FAITHFUL FLATNESS CHEAT SHEET

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The cheat sheet is uneven in its level of presentation. Some elementary things are proved. Others aren't and in at least one instance you're asked to look up EGA (though the proof is not all that difficult). I take pains to define faithful flatness. And yet the notion of a proper map is considered "well-known". I will probably update the cheat sheet every now and then. This is meant to give basic facts about faithful flat maps (but not advanced stuff like descent).

1. Definition

This is more or less the same as what we had in the second lecture on the Picard scheme (i.e., the lecture on Sep 5, unfortunately dated Sep 6 in the handwritten notes). Recall that a map of (commutative) rings $A \to B$ is said to be flat if the functor $(\_) \otimes_A B$ (from Mod_A to Mod_B) is exact. The notion extends to maps of schemes $f\colon X \to Y$ in two ways, namely, (a) for every $x \in X$, the natural map of rings $\mathscr{O}_{Y,f(x)} \to \mathscr{O}_{X,x}$ is flat, and (b) the functor f^* from quasi-coherent $\mathscr{O}_Y$-modules to quasi-coherent $\mathscr{O}_X$-modules is exact. Fortunately the two notions coincide, and both (clearly) coincide with a third possible extension of the notion of flatness (from rings to schemes), namely that for every pair of affine open subschemes $U = \operatorname{Spec} B \subset X$ and $V = \operatorname{Spec} A \subset Y$ such that U maps to V under f, the map $A \to B$ is flat.

1.1. **Stability under base change and composites.** Flatness is stable under base change. Indeed we are quickly reduced to the affine case. If $A \to B$ is flat and $A \to A'$ is a map of rings (always commutative, always with multiplicative identity, and always unital, i.e. $1_A \mapsto 1_B$), and $B' := A' \otimes_A B$, then we have to show that the natural map $A' \to B'$ (induced by $A \to B$) is flat. But,

$$(\_) \otimes_{A'} B' = (\_) \otimes_{A'} (A' \otimes_A B) = (\_) \otimes_A B$$

and the functor on the extreme right is exact, whence so is the functor on the extreme left.

Next, if $X \xrightarrow{f} Y \xrightarrow{g} Z$ is a pair of flat maps, then f^* and g^* are exact, whence so is the composite functor f^*g^*. Now, $f^*g^* = (gf)^*$. It follows that the composite of flat maps is flat.

1.2. **Faithful flatness.** Here is the main result which defines faithful flatness (see also Vistoli's notes in [FGA-ICTP]):

Proposition 1.2.1. [M, Thms. 7.2 and 7.3] *Let $A \to B$ be a map of rings. The following are equivalent:*

(1) *B is flat over A and* $\operatorname{Spec} B \to \operatorname{Spec} A$ *is surjective.*[1]

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[1]As a map of sets.

(2) *A sequence of A-modules*

$$(\mathbf{E}) \qquad M' \to M \to M''$$

is exact if and only if $(\mathbf{E}) \otimes_A B$ *is exact.*

(3) *A homomorphism of A-modules* $M \to M'$ *is injective if and only if the associated homomorphism* $M' \otimes_A B \to M \otimes_A B$ *is injective.*

(4) *B is flat over A, and an A-module M is zero if and only if* $M \otimes_A B = 0$.

(5) *B is flat over A, and* $\mathfrak{m}B \neq B$ *for all maximal ideals* $\mathfrak{m}$ *of A.*

Definition 1.2.2. A map of rings $A \to B$ is said to be *faithfully flat* if it satisfies any of the equivalent conditions of Proposition 1.2.1. A map of schemes $X \to Y$ is said to be faithfully flat if it is flat and surjective (as a map of sets).

Remark 1.2.3. One can show (along the lines of Proposition 1.2.1) that a map of schemes $f\colon X \to Y$ being faithfully flat is equivalent to any of the following:

(1) A sequence $\mathbf{E}$ of quasi-coherent $\mathscr{O}_Y$-modules

$$(\mathbf{E}) \qquad \mathscr{F}' \to \mathscr{F} \to \mathscr{F}''$$

is exact if and only if $f^*(\mathbf{E})$ is exact.

(2) A map $\theta\colon \mathscr{F} \to \mathscr{F}'$ of quasi-coherent $\mathscr{O}_Y$-modules is injective if and only if $f^*\theta$ is injective.

(3) The map f is flat and $f^*\mathscr{F} = 0$ only if $\mathscr{F} = 0$ for $\mathscr{F}$ a quasi-coherent $\mathscr{O}_Y$-module.

2. First properties

Very clearly,

- The composite of faithfully flat maps is faithfully flat.
- Any base change of a faithfully flat map is again faithfully flat. Indeed suppose we have a cartesian square

$$\begin{array}{ccc} X' & \xrightarrow{v} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow[u]{} & Y \end{array}$$

  with f faithfully flat. We have already shown that f' is flat. One only has to check that it is surjective. To that end, for any point z of a scheme Z, let $k(z)$ denote the residue field at z, namely the field $\mathscr{O}_{Z,z}/\mathfrak{m}_z$ where $\mathfrak{m}_z$ is the maximal ideal of $\mathscr{O}_{Z,z}$. Suppose $y' \in Y$. Let $y = u(y') \in Y$. Since f is surjective, we have a point $x \in X$ such that $f(x) = y$. Now, both $k(y')$ and $k(x)$ are field extensions of $k(y)$. Let $R = k(y') \otimes_{k(y)} k(x)$. Note that we have a canonical map $\operatorname{Spec} R \to Y'$ given by the composite $\operatorname{Spec} R \to \operatorname{Spec} k(y') \xrightarrow{y'} Y'$, and similarly a canonical map $\operatorname{Spec} R \to X$ factoring through $\operatorname{Spec} k(x) \xrightarrow{x} X$. Therefore, by definition of a fibre product, we have a unique map $g\colon \operatorname{Spec} R \to X'$, such that the image of $\operatorname{Spec} R$ in Y' under $f' \circ g$ is y'. Thus f' is surjective. In less esoteric (and less rigorous) terms, the point $(x, y') \in X \times_Y Y' = X'$ maps to y' under f'. (The last line justifies the "Very clearly" bit at the beginning of this section.)

2.1. **Openess properties.** One can show that if $A \to B$ is *finitely presented*—i.e., if B is the quotient (as an A-algebra) of a polynomial ring $A[X_1, \dots, X_n]$ over A, by a finitely generated ideal $I \subset A[X_1, \dots, X_n]$)—then the map $\operatorname{Spec} B \to \operatorname{Spec} A$ is an open map. This can be found for example in [M]. The finite presentation is important as the following example shows. Let $A = k[X]$, k a field. Let $U = \operatorname{Spec} A$. Let $V := \operatorname{Spec} \mathscr{O}_{U,u}$ where u is a closed point of U. Then the natural map $V \to U$ is flat. The generic point of V is an open subset of V. Its image in U is the generic point of U which is not an open subset of U for its complement is the set of all maximal ideals of A, which is known to be infinite from Euclid's days (O.K.—I know Euclid did not know about prime and maximal ideals, but that old proof for infinite primes works here too).

However all is not lost. If $X \to Y$ is faithfully flat and quasi-compact then a subset of Y is open if and only if its inverse image in X is open. In other words, in this case (even though the map in question is not necessarily finitely presented or even locally finitely presented), Y has the topology induced by X, i.e., it has the quotient topology induced by the equivalence relation given by the surjective map f. As usual, quasi-compactness is used to reduce to the case of a map of affine schemes, i.e., to the case of a map of rings. We summarize these results below without proofs. In what follows, a locally finitely presented map $f \colon X \to Y$ is what you think it is, namely, around every point of X we can find an affine open subscheme which maps into an affine open subscheme of Y in a finitely presented way.

Proposition 2.1.1. *Let $f \colon X \to Y$ be a flat map of schemes.*

(1) [EGA $\mathrm{IV}_{\mathrm{II}}$, Proposition 2.4.6] *If f is locally finitely presented, then f is an open map.*
(2) [EGA $\mathrm{IV}_{\mathrm{II}}$, Corollaire 2.3.12] *If f is faifthfully flat and quasi-compact, then a subset U of Y is open if and only if $f^{-1}(U)$ is open in X.*

2.2. **Base change.** The main results one uses are summarized in the following proposition.

Proposition 2.2.1. *Suppose*

$$(\square) \qquad \begin{array}{ccc} X' & \xrightarrow{u'} & X \\ {\scriptstyle \varphi'}\downarrow & & \downarrow{\scriptstyle \varphi} \\ Y' & \xrightarrow[u]{} & Y \end{array}$$

is a cartesian square of schemes with u faithfully flat. Then

(1) *φ' flat $\iff$ φ flat.*
(2) *φ' faithfully flat $\iff$ φ faithfully flat.*

If further u is either quasi-compact or locally of finite presentation, then

(iii) *φ' proper $\iff$ φ proper.*

Proof. We only have to prove the implications in the ($\Rightarrow$) direction, the other way being obvious.

(i) Suppose φ' is flat and

$$(\mathbf{E}) \qquad \mathcal{A}' \to \mathcal{A} \to \mathcal{A}''$$

is an exact sequence of quasi-coherent $\mathscr{O}_Y$-modules. The composite $\varphi\circ u'$ equals $u\circ\varphi'$, and the latter (whence the former) is flat, being the composite of flat maps. It follows that ${u'}^*\varphi^*(\mathbf{E})$ is exact. Now u' is faithfully flat, and hence $\varphi^*(\mathbf{E})$ is exact, proving (i).

(ii) Suppose φ' is faithfully flat. Then $u\circ\varphi'$ is surjective (since u and φ' are). In other words, $\varphi\circ u'$ is surjective. This implies φ is surjective. By (i) φ is flat. Thus φ is faithfully flat.

(iii) Now assume φ' is proper and u is quasi-compact[2] (in addition to being faithfully flat). We first show that φ is closed. This will prove that φ is universally closed, since any base change map $T\to Y$ induces a cartesian square $(\square)_T$ which is the base change of $(\square)$

$$(\square)_T \qquad \begin{array}{ccc} X'_T & \xrightarrow{u'_T} & X \\ {\scriptstyle\varphi'_T}\downarrow & & \downarrow{\scriptstyle\varphi_T} \\ Y'_T & \xrightarrow[u_T]{} & Y_T \end{array}$$

and $(\square)_T$ satisfies the same hypotheses as $(\square)$.

So suppose $C\subset X$ is a closed subset. We claim that $\varphi'({u'}^{-1}(C)) = u^{-1}(\varphi(C))$. Assuming this, clearly $u^{-1}(\varphi(C))$ is closed in Y', since $\varphi'({u'}^{-1}(C))$ is, φ' being proper. Now u is faithfully flat and quasi-compact, whence Y has the quotient topology induced by u. It follows that $\varphi(C)$ is closed.

It remains to show that $\varphi'({u'}^{-1}(C)) = u^{-1}(\varphi(C))$. Now

$$\varphi'({u'}^{-1}(C)) = \{y'\in Y' \mid \exists\, x'\in X' \text{ such that } y'=\varphi'(x') \text{ and } u'(x')\in C\}$$

and

$$u^{-1}(\varphi(C)) = \{y'\in Y' \mid \exists\, c\in C \text{ such that } u(y')=\varphi(c)\}.$$

So suppose $y'\in\varphi'({u'}^{-1}(C))$. Pick $x'\in X'$ such that $c:=u'(x')\in C$ and $\varphi'(x')=y'$. Clearly $u(y')=\varphi(c)$, i.e., $y'\in u^{-1}(\varphi(C))$.

Conversely suppose $y'\in u^{-1}(\varphi(C))$. Pick $c\in C$ such that $u(y')=\varphi(c)=y$ (say). The short answer is: Set $x'=(c,y')\in X\times_Y Y' = X'$ and note that $\varphi'(x')=y'$ and $u'(x')=c\in C$, giving $y'\in\varphi'({u'}^{-1}(C))$. A slightly more precise answer is as follows. The residue fields $k(c)$ and $k(y')$ are both extensions of the residue field $k(y)$. Let K be a common field extension of $k(c)$ and $k(y')$. The natural maps $\zeta\colon \operatorname{Spec} K\to Y'$ and $\xi\colon \operatorname{Spec} K\to C$ induced by $k(y')\to K$ and $k(c)\to K$ are such that $u\circ\zeta=\varphi\circ\xi$, whence by the universal property of fibre products we get a map $\operatorname{Spec} K\to X'$. Let $x'\in X'$ be the image of this map. Clearly $\varphi'(x')=y'$ and $u'(x')=c\in C$. Thus $y'\in\varphi'({u'}^{-1}(C))$.

The proof that φ is of finite type if φ' is as follows (this is a sketch): The question reduces to showing that if A is a ring, B an A-algebra, A' a faithfully flat A algebra and $B':=B\otimes_A A'$, then B is finitely generated as an A-algebra if (and clearly only if) B' is finitely generated as an A'-algebra. Now we can find a directed system (B_α) of A-sub-algebras of B which are finitely generated and such that $B=\varinjlim_\alpha B_\alpha$ (in fact the system of all such sub-algebras of B has this property, as is easy to show). Since, tensor products commute with direct limits, we have

[2]The same proof works for u locally of finite presentation. The main point is that in either case, the target has the quotient topology induced by the source.

$B' = \varinjlim_{\alpha} (B_\alpha \otimes_A A')$. Note that $B'_\alpha := B_\alpha \otimes_A A'$ is an A'-sub-algebra of B', since flatness preserves injectivity of maps on tensoring. Now suppose (x'_i) is a system of finite generators of the A'-algebra B'. There is some index α such that all the x'_i belong to B'_α, since the x'_i are only finite in number, and hence $B'_\alpha = B'$ for this α. Thus the inclusion map $B_\alpha \hookrightarrow B$ becomes an isomorphism on applying $(\_) \otimes_A A'$. Faithful flatness of A' then implies that $B_\alpha = B$. □

References

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