \cdots \otimes_A B}_{r \text{ times}}$.

(ii) $\alpha_M \colon M \to B \otimes_A M$, $m \mapsto 1 \otimes m$.

2.2. Descent data. Fix an A-algebra B as above. Every B-module N gives rise to two $B^{\otimes 2}$-modules, namely

(i) $N \otimes_A B$ with module structure $(b_1 \otimes b_2)(n \otimes b) = (b_1 n) \otimes (b_2 b)$;
(ii) $B \otimes_A N$ with module structure $(b_1 \otimes b_2)(b \otimes n) = (b_1 b) \otimes (b_2 n)$.

Similarly we have three $B^{\otimes 3}$-modules, namely $N \otimes_A B \otimes_A B$, $B \otimes_A N \otimes_A B$, and $B \otimes_A B \otimes_A N$, the $B^{\otimes 3}$-module structures being obvious and along the lines of the $B^{\otimes 2}$-module structures described above. Suppose we have a $B^{\otimes 2}$-map

$$\psi \colon N \otimes_A B \to B \otimes_A N.$$

We have three maps induced by ψ described as follows:

(2.2.1)

$$\psi_1 \colon B \otimes_A N \otimes_A B \to B \otimes_A B \otimes_A N; \qquad \psi_1 = \mathrm{id}_B \otimes \psi,$$
$$\psi_2 \colon N \otimes_A B \otimes_A B \to B \otimes_A B \otimes_A N; \qquad \psi_2 = (\mathrm{id}_B \otimes \iota_N) \circ (\psi \otimes \mathrm{id}_B) \circ (\mathrm{id}_N \otimes \iota_B),$$
$$\psi_3 \colon N \otimes_A B \otimes_A B \to B \otimes_A N \otimes_A B; \qquad \psi_3 = \psi \otimes \mathrm{id}_B.$$

Note that ψ_2 is the map $n \otimes b_1 \otimes b \mapsto \sum_\alpha b^*_\alpha \otimes b_1 \otimes n^*_\alpha$, where $\sum_\alpha b^*_\alpha \otimes n^*_\alpha = \psi(n \otimes b)$.

Definition 2.2.2. Let $N \in \mathrm{Mod}_B$. A *descent datum* on N is an isomorphism $\psi \colon N \otimes_A B \xrightarrow{\ \sim\ } B \otimes_A N$ such that with ψ_1, ψ_2, ψ_3 as in (2.2.1), we have

$$\psi_2 = \psi_1 \circ \psi_3$$

as maps from $N \otimes_A B \otimes_A B$ to $B \otimes_A B \otimes_A N$. (This is the so-called *cocycle rule.*) *The category of B-modules with descent data (for A)* is the category $\mathrm{Mod}_{A \to B}$ whose objects are pairs (N, ψ) with $N \in Mod_B$ and ψ a descent datum, and whose morphisms $(N,\,\psi) \xrightarrow{\beta} (N',\,\psi')$ are B-maps $\beta\colon N \to N'$ such that the diagram

$$\begin{array}{ccc} N \otimes_A B & \xrightarrow{\psi} & B \otimes_A N \\ {\scriptstyle \beta\otimes \mathrm{id}_B}\downarrow & & \downarrow{\scriptstyle \mathrm{id}_B\otimes\beta} \\ N' \otimes_A B & \xrightarrow[\psi']{} & B \otimes_A N' \end{array}$$

commutes.

Given an A-module M, there is a very natural descent datum on $B \otimes_A M$, namely the map

$$\psi_M\colon (B \otimes_A M) \otimes_A B \to B \otimes_A (B \otimes_A M)$$

given by $b \otimes m \otimes b' \mapsto b \otimes b' \otimes m$.

Proposition 2.2.3. $(B \otimes_A M,\, \psi_M) \in \mathrm{Mod}_{A\to B}$. *Moreover, if* $M \to M'$ *is an* A*-map then the induced map* $\beta\colon B \otimes_A M \to B \otimes_A M'$ *defines a map in* $\mathrm{Mod}_{A\to B}$.

This is an easy (and obvious) computation, which we leave to the reader. Thus the assignment $M \mapsto (B \otimes_A M,\, \psi_M)$ gives us a functor

$$F\colon \mathrm{Mod}_A \to \mathrm{Mod}_{A\to B}.$$

The theorem of faithful flat descent for affine schemes, i.e. the theorem that follows, says that this assignment is an equivalence of categories.

Theorem 2.2.4. *Suppose* B *is faithfuly flat over* A*. Then the functor* $F\colon \mathrm{Mod}_A \to \mathrm{Mod}_{A\to B}$ *defined above is an equivalence of categories.*

We will prove Theorem 2.2.4 in the next lecture. *Loc.cit.* asserts that for a B-module N to be of the form $B \otimes_A M$ for some A-module M, it is necessary and sufficent for N to carry a descent datum $\psi\colon N \otimes_A B \xrightarrow{\sim} B \otimes_A N$. In this case the module $M \in \mathrm{Mod}_A$ is unique up to isomorphism. In fact, as we will see later,

$$M = \{n \in N \,|\, 1 \otimes n = \psi(n \otimes 1)\}.$$

The proof of *loc.cit.* is not difficult, being essentially a familiar Čech cohomology argument, suitably modified to the faithfully flat situation.

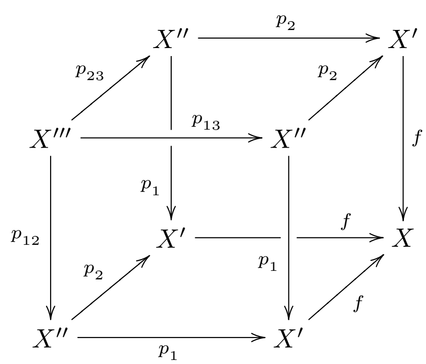
3. Descent for quasi-coherent sheaves on a scheme

3.1. For any scheme Z, let q-coh$_Z$ denote the category of quasi-cohenrent $\mathscr{O}_Z$-modules. Let $f\colon X' \to X$ be a map of schemes. We have a cartesian square (with $X'' = X' \times_X X'$ and $p_1, p_2\colon X'' \rightrightarrows X$ the two projection):

$$\begin{array}{ccc} X'' & \xrightarrow{p_2} & X' \\ {\scriptstyle p_1}\downarrow & \square & \downarrow{\scriptstyle f} \\ X' & \xrightarrow[f]{} & X \end{array}$$

Setting X''' equal to $X' \times_X X' \times_X X'$ and p_{12}, p_{13}, and p_{23} equal to the obvious three projection maps $X''' \to X''$ given by the formulae $(x_1,\, x_2,\, x_3) \mapsto (x_1,\, x_2)$,

$(x_1, x_2, x_3) \mapsto (x_1, x_3)$, and $(x_1, x_2, x_3) \mapsto (x_2, x_3)$ respectively, we have the following three dimensional diagram with each of the six faces being cartesian:



The definitions of the various p_{ij} then imply:

$$p_1 \circ p_{12} = p_1 \circ p_{13}$$
$$p_2 \circ p_{12} = p_1 \circ p_{23}$$
$$p_2 \circ p_{13} = p_2 \circ p_{23}.$$

Definition 3.1.1. Let $\mathscr{G} \in$ q-coh$_{X'}$. A descent datum (on $\mathscr{G}$) with respect to $X' \xrightarrow{f} X$ is an isomorphism

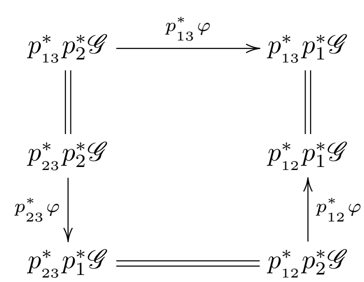
$$\varphi\colon p_2^*\mathscr{G} \xrightarrow{\sim} p_1^*\mathscr{G}$$

such that the cocycle relation

(3.1.1.1) $$p_{12}^*(\varphi) \circ p_{23}^*(\varphi) = p_{13}^*(\varphi)$$

holds, i.e., such that the diagram

(3.1.1.2)



commutes.

One can, in an obvious way, make a category out of quasi-coherent sheaves on X' with descent data withe respect to f. Denote by q-coh$_{X'\to X}$ (or q-coh$_f$) the category whose objects are pairs $(\mathscr{G}, \varphi)$, where $\mathscr{G} \in$ q-coh$_{X'}$ and φ is a descent datum on $\mathscr{G}$. Moreover, with above notations, if $\mathscr{F} \in$ q-coh$_X$, then the natural isomorphism (in fact an identity by our conventions implicit in (3.1.1.2) above)

$$\varphi^f_{\mathscr{F}}(= \varphi_{\mathscr{F}})\colon p_2^*f^*\mathscr{F} \xrightarrow{\sim} p_1^*f^*\mathscr{F}$$

is a descent datum on $f^*\mathscr{F}$. Clearly the assignment $\mathscr{F} \mapsto (f^*\mathscr{F}, \varphi_{\mathscr{F}})$ is functorial in $\mathscr{F} \in$ q-coh$_X$. Let

(3.1.2) $$F\colon \text{q-coh}_X \to \text{q-coh}_{(X'\to X)}$$

denote the resulting functor. Theorem 2.2.4 can be (obviously) restated as: *Let $f\colon X' \to X$ be a faithfully flat map of* **affine schemes**. *Then the functor F in* (3.1.2) *is an equivalence of categories.*

3.2. To generalize the above statement to situations beyond maps of affine schemes, we need to recall that a scheme is called quasi-compact if every open cover has a finite subcover. Every affine scheme (noetherian or not) is quasi-compact. A map of schemes $f\colon X \to Y$ is said to be quasi-compact if the inverse image of a quasi-compact set is quasi-compact. We will show later in this lecture that the natural transformation $F\colon \text{q-coh}_X \to \text{q-coh}_{(X'\to X)}$ of (3.1.2) is an equivalence of categories whenever $f\colon X' \to X$ is a faithfully flat *quasi-compact* map. The quasi-compactness assumption allows us to reduce to the affine case. As a first step we will now show that F is a fully faithful functor under these hypotheses. This means that if $f\colon X' \to X$ is quasi-compact and faithfully flat, $\mathscr{F}$ and $\mathscr{G}$ quasi-coherent sheaves on X, and

$$\beta\colon (f^*\mathscr{F},\, \varphi_{\mathscr{F}}) \to (f^*\mathscr{G},\, \varphi_{\mathscr{G}})$$

a map in $\text{q-coh}_{(X'\to X)}$, then there is a unique map $\alpha\colon \mathscr{F} \to \mathscr{G}$ in q-coh_X such that $f^*\alpha = \beta$. The question is local on X by the uniqueness assertion. Therefore, without loss of generality, we may assume that $X = \operatorname{Spec} A$. Quasi-compactness of f then implies that X' can be covered by a finite number of affine open subschemes $X'_\alpha = \operatorname{Spec} B_\alpha$ of X'. Since the collection of indices α is finite, the scheme $\bar{X}'$ given by $\bar{X}' := \coprod_\alpha X'_\alpha$ is affine. In fact $\bar{X}' = \operatorname{Spec}\left(\prod_\alpha B_\alpha\right)$. Let $\pi\colon \bar{X}' \to X'$ be the natural map and $\bar{f}\colon \bar{X}' \to X$ the composite $f \circ \pi$. In other words we have a commutative diagram

$$\begin{array}{ccc} \bar{X}' & \xrightarrow{\ \pi\ } & X' \\ & \searrow^{\bar{f}} & \downarrow{\scriptstyle f} \\ & & X \end{array}$$

and every arrow in the above diagram is a faithfully flat quasi-compact map. To lighten notation we write Z and $\bar{Z}$ for $X' \times_X X'$ and $\bar{X}' \times_X \bar{X}'$ respectively (rather than the more familiar X'' and $\bar{X}''$) and let $p_1, p_2\colon Z \rightrightarrows X'$, and $\bar{p}_1, \bar{p}_2\colon \bar{Z} \rightrightarrows \bar{X}'$ be the projections. We then have maps $q\colon Z \to X$ and $\bar{q}\colon \bar{Z} \to X$ given by the compsites $q = f \circ p_1 = f \circ p_2$ and $\bar{q} = \bar{f} \circ \bar{p}_1 = \bar{f} \circ \bar{p}_2$. For $\mathscr{F}, \mathscr{G} \in \text{q-coh}_X$ we have the following commutative diagram.

$$\begin{array}{ccccccc} 0 \longrightarrow & \operatorname{Hom}_{\mathscr{O}_X}(\mathscr{F},\, \mathscr{G}) & \xrightarrow{f^*} & \operatorname{Hom}_{\mathscr{O}_{X'}}(f^*\mathscr{F},\, f^*\mathscr{G}) & \xrightarrow{p_1^*-p_2^*} & \operatorname{Hom}_{\mathscr{O}_Z}(q^*\mathscr{F},\, q^*\mathscr{G}) \\ & \Big\| & & \downarrow{\scriptstyle \pi^*} & & \downarrow{\scriptstyle (\pi\times\pi)^*} \\ 0 \longrightarrow & \operatorname{Hom}_{\mathscr{O}_X}(\mathscr{F},\, \mathscr{G}) & \xrightarrow[\bar{f}^*]{} & \operatorname{Hom}_{\mathscr{O}_{\bar{X}'}}(\bar{f}^*\mathscr{F},\, \bar{f}^*\mathscr{G}) & \xrightarrow[\bar{p}_1^*-\bar{p}_2^*]{} & \operatorname{Hom}_{\mathscr{O}_{\bar{Z}}}(\bar{q}^*\mathscr{F},\, \bar{q}^*\mathscr{G}) \end{array}$$

The bottom row is exact since $\bar{f}\colon \bar{X}' \to X$ is a map of affine schemes, whence Theorem 2.2.4 applies. To show that the functor F of (3.1.2) is fully faithful, we have to show that the top row is also exact. Since π and $\pi \times \pi$ are faithfully flat, the downward arrows are injective (see Remark 1.1.3, esp. towards the end), whence the top row is exact.

Assuming Theorem 2.2.4, we are now in a position to prove:

Theorem 3.2.1. *Let $f\colon X' \to X$ be a faithfully flat quasi-compact map of schemes. Then the functor $F\colon \text{q-coh}_X \to \text{q-coh}_{(X'\to X)}$ of (3.1.2) is an equivalence of categories.*

Proof. The question is local on X as can be checked (exercise). Therefore, without loss of generality, we may assume $X = \operatorname{Spec} A$. Since f is quasi-compact and affine schemes are quasi-compact, as before X' is quasi-compact, and we can cover X' by a finite number of affine open subschemes $X'_\alpha = \operatorname{Spec} B_\alpha$ of X'. As before, let $\bar{X}'$ be the affine scheme $\bar{X}' := \coprod_\alpha X'_\alpha$. Let π, $\bar{f}$, p_i, $\bar{p}_i$, $i = 1, 2$, q, and $\bar{q}$ be as in the proof of the full faithfulness of F above.

We have canonical maps

$$\bar{X}' \times_{X'} \bar{X}' \xrightarrow{\delta} \bar{X}' \times_X \bar{X}' \xrightarrow{\pi\times\pi} X' \times_X X'.$$

If $(\mathscr{G}, \varphi) \in \text{q-coh}_{(X'\to X)}$, then one checks that $(\pi^*\mathscr{G}, (\pi\times\pi)^*\varphi)$ is a descent datum for the map $\bar{f}\colon \bar{X}' \to X$. Moreover, using the fact that φ restricted to the diagonal $X' \hookrightarrow X' \times_X X'$ is the identity map on $\mathscr{G}$, one checks that

$$\delta^*(\pi\times\pi)^*\varphi = \varphi^\pi_{\mathscr{G}}. \tag{3.2.1.1}$$

Since $\bar{f}$ is a map of affine schemes, by Theorem 2.2.4[2], the quasi-coherent sheaf $\pi^*\mathscr{G}$ descends to $\mathscr{F} \in \text{q-coh}_X$, and we can identify $(\bar{f}^*\mathscr{F}, \varphi^{\bar{f}}_{\mathscr{F}})$ with the descent datum $(\pi^*\mathscr{G}, (\pi\times\pi)^*\varphi)$. Applying δ^* to this identification, we get an identification of descent data with respect to the map $\pi\colon \bar{X}' \to X'$. In greater detail, the descent datum $\delta^*\varphi^{\bar{f}}_{\mathscr{F}}$ on $\bar{f}^*\mathscr{F} (= \pi^*f^*\mathscr{F})$ identifies with $\delta^*(\pi\times\pi)^*\varphi$ on $\pi^*\mathscr{G}$. Now, $\delta^*\varphi^{\bar{f}}_{\mathscr{F}} = \varphi^\pi_{f^*\mathscr{F}}$. Using this and (3.2.1.1) we obtain an isomorphism $f^*\mathscr{F} \xrightarrow{\ \sim\ } \mathscr{G}$ by the full faithfulness of the functor $\mathscr{K} \mapsto (\pi^*\mathscr{K}, \varphi^\pi_{\mathscr{K}})$ on $\text{q-coh}_{X'}$ as proven in the discussion at the beginning of Subsection 3.2.[3] It remains to identify $\varphi^f_{\mathscr{F}}$ with φ under the just deduced isomorphism $f^*\mathscr{F} \xrightarrow{\ \sim\ } \mathscr{G}$. According Remark 1.1.3 this can be checked after applying $(\pi\times\pi)^*$ since $\pi\times\pi$ is faithfully flat. Doing this yields the original identification $(\bar{f}^*\mathscr{F}, \varphi^{\bar{f}}_{\mathscr{F}}) \xrightarrow{\ \sim\ } (\pi^*\mathscr{G}, (\pi\times\pi)^*\varphi)$, whence $(f^*\mathscr{F}, \varphi^f_{\mathscr{F}}) \xrightarrow{\ \sim\ } (\mathscr{G}, \varphi)$. It is clear that the process $(\mathscr{G}, \varphi) \mapsto \mathscr{F}$ is functorial in $(\mathscr{G}, \varphi) \in \text{q-coh}_{(X'\to X)}$ and by its "construction" gives a pseudo-inverse to the functor F. □

Thus it only remains for us to prove the affine case of Theorem 3.2.1 (i.e. Theorem 2.2.4) for us to complete its proof. This, as we stated earlier, will be done in the next lecture.

References

[FGA] A. Grothendieck, *Fondements de la Géométrie Algébrique*, Sém, Bourbaki, exp. no° 149 (1956/57), 182 (1958/59), 190 (1959/60), 195(1959/60), 212 (1960/61), 221 (1960/61), 232 (1961/62), 236 (1961/62), Benjamin, New York, (1966).

[EGA] ——— and J. Dieudonné, *Élements de géométrie algébrique I*, Grundlehren Vol **166**, Springer, New York (1971).

[FGA-ICTP] B. Fantechi, L. Göttsche, L. Illusie, S.L. Kleiman, N. Nitsure, A. Vistoli, *Fundamental Algebraic Geometry, Grothendieck's FGA explained*, Math. Surveys and Monographs, Vol **123**, AMS (2005).

[BLR] S. Bosch, W. Lütkebohmert, M. Raynaud, *Néron Models*, Ergebnisse Vol **21**, Springer-Verlag, New York, 1980.

[M] H. Matsumura, *Commutative Ring Theory*, Cambridge Studies **89**.

[2] To be proved in the next lecture

[3] Note that π is quasi-compact and faithfully flat.