PICARD-I: OVERVIEW

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1. Basic Definitions

1.1. For any scheme Z, $\mathrm{Pic}(Z)$ is defined to be the group of isomorphism classes of line bundles on Z (the group operation being tensor product).

We will fix a base scheme S (*not necessarily noetherian!*) and $\mathbb{S}\mathrm{ch}_{/S}$ will denote the category of S-schemes. Recall that this means that the objects of $\mathbb{S}\mathrm{ch}_{/S}$ are maps $X \to S$ and a morphism between two objects, say $X \to S$ and $Y \to S$, in $\mathbb{S}\mathrm{ch}_{/S}$ is a commutative diagram in the category of schemes:

$$\begin{array}{ccc} X & \longrightarrow & Y \\ & \searrow & \downarrow \\ & & S \end{array}$$

In practice, if $X \to S$ is an object of $\mathbb{S}\mathrm{ch}_{/S}$, we often write $X \in \mathbb{S}\mathrm{ch}_{/S}$, rather than $(X \to S) \in \mathbb{S}\mathrm{ch}_{/S}$, especially if the underlying map from X to S is understood from the context (for example when X is a k-variety, where k is a field, and $S = \mathrm{Spec}(k)$). With this understanding, if $X \in \mathbb{S}\mathrm{ch}_{/S}$, the underlying map $X \to S$ is called the *structural map*. If S is affine, say $S = \mathrm{Spec}(A)$ then we often write $\mathbb{S}\mathrm{ch}_{/A}$ instead of $\mathbb{S}\mathrm{ch}_{/S}$ and refer to X as an A-scheme rather than as an S-scheme. For two S-schemes X and T, X_T is the T-scheme given by the formula

$$X_T := X \times_S T. \tag{1.1.1}$$

(More precisely: X_T is the T-scheme $X \times_S T \xrightarrow{\text{projection}} T$.)

The category of abelian groups will be denoted by the symbol $\mathbf{Ab}$.

Definition 1.1.2. (The absolute Picard functor) Let $X \in \mathbb{S}\mathrm{ch}_{/S}$. The *absolute Picard functor* $\mathrm{Pic}_X \colon \mathbb{S}\mathrm{ch}_{/S} \to \mathbf{Ab}$ is the (contravariant) functor given by the formula

$$\mathrm{Pic}_X(T) := \mathrm{Pic}(X \times_S T) = \mathrm{Pic}(X_T).$$

Definition 1.1.3. (The relative Picard functor) Let $X \in \mathbb{S}\mathrm{ch}_{/S}$. The *relative Picard functor* $\mathscr{P}ic_{X/S} \colon \mathbb{S}\mathrm{ch}_{/S} \to \mathbf{Ab}$ is the (contravariant) functor given by the formula

$$\mathscr{P}ic_{X/S}(T) := \frac{\mathrm{Pic}(X \times_S T)}{\mathrm{Pic}(T)} = \frac{\mathrm{Pic}_X(T)}{\mathrm{Pic}(T)} \left(= \frac{\mathrm{Pic}(X_T)}{\mathrm{Pic}(T)} \right).$$

The absolute Picard $\mathrm{Pic}_X(T)$ gives us a family of line bundles on X parameterized by T. However if ξ is such a family (i.e., if $\xi \in \mathrm{Pic}_X(T)$) then $\xi \otimes p_T^* L \in \mathrm{Pic}_X(T)$ gives the "same" family, where $p_T \colon X_T \to T$ is the structural map of the T-scheme X_T, and $L \in \mathrm{Pic}(T)$. Indeed, $p_T^* L|_{X_t}$ is trivial on each fiber $X_t := X_T \times_T \mathrm{Spec} k(t)$

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of p_T, with $t \in T$. This is why it is more natural to consider the relative Picard functor.

As expected, if $S = \mathrm{Spec}(A)$, then we often write $\mathscr{P}ic_{X/A}$ for $\mathscr{P}ic_{X/S}$.

Our goal is to represent the functor $\mathscr{P}ic_{X/S}$ under certain special conditions on the structural map $X \to S$.[1] This means the following: there is an S-scheme $\mathbf{Pic}_{X/S}$ as well as a line bundle L_u on $X \times_S \mathbf{Pic}_{X/S}$ such that if $\xi \in \mathscr{P}ic_{X/S}(T)$ for some $T \in \mathbb{S}\mathrm{ch}_{/S}$, then there exists a unique map—the so called *classifying map*—$g\colon T \to \mathbf{Pic}_{X/S}$ such that the isomorphism class ξ_u^g of the line bundle $(1 \times g)^* L_u$ is equivalent to ξ, i.e., the image of $\xi_u^g \in \mathrm{Pic}(X_T)$ in $\mathscr{P}ic_{X/S}(T) = \mathrm{Pic}(X_T)/\mathrm{Pic}(T)$ is ξ. In other words, we want an isomorphism of functors:

$$\mathrm{Hom}_{\mathbb{S}\mathrm{ch}_{/S}}(-,\, \mathbf{Pic}_{X/S}) \xrightarrow{\ \sim\ } \mathscr{P}ic_{X/S}$$

such that $\mathbf{1} \in \mathrm{Hom}_{\mathbb{S}\mathrm{ch}_{/S}}(\mathbf{Pic}_{X/S},\, \mathbf{Pic}_{X/S})$ maps to $\xi_u \in \mathscr{P}ic_{X/S}(\mathbf{Pic}_{X/S})$ where ξ_u is the image of $[L_u] \in \mathrm{Pic}_X(\mathbf{Pic}_{X/S})$ in $\mathscr{P}ic_{X/S}(\mathbf{Pic}_{X/S})$. Here $[L_u]$ denotes the isomorphism class of the line bundle L_u.

2. Overview

In what follows, (Sets) will denote the category of sets.

2.1. Recall that a (contravariant) functor $F\colon \mathscr{C} \to (\mathrm{Sets})$ on a category $\mathscr{C}$ is called *representable* if there exists an object $X \in \mathscr{C}$ such that there is a functorial isomorphism

$$h_X := \mathrm{Hom}_{\mathscr{C}}(-,\, X) \xrightarrow{\ \sim\ } F.$$

Suppose this is true and set $\theta \in F(X)$ equal to the image of $\mathbf{1}_X \in \mathrm{Hom}_{\mathscr{C}}(X,\, X) = h_X(X)$ under the above isomorphism. Then the pair $(X,\, \theta)$ is said to *represent* F.

In fact, given any pair $(X,\, \theta)$ with $X \in \mathscr{C}$ and $\theta \in F(X)$ one has an induced map of functors

$$\hat{\theta}\colon h_X \to F$$

uniquely characterized by the following property: for $Y \in \mathscr{C}$, and $f \in h_X(Y)$ (i.e., $Y \xrightarrow{f} X$),

$$\hat{\theta}(Y)(f) = F(f)(\theta)$$

where $F(f)\colon F(X) \to F(Y)$ is the map induced by $f\colon Y \to X$ and the contravariance of F. In other words, $(X,\, \theta)$ represents F if and only if $\hat{\theta}$ is an isomorphism. We often do not distinguish between:

- X and h_X;
- $\theta \in F(X)$ and $h_X \xrightarrow{\hat{\theta}} F$.

Thus, to say θ is an element of $F(X)$ is to say that we have a "map"

$$\theta\colon X \to F.$$

[1]In general this is not possible—even when X is a projective k-variety, where k is a field—without modifying the relative Picard functor, for, as we shall see later in these lectures, in order for a functor to be representable, it must be a "sheaf" on a Grothendieck topology, namely the fpqc-topology. The trick is to "sheafify" the "presheaf" $\mathscr{P}ic_{X/S}$.

2.2. In the event $F\colon \mathscr{C} \to \mathbf{Grp}$ is a contravariant functor, with $\mathbf{Grp}$ the category of groups, then, with $\mathbf{Grp} \xrightarrow{\pi}$ (Sets) the forgetful functor, one says F is representable if $\pi \circ F$ is representable. If this happens, one can show that the representing object $G \in \mathscr{C}$ is a group-object, under very mild hypothesis ($\mathscr{C}$ should have products and F should respect products). In particular, if $\mathscr{P}ic_{X/S}$ is representable, then the representing object is necessarily a group scheme, in fact a commutative group scheme. The above statement applies to any sheafification of $\mathscr{P}ic_{X/S}$ which is also $\mathbf{Ab}$-valued.

2.3. If $X \in \mathbb{S}\mathrm{ch}_{/S}$ then one can show (as we will in later lectures) that X (i.e., h_X) is a "sheaf" of sets on the Zariski, étale, fppf, and fpqc topologies on S—which topologies will be defined later. Therefore for any contravariant (Sets)-valued functor to be representable it too needs to be a sheaf in all these topologies. Unfortunately $\mathscr{P}ic_{X/S}$ as defined need not be a sheaf even in the Zariski topology, not even if $S = \mathrm{Spec}(k)$, k a field and X a smooth projective curve. It is a sheaf in the Zariski topology however if X has a k-rational point. In this case, it is actually a sheaf in all the topologies mentioned above.

2.4. I will say more on these topologies later. For now, we simply give a quick overview of what is required. The functor $\mathscr{P}ic_{X/S}$ is a presheaf for any Grothendieck topology on $\mathbb{S}\mathrm{ch}_{/S}$—by definition—since it is a contravariant functor. One can sheafify to get various Picard functors:

$$(\wp) \qquad (\mathscr{P}ic_{X/S})_{(\mathrm{Zar})}, (\mathscr{P}ic_{X/S})_{(\text{ét})}, (\mathscr{P}ic_{X/S})_{(\mathrm{fppf})}, (\mathscr{P}ic_{X/S})_{(\mathrm{fpqc})}.$$

Now the topologies Zar, ét, fpqc, fppf are succesively finer than their predecessors and moreover, each of the above functors is the sheafification of any of its predecessors in the appropriate topology[2]. If any one of them is representable, then it is a sheaf in all the topologies, whence equal to its sheafification in any succeeding topology in the hierarchy (ranked from coarser to finer): Zar, ét, fppf, fpqc. It follows of that if any functor in ($\wp$) is representable then (a) it is equal to all the succeeding functor and therefore (b) all the succeeding functors are representable. In this case the common representing group-scheme is called the *Picard scheme* and is denoted $\mathbf{Pic}_{X/S}$. The main point being made is this: while there are four functors (five, counting the absolute Picard functor Pic_X), there is at most one Picard scheme $\mathbf{Pic}_{X/S}$. We state this formally as a definition.

Definition 2.4.1. We will say that the *Picard scheme* $\mathbf{Pic}_{X/S}$ *exists* if some functor in ($\wp$) is representable (and hence, so are all succeeding functors in ($\wp$)). In this case $\mathbf{Pic}_{X/S}$ will denote the common scheme representing this functor and all its successors in ($\wp$).

2.5. In order to understand the topologies mentioned above—we are mainly interested in the fppf topology—and sheaves on them, we will have to spend time on faithful flat descent.

2.6. Hilbert schemes enter into the construction in two essential ways:

(i) To find a space of effective relative Cartier divisors for $X \to S$.
(ii) To construct the quotient of a quasi-projective S-scheme X by a flat, proper, equivalence relation.

[2] In fact each of them is the sheafification of the absolute Picard functor Pic_X in the appropriate topology

3. Main Theorems

The Main theorems concerning the existence of $\mathbf{Pic}_{X/S}$ are given below. Terms such as finitely presented and locally finitely presented will be defined in later lectures. The usual practice is to represent $\mathscr{P}ic^{(\mathrm{fppf})}_{X/S}$, and this is how it is done in [BLR].

Theorem 1. [Grothendieck, FGA] *Let $X \xrightarrow{f} S$ be flat, projective, finitely presented map of schemes with reduced and irreducible geometric fibers. Then $\mathbf{Pic}_{X/S}$ exists as a separated quasi-projective S-scheme which is locally of finite presentation.*

Theorem 2. [Mumford, unpublished] *Let $X \xrightarrow{f} S$ be flat, projective, finitely presented map of schemes with reduced geometric fibers such that the irreducible components of the ordinary fibers are geometrically irreducible. Then $\mathbf{Pic}_{X/S}$ exists as a possibly non-separated S-scheme which is locally of finite presentation.*

Before stating the next theorem (and this is the version we will prove) due to Altmann and Kleiman, we need some notations and conventions nailed down. Suppose $X \xrightarrow{f} S$ is flat and strongly projectibe. Given $\xi \in \mathscr{P}ic_{X/S}$, T connected, we have a line bundle L_ξ on X_T. Since X_T is flat over T, the Hilbert polynomial of $L_\xi|_{X_t}$ is independent of $t \in T$ (since T connected). Denote this polynomial $\mathbf{\Phi}_\xi$. For any polynomial $\mathbf{\Phi} \in \mathbb{Q}[t]$, let

$$\mathscr{P}ic^{\mathbf{\Phi}}_{X/S}(T) = \{\xi \in \mathscr{P}ic_{X/S}(T) \,|\, \mathbf{\Phi}_\xi = \mathbf{\Phi}\}.$$

$\mathscr{P}ic^{\mathbf{\Phi}}_{X/S}$ is clearly a contravariant (Sets)-valued functor on $\mathbb{S}\mathrm{ch}_{/S}$, in fact it is a subfunctor of $\mathscr{P}ic_{X/S}$. One checks, easily, that

$$\mathscr{P}ic_{X/S}(T) = \coprod_{\mathbf{\Phi}} \mathscr{P}ic^{\mathbf{\Phi}}_{X/S}(T).$$

As in ($\wp$) we have various sheafifications of the functor $\mathscr{P}ic^{\mathbf{\Phi}}_{X/S}$. If any one of them is representable, the representing object is denoted $\mathbf{Pic}^{\mathbf{\Phi}}_{X/S}$.

Theorem 3. [Altmann-Kleiman] *Suppose S is quasi-compact and $f\colon X \to S$ is flat, strongly projective, with reduced and irreducible geometric fibers. Then for any $\mathbf{\Phi} \in \mathbb{Q}[t]$, $\mathbf{Pic}^{\mathbf{\Phi}}_{X/S}$ exists as a strongly quasi-projective S-scheme. Moreover, the relative Picard scheme for X/S also exists and*

$$\mathbf{Pic}_{X/S} = \coprod_{\mathbf{\Phi}} \mathbf{Pic}^{\mathbf{\Phi}}_{X/S}.$$

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