Grothendieck Duality and Transitivity for Formal Schemes

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Abstract. The foundations of Grothendieck Duality that we pick is the one initiated by Deligne in [D1]. The principal aim of this monograph is to use these foundations to give concrete versions of residues and traces of differential forms. Formal schemes enter into the picture (via residues) even if one is only interested in ordinary schemes. In Part 1, we shore up the abstract theory in the foundations and in Part 2, we use these results to give our concrete answers. In a little greater detail, for a proper map $f\colon X \to Y$ of noetherian ordinary schemes, one has a well-known natural transformation, $\mathbf{L}f^*(-) \overset{\mathbf{L}}{\otimes} f^!\mathscr{O}_Y \to f^!$, obtained via the projection formula, which extends, using Nagata's compactification, to the case where f is separated and of finite type. In the first part of the monograph we extend this transformation to the situation where f is a pseudo-finite-type map of noetherian formal schemes which is a composite of compactifiable maps, and show it is compatible with the pseudofunctorial structures involved. This natural transformation has implications for the abstract theory of residues and traces, giving Fubini type results for iterated maps. In the second part of the monograph these abstractions are rendered concrete. Briefly, for a smooth map between noetherian schemes, Verdier relates the top relative differentials of the map with the twisted inverse image functor "upper shriek" [V]. We show that the associated traces for smooth proper maps can be rendered concrete by showing that the resulting theory of residues satisfy the residue formulas (R1)–(R10) in Hartshorne's *Residues and Duality* [RD]. We show that the resulting abstract transitivity map relating the twisted image functors for the composite of two smooth maps satisfies an explicit formula involving differential forms. We also give explicit formulas for traces of differential forms for finite flat maps (arising from Verdier's isomorphism) between schemes which are smooth over a common base, and use this to relate Verdier's isomorphism to Kunz and Waldi's regular differentials. These results also give concrete realisations of traces and residues for Lipman's fundamental class map via the results of Lipman and Neeman [LN2] relating the fundamental class to Verdier's isomorphism.

Contents

Introduction

The foundations for Grothendieck Duality (abbreviated to GD for the rest of this book) as developed and used in Hartshorne's classic book [**RD**], and in Conrad's book [**C1**], are based on residual complexes. In this approach, the functor $f^!$, for a suitable finite-type map f, as well as its attendant trace $\mathrm{Tr}_f : \mathbf{R}f_*f^! \to \mathbf{1}$ (when f is proper) have a certain concreteness built into their construction. One then has to work out a large array of compatibilites between the various concrete representations of $f^!$ and Tr_f and there often are different concrete representations of these for the same map, e.g., a finite map which also factors as a closed immersion followed by a smooth map.

In a different direction, in his appendix to [**RD**], Deligne initiated an approach to GD which is conceptually attractive. From this point of view, $f^!$ for a proper map f is the right adjoint to $\mathbf{R}f_*$, and exists for very general category-theoretic reasons. These foundations have been worked on, extended, and new techniques have been introduced over the years by Lipman, Neeman, and their collaborators. Residual complexes and dualizing complexes are not needed to build GD in this approach. We mention [**D1**], [**D2**], [**D2'**], [**V**] for literature on this approach before the 1980s, and recent work found in [**Ne1**], [**Ne2**], which do much to extend (via a conceptually different approach to finding right adjoints) the work initiated by Deligne and Verdier to more general situations, often bypassing the old annoying hypotheses on boundedness for the existence of $f^!$ or for its base change. The stable version of these can be found in Lipman's elegant and carefully written book [**L4**]. We also mention Neeman's recent manuscript [**Ne4**] which gives a coherent account of the difficulties and the recent simplifications of many matters.

The aim of this book is to use the foundations of GD initiated by Deligne to get concrete results concerning traces, residues, and transitivity. Since we rely on formal schemes as a way to our results on maps between ordinary schemes, we have been influenced enormously by the work of Alonso Tarrío, Jeremías López, and Lipman in [**AJL1**], [**AJL2**], and [**AJL3**].

Part 1 of the book concentrates on shoring up the abstract machinery of GD for this purpose, and Part 2 on obtaining concrete formulas involving differential forms. Examples of the latter are the residue formulas (R1)–(R10) stated without proof in [**RD**, III, §9]. These ten formulas were proved by Conrad in [**C1**], but using the foundations in [**RD**] rather than the ones mentioned in the opening paragraph above.

The results in Part 2 are easier if we allow ourselves to work with completions of finite type relative schemes $f : X \to Y$ along subschemes of X which are finite (and often flat) over Y, provided the abstract results in Part 1 are proved for such completions. It is, in the main, no more difficult to work with general (noetherian) formal schemes, than with formal schemes which arise as completions. This is precisely what we do in Part 1 and in a substantial part of Part 2. The schemes in this book are noetherian formal schemes.

Abstract matters

The two main themes of Part 1 are:

(a) traces associated with psuedo-proper maps $f\colon \mathscr{X} \to \mathscr{Y}$ (i.e., maps which are proper modulo ideals of definition of $\mathscr{X}$ and $\mathscr{Y}$); and
(b) transitivity.

Traces, at least for smooth maps, are analogues of integrals (or of integration along fibres), although the definitions, in the abstract theory, can obscure this. If for simplicity we stick to ordinary schemes, the trace associated to a proper map f is the co-unit associated to the adjoint pair $(\mathbf{R}f_*, f^!)$. Closely related to traces are residues, which we define in the abstract setting of Part 1 for Cohen-Macaulay maps between ordinary schemes. Traces and residues for such maps are studied in Chapter 3. Roughly, if $f\colon X \to Y = \operatorname{Spec} A$ is a proper Cohen-Macaulay map whose non-empty fibres are all of dimension d, and Z is a closed subcheme of X which is finite and flat over Y, then our abstract residue map is the composite $\mathrm{H}^d_Z(X, \omega_{X/Y}) \to \mathrm{H}^d(X, \omega_{X/Y}) \to A$. Here $\omega_{X/Y}$ is the relative dualizing sheaf $H^{-d}(f^!\mathscr{O}_Y)$, the first arrow is the natural one, and the second arrow is induced by the abstract trace (or a version of it). Formal schemes have the advantage that residues can be regarded as traces for the pseudo-proper (in fact pseudo-finite) map $\widehat{f}\colon \mathscr{X} \to Y$ which is the completion of f along the closed subscheme Z. The mantra is (as in complex variables): *residues determine integrals*. Local duality gives substance to that mantra.

Transitivity is a loose term for the study of relationships between the various duality functors associated to the composite of two or more maps. For example, if $f\colon X \to Y$ and $g\colon Y \to Z$ are a pair of proper maps between ordinary schemes, one aspect of transitivity is the study of a map $\chi\colon \mathbf{L}f^*g^!\mathscr{O}_Z \overset{\mathbf{L}}{\otimes} f^!\mathscr{O}_Y \longrightarrow (gf)^!\mathscr{O}_Z$ that arises from universal properties of the pseudofunctor $(-)^!$, also known informally as "upper shriek". Briefly $\chi(=\chi_{[g,f]})$ is the map which makes a Fubini type theorem for traces (the analogues of integrals) associated to f, g, and gf true. As shown in [**L4**], one can relax the requirement that f and g are proper. If we allow more than two maps, relationships between the various transitivity are proved in *ibid.* Extending these results to iterated maps of pseudo-finite type maps of formal schemes requires considerable effort and these matters are studied in Chapter 5. In Chapter 6 we look at transitivity for Cohen-Macaulay maps, culminating in formulas of the form

$$\mathbf{res}^{\#}_{W_2}\begin{bmatrix}\mathbf{res}^{\#}_{W_1}\begin{bmatrix}\nu\\ \mathbf{v}\end{bmatrix}\mu\\ \mathbf{u}\end{bmatrix} = \mathbf{res}^{\#}_{W_1\cap f^{-1}(W_2)}\begin{bmatrix}\chi(\mu\otimes\nu)\\ \mathbf{v},\mathbf{u}\end{bmatrix}$$

(see (6.3.2) and (6.3.3)). Here X, Y, Z are affine, $\mathbf{u} = (u_1, \dots, u_m)$ is an m-tuple of sections of $\mathscr{O}_Y$ cutting out a closed subscheme W_2 of Y, $\mathbf{v} = (v_1, \dots, v_n)$ an n-tuple of sections of $\mathscr{O}_X$ cutting out a closed subscheme W_1 of X, where m is the relative dimension of g and m that of f. The symbols μ and ν are sections of the relative dualizing sheaves of g and f respectively. The residues $\mathbf{res}^{\#}_{W}\left[\begin{smallmatrix}a\\ b\end{smallmatrix}\right]$ above are the abstract residues. Concrete forms of these, with μ and ν being suitable differential forms, are studied in Part 2, about which we say a few words below.

Differential forms and Grothendieck Duality

Given the highly abstract methods of construction, and definitions based on universal properties, the question arises:

> *To what extent can we render concrete realisations of the various constructions occurring in this version of GD?*

The issue of concrete representations of $f^!$ (for our preferred version of GD) was addressed partially, soon after [**D1**] appeared, by Verdier when f is smooth [**V**]. The answer is $f^! \cong f^*(-) \otimes_{\mathscr{O}_X} \Omega^n_{X/Y}[n]$ where n is the relative dimension of $f\colon X \to Y$. This isomorphism in turn depends on the concrete representation $i^! \cong \mathbf{L}i^*(-) \otimes \wedge^d_{\mathscr{O}_U} \mathscr{N}[-d]$ (via the fundamental local isomorphism) for a regular immersion $i\colon U \hookrightarrow V$ of codimension d, with $\mathscr{N}$ the normal bundle of U in V. Verdier's answer for smooth maps is only a partial answer because the associated trace map (when f is proper)

$$\mathrm{tr}_f\colon \mathrm{R}^n f_*(\Omega^n_{X/Y}) \longrightarrow \mathscr{O}_Y,$$

denoted $\int_f$ in [**V**],[1] is seemingly intractable via this approach. In fact tr_f has not been worked out in the literature even when Y is the spectrum of a field k. However, from the abstract properties of $f^!$ and the fact that $f^!\mathscr{O}_Y$ is concentrated in degree $-n$ (for example by Verdier's isomorphism), the pair $(\Omega^n_{X/Y}, \mathrm{tr}_f)$ is easily seen to represent the functor $\mathrm{Hom}_Y(\mathrm{R}^n f_*(-), \mathscr{O}_Y)$ on quasi-coherent sheaves on X when f is proper (the only situation where tr_f is defined), and in fact tr_f and the composite $\mathbf{R}f_*\Omega^n_{X/Y}[n] \xrightarrow{\ \sim\ } \mathbf{R}f_*f^!\mathscr{O}_Y \xrightarrow{\mathrm{Tr}_f(\mathscr{O}_Y)} \mathscr{O}_Y$ determine each other.

When Z is a closed subscheme of X, finite over Y, defined locally by an $\mathscr{O}_X$-sequence, and $\mathscr{E}xt^i_f(\mathscr{O}_Z, -)$ the i^{th} right-derived functor of $f_*\mathscr{H}om_X(\mathscr{O}_Z, -)$, Verdier asserts (see top of p. 400 of [**V**]) that the composite

$$\mathscr{E}xt^n_f(\mathscr{O}_Z, \Omega^n_{X/Y}) \longrightarrow \mathrm{R}^n f_*(\Omega^n_{X/Y}) \xrightarrow{\mathrm{tr}_f} \mathscr{O}_Y \tag{1}$$

is governed by the residue symbol of [**RD**, Chap. III, § 9].[2] It is certainly true that if this is so, following (essentially) the argument given in [**V**, bottom of p. 399], the trace map tr_f can be realised in an explicit way. However, the proof that (1) (denoted Res_Z in [**V**]) is governed by the residue symbol is not there in the literature. In the over 50 years that have passed since Verdier's assertion, it has been recognised by experts that this is in fact a non-trivial problem (see our quote of Conrad below). One difficulty is the assertion (R4) in [**V**, p. 400], namely that (1) commutes with arbitrary base change. This needs, at the very least, for one to show that the isomorphism $f^!\mathscr{O}_Y \cong \Omega^n_{X/Y}[n]$ of Verdier commutes with arbitrary base change, in a sense we will make more precise in a moment. This compatibility with base change was only established in 2004 by the second author [**S2**]. In slightly greater detail, here is what the compatibility entails. Suppose $u\colon Y' \to Y$ is a map and $g\colon X \times_Y Y' \to Y'$ and $v\colon X \times_Y Y' \to X$ are the two projections. To show

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[1] The map tr_f is $\mathrm{H}^0(-)$ applied to the composite $\mathbf{R}f_*\Omega^n_{X/Y}[n] \xrightarrow{\ \sim\ } \mathbf{R}f_*f^!\mathscr{O}_Y \xrightarrow{\mathrm{Tr}_f(\mathscr{O}_Y)} \mathscr{O}_Y$, where the first arrow is Verdier's isomorphism.

[2] In [**RD**], proofs of the assertions about the residue symbol are not given. They are provided later by Conrad [**C1**], the construction and definition of various traces being those developed in [**RD**]. They do not apply to our situation since we use a different foundation for GD.

the compatibility of Verdier's isomorphism $f^!\mathscr{O}_Y \cong \Omega^n_{X/Y}[n]$ with arbitrary base change, first one needs to show that there is an isomorphism

$$\theta^f_u\colon v^*f^!\mathscr{O}_Y \xrightarrow{\ \sim\ } g^!\mathscr{O}_{Y'}$$

for our smooth f (even when it is not proper). This is a delicate point, especially if one demands that in the proper case $\mathrm{Tr}_f(\mathscr{O}_Y)$ should be compatible with this base change isomorphism (remember, Tr_f in Deligne's approach, is defined as a co-adjoint unit and is not explicit), and that the isomorphism is also compatible with open immersions into X. After this is established, one has to check that this base change isomorphism θ^f_u when grafted on to Verdier's isomorphisms for f and for g give the canonical isomorphism of differential forms. It is easier to carry out the first part in the slightly more general situation of f being Cohen-Macaulay, and this is one of the main results of [**S2**]. In [**C1**], the base change isomorphism θ^f_u is proven using the foundations of GD in [**RD**]. However, the isomorphism between $f^!\mathscr{O}_Y$ and $\Omega^n_{X/Y}[n]$ in [**RD**] and [**C1**] is by fiat, and it is not clear that it is the same as Verdier's isomorphism. In other words, it is not clear that the trace $\mathrm{R}^nf_*(\Omega^n_{X/Y}) \to \mathscr{O}_Y$ built using the foundations of GD in [**RD**] is the same as the one that arises when using the foundations initiated in [**D1**]. In fact, we are back to the frustrating detail that we do not know the tr_f explicitly when we work with the foundations initiated in [**D1**]. Even with the compatibility of Verdier's isomorphism with arbitrary base change in hand, showing that (1) is governed by the residue symbol of [**RD**, Chap. III, § 9] is not trivial. In fact it takes all of Part 2. We can do no better than quote Conrad from his introduction to his book [**C1**] (using however our labelling of the citations given there):

> "...The methods in [**V**] take place in derived categories with "bounded below" conditions. This leads to technical problems for a base change such as $p\colon \mathrm{Spec}(A/\mathfrak{m}) \hookrightarrow \mathrm{Spec}(A)$ with $(A, \mathfrak{m})$ a non-regular local ring, in which case the right exact p^* does not have finite homological dimension (so $\mathbf{L}p^*$ does not make sense as a functor between "bounded below" derived categories). Moreover, Deligne's construction of the trace map in [**RD**, Appendix], upon which [**V**] is based, is so abstract that it is a non-trivial task to relate Deligne's construction to the sheaf $\mathrm{R}^nf_*(\Omega^n_{X/Y})$. However, a direct relation between the duality theorem and differential forms is essential for many important calculations (e.g., [**Maz**, §6, §14(p.121)])."

The prime object of study in Part 2 of this book is Verdier's isomorphism [**V**, p. 397, Thm. 3]

$$\Omega^n_{X/Y}[n] \xrightarrow{\ \sim\ } f^!\mathscr{O}_Y$$

for a smooth separated morphism $f\colon X \to Y$ of ordinary schemes of relative dimension n. Strictly speaking, the isomorphism in *loc.cit.* is from $f^!\mathscr{O}_Y$ to $\Omega^n_{X/Y}[n]$, and thus, we are talking about the inverse of the map in *loc.cit.* In view of recent results of Lipman and Neeman, this is the *fundamental class map* c_f *associated with* f [**LN2**, p. 152, (4.4.1)], In [**L3**], Lipman outlines a programme for a global residue theorem via the fundamental class map (see [*ibid.*, § 5.5 and § 5.6]). Part 2 is intimately related to that programme via the just mentioned results of Lipman and Neeman. However, we do not use the results on the fundamental class map of *ibid.* Since the isomorphism we use (between $\Omega^n_{X/Y}[n]$ and $f^!\mathscr{O}_Y$) is that described by Verdier, we call it the Verdier isomorphism rather than the fundamental class.

We also recommend Beauville's expository paper [**Be**] for an overview (without proofs) of residues, especially for the concrete expressions for them. Our attention was drawn to it recently by Joe Lipman.

We elaborate on these points in the overview to Part 2 in Chapter 7.

Part 1

The abstract theory

CHAPTER 1

Overview for Part 1

All schemes are assumed to be noetherian. They could be formal or ordinary. For any formal scheme $\mathscr{X}$, $\mathcal{A}(\mathscr{X})$ is the category of $\mathscr{O}_{\mathscr{X}}$-modules and $\mathbf{D}(\mathscr{X})$ its derived category. The torsion functor $\varGamma'_{\mathscr{X}}$ on $\mathscr{O}_{\mathscr{X}}$-modules is defined by the formula

$$\varGamma'_{\mathscr{X}} := \varinjlim_{n} \mathscr{H}om_{\mathscr{O}_{\mathscr{X}}}(\mathscr{O}_{\mathscr{X}}/\mathscr{I}^n, -)$$

where $\mathscr{I}$ is any ideal of definition of the formal scheme $\mathscr{X}$. A *torsion module* $\mathscr{F}$ is an object in $\mathcal{A}(\mathscr{X})$ such that $\varGamma'_{\mathscr{X}}\mathscr{F} = \mathscr{F}$. The category $\mathcal{A}_{\mathrm{c}}(\mathscr{X})$ (resp. $\mathcal{A}_{\vec{\mathrm{c}}}(\mathscr{X})$, resp. $\mathcal{A}_{\mathrm{qc}}(\mathscr{X})$, resp. $\mathcal{A}_{\mathrm{qct}}(\mathscr{X})$) is the category of coherent (resp. direct limit of coherent, resp. quasi-coherent, resp. quasi-coherent and torsion) $\mathscr{O}_{\mathscr{X}}$-modules. $\mathbf{D}_{\mathrm{c}}(\mathscr{X})$ (resp. $\mathbf{D}_{\vec{\mathrm{c}}}(\mathscr{X})$, resp. $\mathbf{D}_{\mathrm{qc}}(\mathscr{X})$, resp. $\mathbf{D}_{\mathrm{qct}}(\mathscr{X})$) is the subcategory of $\mathbf{D}(\mathscr{X})$ of complexes having homology in $\mathcal{A}_{\mathrm{c}}(\mathscr{X})$ (resp. $\mathcal{A}_{\vec{\mathrm{c}}}(\mathscr{X})$, resp. $\mathcal{A}_{\mathrm{qc}}(\mathscr{X})$, resp. $\mathcal{A}_{\mathrm{qct}}(\mathscr{X})$), while $\mathbf{D}^*_{\mathrm{c}}(\mathscr{X})$, (resp. $\mathbf{D}^*_{\vec{\mathrm{c}}}(\mathscr{X})$, resp. $\mathbf{D}^*_{\mathrm{qc}}(\mathscr{X})$, resp. $\mathbf{D}^*_{\mathrm{qct}}(\mathscr{X})$) for $*$ in $\{b, +, -\}$ denotes the corresponding full subcategory whose homology is additionally, bounded, or bounded below, or bounded above, accordingly.

We will be using the notions of pseudo-proper, pseudo-finite-type, pseudo-finite maps used in [**AJL2**]. For example if $f\colon \mathscr{X} \to \mathscr{Y}$ is a map of formal schemes, it is *pseudo-proper* if the map of ordinary schemes $f_0\colon X \to Y$ obtained by quotienting the structure sheaves $\mathscr{O}_{\mathscr{X}}$ and $\mathscr{O}_{\mathscr{Y}}$ by ideals of definition for $\mathscr{X}$ and $\mathscr{Y}$, is proper. If this is true for one pair of defining ideals $(\mathscr{I}, \mathscr{J})$ with $\mathscr{J} \subset \mathscr{O}_{\mathscr{Y}}$ and $\mathscr{J}\mathscr{O}_{\mathscr{X}} \subset \mathscr{I} \subset \mathscr{O}_{\mathscr{X}}$, then it is true for all such pairs. The definitions of pseudo-finite-type, pseudo-finite are analogous.

We assume familiarity with the viewpoint of duality that was initiated by Deligne in [**D1**], especially as laid out by Lipman in [**L4**, Chapter 4]. In particular, we assume familiarity with the main properties of the *twisted inverse-image pseudofunctor* $-^{\mathbf{!}}$. This is a $\mathbf{D}^+_{\mathrm{qc}}$-valued contravariant pseudofunctor on the category of schemes and separated essentially finite-type maps.

We begin by recalling the notion of a (contravariant) pseudofunctor. A *pseudofunctor* $(-)^{\triangle}$ (or simply $-^{\triangle}$) on a category $\mathscr{C}$ is an assignment of categories $X^{\triangle}$, one for each $X \in \mathscr{C}$, such that for each map $f\colon X \to Y$ in $\mathscr{C}$, we have a functor $f^{\triangle}\colon Y^{\triangle} \to X^{\triangle}$, for each $X \in \mathscr{C}$, an isomorphism $\eta^{\triangle}_X\colon \mathbf{1}^{\triangle}_X \xrightarrow{\ \sim\ } \mathbf{1}_{X^{\triangle}}$, for each pair of composable maps $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathscr{C}$, an isomorphism of functors $C^{\triangle}_{f,g}\colon (gf)^{\triangle} \xrightarrow{\ \sim\ } f^{\triangle}g^{\triangle}$, such that for a third map $h\colon Z \to S$ in $\mathscr{C}$, "associativity" holds, i.e., the following diagram commutes,

$$\begin{tikzcd}
f^{\triangle}g^{\triangle}h^{\triangle} \arrow[r, "\sim", "C^{\triangle}_{f,g}"'] \arrow[d, "C^{\triangle}_{g,h}"', "\wr"] & (gf)^{\triangle}h^{\triangle} \arrow[d, "\wr"', "C^{\triangle}_{gf,h}"] \\
f^{\triangle}(hg)^{\triangle} \arrow[r, "\sim", "C^{\triangle}_{f,hg}"'] & (hgf)^{\triangle}.
\end{tikzcd}$$

and for the compositions $\mathbf{1}_Y f = f = f\mathbf{1}_X$, the isomorphisms $\eta_{-}^{\triangle}$ and $C_{-,-}^{\triangle}$ are compatible in the obvious way. However, as we shall see, we may need to relax the condition that $\eta_X^{\triangle}\colon \mathbf{1}_X^{\triangle} \to \mathbf{1}_{X^{\triangle}}$ be an isomorphism for objects $X \in \mathscr{C}$. In such a case, if other conditions are satisfied, $-^{\triangle}$ is called a *pre-pseudofunctor*. For simplicity, we may often call these naturally occurring pre-pseudofunctors as pseudofunctors.

1.1. Transitivity

The principal aim of Part 1 of this book is to extend results in [**L4**, § 4.9] to the situation of formal schemes. An example of the kind of result in *loc.cit.* that would interest us is the existence of a map

$$\mathbf{L}f^*g^!\mathscr{O}_Z \overset{\mathbf{L}}{\otimes}_{\mathscr{O}_X} f^!\mathscr{O}_Y \longrightarrow (gf)^!\mathscr{O}_Z \tag{1.1.1}$$

for a pair of finite-type separated maps $X \xrightarrow{f} Y \xrightarrow{g} Z$. This is closely related to the results in [**LS**]. Let us discuss (1.1.1) in some detail to orient the reader to the kind of questions we are interested in as well as the difficulties involved in answering them for formal schemes.

Recall that if $h\colon V \to W$ is a proper map of schemes, then the *twisted inverse image functor* $h^!\colon \mathbf{D}_{\mathrm{qc}}^+(W) \to \mathbf{D}_{\mathrm{qc}}^+(V)$ is a right adjoint to $\mathbf{R}h_*\colon \mathbf{D}_{\mathrm{qc}}^+(V) \to \mathbf{D}_{\mathrm{qc}}^+(W)$. We therefore have the co-adjoint unit

$$\mathrm{Tr}_h\colon \mathbf{R}h_*h^! \to \mathbf{1}_{\mathbf{D}_{\mathrm{qc}}^+(W)}, \tag{1.1.2}$$

the so-called *trace map* for h, such that the map

$$\mathrm{Hom}_{\mathbf{D}_{\mathrm{qc}}^+(V)}(\mathscr{F},\, h^!\mathscr{G}) \to \mathrm{Hom}_{\mathbf{D}_{\mathrm{qc}}^+(W)}(\mathbf{R}h_*\mathscr{F}, \mathscr{G})$$

given by $\varphi \mapsto \mathrm{Tr}_h(\mathscr{G})\circ\mathbf{R}h_*(\varphi)$ is an isomorphism [**L4**, p.204, Theorem 4.8.1 (i)]. Next, if h is étale, by part (ii) of *loc.cit.*, we have $h^! = h^*$. Note that if $\mathscr{F} \in \mathbf{D}_{\mathrm{qc}}(X)$ and $\mathscr{G} \in \mathbf{D}_{\mathrm{qc}}(Y)$, we have the bifunctorial *projection isomorphism* of [**L4**, p.139, Proposition 3.9.4]

$$\mathscr{G} \overset{\mathbf{L}}{\otimes}_{\mathscr{O}_Y} \mathbf{R}h_*\mathscr{F} \ \xrightarrow{\ \sim\ }\ \mathbf{R}h_*(\mathbf{L}h^*\mathscr{G} \otimes_{\mathscr{O}_X} \mathscr{F}). \tag{1.1.3}$$

Finally, recall that if h is separated and of finite type, by a famous theorem of Nagata [**N**] one can find a compactification of h, i.e., a factorization $h = \bar{h}\circ i$ with i an *open immersion* and $\bar{h}$ a *proper* map. Thus, after choosing such a compactification, one could define $h^!$ for such maps, by the formula $h^! = i^*\bar{h}^!$. That this is independent of the compactification chosen is proven in [**D1**]. The technical difficulties encountered carrying out this program are formidable, and form the content of [**D1**], [**D2**], [**D2'**], and [**L4**, Chapter 4].

The map (1.1.1) is described as follows. First suppose f and g are proper. One has a natural map

$$\mathbf{R}(gf)_*(\mathbf{L}f^*g^!\mathscr{O}_Z \overset{\mathbf{L}}{\otimes}_{\mathscr{O}_X} f^!\mathscr{O}_Y) \longrightarrow \mathscr{O}_Z \tag{1.1.4}$$

given by the composite

$$\begin{aligned}\mathbf{R}(gf)_*(\mathbf{L}f^*g^!\mathscr{O}_Z \overset{\mathbf{L}}{\otimes}_{\mathscr{O}_X} f^!\mathscr{O}_Y) &\xrightarrow{\quad\sim\quad} \mathbf{R}g_*\mathbf{R}f_*(\mathbf{L}f^*g^!\mathscr{O}_Z \overset{\mathbf{L}}{\otimes}_{\mathscr{O}_X} f^!\mathscr{O}_Y)\\ &\xrightarrow{(1.1.3)^{-1}} \mathbf{R}g_*(g^!\mathscr{O}_Z \overset{\mathbf{L}}{\otimes}_{\mathscr{O}_Y} \mathbf{R}f_*f^!\mathscr{O}_Y)\\ &\xrightarrow{\mathbf{R}g_*(\mathbf{1}\otimes\mathrm{Tr}_f)} \mathbf{R}g_*g^!\mathscr{O}_Z\\ &\xrightarrow{\quad\mathrm{Tr}_g\quad} \mathscr{O}_Z.\end{aligned}$$

Since $(gf)^!$ is right adjoint to $\mathbf{R}(gf)_*$, the map (1.1.4) gives rise to (1.1.1) as the unique map such that $\mathrm{Tr}_{gf}(\mathscr{O}_Z)\circ\mathbf{R}(gf)_*(1.1.1) = (1.1.4)$.

If f or g is not proper, one can find a compactification of gf, say $gf = F\circ j$, such that $F = \bar{g}\circ\bar{f}$, with $\bar{f}$, $\bar{g}$ proper maps, and where these maps embed into a commutative diagram

(1.1.5)
$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] & \overline{X} \arrow[r] \arrow[dl] & \overline{\overline{X}} \arrow[dl, "\bar{f}"] \\ Y \arrow[r] \arrow[d, "g"'] & \overline{Y} \arrow[dl, "\bar{g}"] & \\ Z & & \end{tikzcd}$$

with all horizontal arrows open immersions and all south-west pointing arrows proper, with the composite of the two horizontal arrows on the top row being j. The map (1.1.1) for the pair (f, g) can be defined by "restricting" the corresponding map for $(\bar{f}, \bar{g})$ to X. The map (1.1.1) is independent of the choice of such diagrams — this is the essential content of [**L4**, pp. 231–232, Lemma 4.9.2]. We provide a proof for formal schemes in Proposition 5.2.4 below.

The map (1.1.1) is to be regarded as an abstract form of certain transitivity results for differential forms which are important for duality, e.g., property (R4) of residues stated in [**RD**, p.198]. Briefly, if $h\colon V \to W$ is a smooth map of relative dimension n, then one can show that there is an isomorphism

$$\omega_h[n] \xrightarrow{\ \sim\ } h^!\mathscr{O}_W, \tag{1.1.6}$$

where $\omega_h = \Omega^n_{V/W}$ is the n^{th} exterior power of the $\mathscr{O}_V$-module of relative differential forms $\Omega^1_{V/W}$ for the map h. There are many descriptions of such isomorphisms (see [**V**, p. 397, Theorem 3], [**HK2**, p. 84, Duality theorem], [**HS**, p. 750, Duality Theorem]; the general hypothesis in the last two cited papers is that the base scheme has no embedded points). Thus if f and g are smooth of relative dimensions, say, m and n respectively, then, upon taking homology in degree $m+n$, (1.1.1) induces a map of coherent $\mathscr{O}_X$-modules,

$$f^*\omega_g \otimes_{\mathscr{O}_X} \omega_f \to \omega_{gf}. \tag{1.1.7}$$

The above map is an isomorphism since (1.1.1) is in this special case, f being a perfect map. How this compares with the usual isomorphism between $f^*\omega_g \otimes_{\mathscr{O}_X} \omega_f$ and ω_{gf} depends on the choice of (1.1.6) which in turn depends on the choice of concrete trace maps of the form $\mathbf{R}h_*\omega_h[n] \to \mathscr{O}_W$. For the one implicit in [**HS**], the problem is studied in [**LS**] (see also the correction). In Part 2 we will show that when (1.1.6) is chosen to be the isomorphism given by Verdier in [**V**, p. 397, Thm. 3], the map (1.1.7) is the map given locally by $f^*(\mu) \otimes \nu \mapsto \nu \wedge f^*(\mu)$ where the notation is self-explanatory.

The problem of finding the concrete expression for (1.1.7) given (1.1.6) is best attacked by expanding the scope of our study from ordinary schemes to formal schemes. One reason for this is that maps such as (1.1.1) are compatible (in a precise sense) with completions along closed subschemes in X, Y, and Z. This allows for far greater flexibility than the method of compactifying and restricting. From this larger point of view, property (R4) of residues in [**RD**, p. 198] is a concrete manifestation of (1.1.1) for maps between formal schemes.

We lay the foundations for all of this in this part of the book. The generalization of parts of §4.9 of [**L4**] (in particular of the map (1.1.1) above) is carried out in section Chapter 5, especially in Proposition 5.2.4. As for property (R4) for residues, an abstract form of it for Cohen-Macaulay maps is proved by us in Proposition 6.3.1 below. We draw the reader's attention especially to formulas (6.3.2) and (6.3.3) which follow from *loc.cit.*

If we move to formal schemes which are not necessarily ordinary, [**AJL2**, Theorem 6.1] assures us of the existence of a right adjoint to $\mathbf{R}f_*\colon \mathbf{D}_{\mathrm{qct}}(\mathscr{X}) \to \mathbf{D}_{\mathrm{qct}}(\mathscr{Y})$ for a pseudo-proper map (and more) $f\colon \mathscr{X} \to \mathscr{Y}$ and this right adjoint is denoted $f^!$. However, given a general separated pseudo-finite-type map f, we are no longer assured that f has a compactification, i.e., we are no longer assured that we have a factorization $f = \bar{f} \circ i$ with i an open immersion, and $\bar{f}$ pseudo-proper.[1] We are therefore forced to work in the category $\mathbb{G}$ whose objects are formal schemes and whose morphisms are composites of "compactifiable" maps. In [**Nay**] the first author shows that we have a pre-pseudofunctor $-^!$ on $\mathbb{G}$, which generalizes what we have for the category of finite-type separated maps on ordinary schemes (see Chapter 2).

If k is a field and $A = k[|X_1, \dots, X_d|]$ the ring of power series over k in d analytically independent variables, $\mathfrak{m}$ the maximal ideal of A, $\mathscr{X}$ the formal spectrum of A with its $\mathfrak{m}$-adic topology, and $\mathscr{Y}$ the spectrum of k, then the natural map $f\colon \mathscr{X} \to \mathscr{Y}$ is pseudo-proper and $f^!\mathscr{O}_{\mathscr{Y}}$ is the torsion sheaf obtained by sheafifying the A-module $\mathrm{H}^d_{\mathfrak{m}}(\omega_A)$ where ω_A is "the" canonical module of A. From here to recovering local duality for the complete local ring A requires a more careful examination of the relationship between $f^!\mathscr{O}_{\mathscr{Y}}$ and the canonical module ω_A. As it turns out, ω_A, a finitely generated A-module, can be recovered from the torsion module associated to $f^!\mathscr{O}_{\mathscr{Y}}$. More generally, for a pseudo-proper map $f\colon \mathscr{X} \to \mathscr{Y}$ between formal schemes and an object $\mathscr{G} \in \mathbf{D}^+_{\mathrm{c}}(\mathscr{Y})$, there is a deep relationship between $f^!\mathscr{G} \in \mathbf{D}_{\mathrm{qct}}(\mathscr{X})$ and an associated object in $\mathbf{D}^+_{\mathrm{c}}(\mathscr{X})$, of which the above mentioned relationship between $\mathrm{H}^d_{\mathfrak{m}}(\omega_A)$ and ω_A is an example. This necessitates the development of a second twisted inverse image functor $f^{\#}$ related to $f^!$. The twisted inverse image $f^{\#}$ was introduced by Alonso, Jéremías, and Lipman in [**AJL2**] and the relationship between $f^!$ and $f^{\#}$ is one of the many important portions of that work.

Part 1 is mainly concerned with a pre-pseudofunctor $-^{\#}$ on the category $\mathbb{G}$ such that for f pseudo-proper, $f^{\#}$ is the functor mentioned in the last paragraph. The pseudofunctor $-^{\#}$ is one of two ways that the twisted inverse image pseudofunctor $-^!$ on the category of ordinary schemes and separated finite-type maps generalizes

[1] On the other hand, we do not have an example of a separated pseudo-finite-type map which does not have a compactification.

to $\mathbb{G}$, the other being the pre-pseudofunctor $-^!$ on $\mathbb{G}$ discussed above. The map (1.1.1) can be defined for formal schemes with $f^!$ and $g^!$ replaced by $f^\#$ and $g^\#$. The principal technical issue which creates complications is the lack of diagrams like (1.1.5) into which a pair of maps $\mathscr{X} \xrightarrow{f} \mathscr{Y} \xrightarrow{g} \mathscr{Z}$ embed. In the next sub-section we give brief introduction to $-^!$ and $-^\#$ on $\mathbb{G}$.

1.2. Two twisted inverse images

The duality pseudofunctors $-^\#$ and $-^!$ on $\mathbb{G}$ are explained in Chapter 2 of this book, but for the purposes of this overview we say a few quick words. For a pseudo-proper map $f\colon \mathscr{X} \to \mathscr{Y}$, the functor $f^!\colon \mathbf{D}^+_{\mathrm{qct}}(\mathscr{Y}) \to \mathbf{D}^+_{\mathrm{qct}}(\mathscr{X})$ is right adjoint to $\mathbf{R}f_*\colon \mathbf{D}^+_{\mathrm{qct}}(\mathscr{X}) \to \mathbf{D}^+_{\mathrm{qct}}(\mathscr{Y})$. In fact $f^!$ extends to a larger category $\widetilde{\mathbf{D}}^+_{\mathrm{qc}}(\mathscr{Y})$ which contains $\mathbf{D}^+_{\mathrm{qct}}(\mathscr{Y})$, namely the full subcategory of $\mathbf{D}(\mathscr{Y})$ of objects $\mathscr{G}$ such that $\mathbf{R}\varGamma'_{\mathscr{Y}}(\mathscr{G}) \in \mathbf{D}^+_{\mathrm{qc}}(\mathscr{Y})$, the extended functor being $f^!\circ\mathbf{R}\varGamma'_{\mathscr{Y}}$. Note that this extended $f^!$ continues to take values in $\mathbf{D}^+_{\mathrm{qct}}(\mathscr{X})$. There is another duality functor associated with the pseudo-proper map f, namely $f^\#\colon \widetilde{\mathbf{D}}^+_{\mathrm{qc}}(\mathscr{Y}) \to \widetilde{\mathbf{D}}^+_{\mathrm{qc}}(\mathscr{X})$, which is right adjoint to $\mathbf{R}f_*\mathbf{R}\varGamma'_{\mathscr{X}}\colon \widetilde{\mathbf{D}}^+_{\mathrm{qc}}(\mathscr{X}) \to \widetilde{\mathbf{D}}^+_{\mathrm{qc}}(\mathscr{Y})$.

The functors $f^\#$ and $f^!$ are related via the formulas $f^! \cong \mathbf{R}\varGamma'_{\mathscr{X}} f^\#$ and $f^\# \cong \mathbf{\Lambda}_{\mathscr{X}} f^!$, where $\mathbf{\Lambda}_{\mathscr{X}}(-) = \mathbf{R}\mathscr{H}om(\mathbf{R}\varGamma'_{\mathscr{X}}\mathscr{O}_{\mathscr{X}}, -)$.

As an example, if k is a field, $\mathscr{X}$ the formal spectrum of the power series ring $A = k[|X_1, \dots, X_d|]$ (given the $\mathfrak{m}$-adic topology, where $\mathfrak{m}$ is the maximal ideal of A), $\mathscr{Y} = \operatorname{Spec} k$, and $f\colon \mathscr{X} \to \mathscr{Y}$ the map of formal schemes corresponding to the obvious k-algebra map $k \to k[|X_1, \dots, X_d|]$, then f is pseudo-proper. Identifying A-modules with their associated sheaves on $\mathscr{X}$, and writing $\widehat{\Omega}^d_{A/k}$ for the universally finite module of d-forms for the algebra A/k, we have $f^!(k) = \mathrm{H}^d_{\mathfrak{m}}(\widehat{\Omega}^d_{A/k})[0]$ and $f^\#(k) = \widehat{\Omega}^d_{A/k}[d]$.

This is for pseudo-proper maps, the original setting for defining $f^!$ in [**AJL2**]. However, in [**Nay**], the first author was able to show that $f^!\colon \widetilde{\mathbf{D}}^+_{\mathrm{qc}}(\mathscr{Y}) \to \widetilde{\mathbf{D}}^+_{\mathrm{qc}}(\mathscr{X})$ can be defined when $f\colon \mathscr{X} \to \mathscr{Y}$ is in $\mathbb{G}$, even when it is not pseudo-proper, in such a way that (a) when f is an open immersion, $f^! \cong f^*\mathbf{R}\varGamma'_{\mathscr{Y}} = \mathbf{R}\varGamma'_{\mathscr{X}} f^*$, and (b) such that the resulting variance theory $-^!$ is a pre-pseudofunctor. In fact, $\mathbf{1}^!_{\mathscr{X}} \xrightarrow{\sim} \mathbf{R}\varGamma'_{\mathscr{X}}$, and the latter functor is not isomorphic to the identity functor.

For a map $f\colon \mathscr{X} \to \mathscr{Y}$ in $\mathbb{G}$, we set $f^\# = \mathbf{\Lambda}_{\mathscr{X}}(f^!)$. The source of $f^\#$ is $\widetilde{\mathbf{D}}^+_{\mathrm{qc}}(\mathscr{Y})$ and its target is $\widetilde{\mathbf{D}}^+_{\mathrm{qc}}(\mathscr{X})$, so that $f^\#\colon \widetilde{\mathbf{D}}^+_{\mathrm{qc}}(\mathscr{Y}) \to \widetilde{\mathbf{D}}^+_{\mathrm{qc}}(\mathscr{X})$. If f is pseudo-proper this definition of $f^\#$ agrees with the earlier one (as the functor which is right adjoint to $\mathbf{R}f_*\mathbf{R}\varGamma'_{\mathscr{X}}$). The variance theory $-^\#$ is a pre-pseudofunctor with $\mathbf{1}^\#_{\mathscr{X}} \xrightarrow{\sim} \mathbf{\Lambda}_{\mathscr{X}}$. Our $f^\#$ agrees with the one in [**AJL2**] only when f is pseudo-proper.

Part 1 is organized as follows. The definitions of $-^!$ and $-^\#$ and their first properties are given in Chapter 2. §3.1 and Chapter 3 deal with Cohen-Macaulay maps. The meat of Part 1 is in Chapters 5 and 6. There is an appendix which contains a number of results useful in the main body of the text. We have placed these results in the appendix so that the main narrative is not broken into disconnected bits.

1.3. Notations and basics on formal schemes

We discuss some basic matters on formal schemes and the derived categories of complexes on them. Most of what we say here can be found with more details in [**AJL2**]. Also, for the basic conventions on derived functors we refer to [**L4**].

For any formal scheme $\mathscr{X}$ and any coherent ideal $\mathscr{I}$ in $\mathscr{O}_{\mathscr{X}}$, $\varGamma_{\mathscr{I}}$ denotes the functor that assigns to any $\mathscr{O}_{\mathscr{X}}$-module $\mathscr{F}$, the submodule of sections of $\mathscr{F}$ annihilated locally by some power of $\mathscr{I}$. The torsion functor $\varGamma'_{\mathscr{X}}$ is the one that assigns to any $\mathscr{F}$ the submodule of sections of $\mathscr{F}$ annihilated locally by some open ideal in $\mathscr{O}_{\mathscr{X}}$. Thus for any defining ideal $\mathscr{I}$ for $\mathscr{X}$, $\varGamma'_{\mathscr{X}} = \varGamma_{\mathscr{I}}$. A torsion module is a module $\mathscr{F}$ satisfying $\varGamma'_{\mathscr{X}}\mathscr{F} = \mathscr{F}$.

For any formal scheme $\mathscr{X}$, the abelian categories $\mathcal{A}_?(\mathscr{X})$ for ? in $\{\mathrm{c}, \vec{\mathrm{c}}, \mathrm{qc}, \mathrm{qct}\}$ are defined as in the beginning of this book and the same applies to the definition of derived categories $\mathbf{D}^*_?(\mathscr{X})$.

We use the notation $\mathbf{R}F$ (resp. $\mathbf{L}F$) to denote the right (resp. left) derived functor associated to any (triangulated) functor F between derived categories, and for the derived tensor product we use $\overset{\mathbf{L}}{\otimes}$.

Let $\widetilde{\mathbf{D}}_{\mathrm{qc}}(\mathscr{X})$ denote the triangulated full subcategory of $\mathbf{D}(\mathscr{X})$ whose objects consist of complexes $\mathscr{F}$ such that $\mathbf{R}\varGamma'_{\mathscr{X}}\mathscr{F} \in \mathbf{D}_{\mathrm{qc}}(\mathscr{X})$ (and hence $\mathbf{R}\varGamma'_{\mathscr{X}}\mathscr{F} \in \mathbf{D}_{\mathrm{qct}}(\mathscr{X})$). Similarly, $\widetilde{\mathbf{D}}^+_{\mathrm{qc}}(\mathscr{X})$ denotes the subcategory consisting of complexes $\mathscr{F}$ such that $\mathbf{R}\varGamma'_{\mathscr{X}}\mathscr{F} \in \mathbf{D}^+_{\mathrm{qc}}(\mathscr{X})$. Thus there are full subcategories $\mathbf{D}^+_{\mathrm{qct}}(\mathscr{X}) \subset \mathbf{D}^+_{\mathrm{qc}}(\mathscr{X}) \subset \widetilde{\mathbf{D}}^+_{\mathrm{qc}}(\mathscr{X})$ which are all equal when $\mathscr{X}$ is an ordinary scheme.

The functor $\mathbf{R}\varGamma'_{\mathscr{X}} \colon \mathbf{D}(\mathscr{X}) \to \mathbf{D}(\mathscr{X})$ has a right adjoint given by

$$\mathbf{\Lambda}_{\mathscr{X}}(-) = \mathbf{R}\mathscr{H}om(\mathbf{R}\varGamma'_{\mathscr{X}}\mathscr{O}_{\mathscr{X}}, -).$$

Via canonical maps $\mathbf{R}\varGamma'_{\mathscr{X}} \to 1 \to \mathbf{\Lambda}_{\mathscr{X}}$, both $\mathbf{R}\varGamma'_{\mathscr{X}}$ and $\mathbf{\Lambda}_{\mathscr{X}}$ are idempotent functors and in fact there are natural isomorphisms

$$\mathbf{R}\varGamma'_{\mathscr{X}}\mathbf{R}\varGamma'_{\mathscr{X}} \xrightarrow{\sim} \mathbf{R}\varGamma'_{\mathscr{X}} \xrightarrow{\sim} \mathbf{R}\varGamma'_{\mathscr{X}}\mathbf{\Lambda}_{\mathscr{X}}, \qquad \mathbf{\Lambda}_{\mathscr{X}}\mathbf{R}\varGamma'_{\mathscr{X}} \xrightarrow{\sim} \mathbf{\Lambda}_{\mathscr{X}} \xrightarrow{\sim} \mathbf{\Lambda}_{\mathscr{X}}\mathbf{\Lambda}_{\mathscr{X}}.$$

In particular, $\mathbf{\Lambda}_{\mathscr{X}}(\widetilde{\mathbf{D}}^+_{\mathrm{qc}}(\mathscr{X})) \subset \widetilde{\mathbf{D}}^+_{\mathrm{qc}}(\mathscr{X})$ and therefore, the restriction of $\mathbf{R}\varGamma'_{\mathscr{X}}$ to $\widetilde{\mathbf{D}}^+_{\mathrm{qc}}(\mathscr{X})$ and of $\mathbf{\Lambda}_{\mathscr{X}}$ to $\mathbf{D}^+_{\mathrm{qct}}(\mathscr{X})$ also constitute an adjoint pair.

For $\mathscr{F} \in \mathbf{D}_{\mathrm{c}}(\mathscr{X})$, the canonical map $\mathscr{F} \to \mathbf{\Lambda}_{\mathscr{X}}\mathscr{F}$ is an isomorphism by Greenlees-May duality [**AJL2**, Prop 6.2.1]. More generally, if $\mathbf{D}_{\vec{\mathrm{c}}}(\mathscr{X})$ is the subcategory of $\mathbf{D}_{\mathrm{qc}}(\mathscr{X})$ consisting of complexes whose homology sheaves are direct limits of coherent ones, then the restriction of $\mathbf{\Lambda}_{\mathscr{X}}$ to $\mathbf{D}_{\vec{\mathrm{c}}}(\mathscr{X})$ is isomorphic to the left-derived functor of the completion functor $\Lambda_{\mathscr{X}}$ which assigns to any sheaf $\mathscr{F}$, the inverse limit $\varprojlim_n \mathscr{F}/\mathscr{I}^n\mathscr{F}$ where $\mathscr{I}$ is any defining ideal in $\mathscr{O}_{\mathscr{X}}$. In contrast, note that $\mathbf{R}\varGamma'_{\mathscr{X}}$ does not preserve coherence of homology in general.

Let $f \colon \mathscr{X} \to \mathscr{Y}$ be a map of noetherian formal schemes. Then there are natural isomorphisms (see [**AJL2**, Proposition 5.2.8])

$$\begin{aligned} \mathbf{R}\varGamma'_{\mathscr{X}}\mathbf{L}f^*\mathbf{R}\varGamma'_{\mathscr{Y}} &\xrightarrow{\sim} \mathbf{R}\varGamma'_{\mathscr{X}}\mathbf{L}f^* \xrightarrow{\sim} \mathbf{R}\varGamma'_{\mathscr{X}}\mathbf{L}f^*\mathbf{\Lambda}_{\mathscr{Y}}, \\ \mathbf{\Lambda}_{\mathscr{X}}\mathbf{L}f^*\mathbf{R}\varGamma'_{\mathscr{Y}} &\xrightarrow{\sim} \mathbf{\Lambda}_{\mathscr{X}}\mathbf{L}f^* \xrightarrow{\sim} \mathbf{\Lambda}_{\mathscr{X}}\mathbf{L}f^*\mathbf{\Lambda}_{\mathscr{Y}}. \end{aligned} \tag{1.3.1}$$

Here the first isomorphism in the first line follows easily from the fact that for any coherent ideal $\mathscr{I} \subset \mathscr{O}_{\mathscr{Y}}$, we have $\mathbf{L}f^*\mathbf{R}\varGamma_{\mathscr{I}} \cong \mathbf{R}\varGamma_{\mathscr{I}\mathscr{O}_{\mathscr{X}}}\mathbf{L}f^*$, a fact which can be checked locally using stable Koszul complexes, see (C.5.2) in Appendix below for instance. The remaining isomorphisms in (1.3.1) result from the first one by pre-composing with $\mathbf{\Lambda}_{\mathscr{X}}$ or post-composing with $\mathbf{\Lambda}_{\mathscr{Y}}$.

For $f \colon \mathscr{X} \to \mathscr{Y}$ as above, f^* sends torsion $\mathscr{O}_{\mathscr{Y}}$-modules to torsion $\mathscr{O}_{\mathscr{X}}$-modules and hence we have $\mathbf{L}f^*(\mathbf{D}_{\mathrm{qct}}(\mathscr{Y})) \subset \mathbf{D}_{\mathrm{qct}}(\mathscr{X})$ (in addition to the usual inclusions $\mathbf{L}f^*(\mathbf{D}_{\mathrm{qc}}(\mathscr{Y})) \subset \mathbf{D}_{\mathrm{qc}}(\mathscr{X}), \mathbf{L}f^*(\mathbf{D}_{\mathrm{c}}(\mathscr{Y})) \subset \mathbf{D}_{\mathrm{c}}(\mathscr{X})$). By (1.3.1) we also deduce that $\mathbf{L}f^*(\widetilde{\mathbf{D}}_{\mathrm{qc}}(\mathscr{Y})) \subset \widetilde{\mathbf{D}}_{\mathrm{qc}}(\mathscr{X})$.

Unlike the case of ordinary (noetherian) schemes, $\mathbf{R}f_*$ does not map $\mathbf{D}^+_{\mathrm{qc}}(\mathscr{X})$ (or even $\mathbf{D}^+_{\mathrm{c}}(\mathscr{X})$) inside $\mathbf{D}^+_{\mathrm{qc}}(\mathscr{Y})$ in general. Under additional torsion conditions we do get the desired behaviour. Thus we have $\mathbf{R}f_*(\mathbf{D}^+_{\mathrm{qct}}(\mathscr{X})) \subset \mathbf{D}^+_{\mathrm{qct}}(\mathscr{Y})$ and therefore $\mathbf{R}f_*\mathbf{R}\mathit{\Gamma}'_{\mathscr{X}}(\widetilde{\mathbf{D}}^+_{\mathrm{qc}}(\mathscr{X})) \subset \mathbf{D}^+_{\mathrm{qct}}(\mathscr{Y})$.

For any morphism $f\colon \mathscr{X} \to \mathscr{Y}$ as above and for any closed subset $Z \subset \mathscr{X}$, we set $\mathbf{R}_Z f_* := \mathbf{R}f_*\mathbf{R}\mathit{\Gamma}_Z$ where, for any sheaf $\mathscr{F}$ on $\mathscr{X}$, $\mathit{\Gamma}_Z(\mathscr{F})$ is the subsheaf of sections of $\mathscr{F}$ with support in Z. Likewise we set $\mathrm{R}'_{\mathscr{X}} f_* := \mathbf{R}f_*\mathbf{R}\mathit{\Gamma}'_{\mathscr{X}}$. For any integer r we use $\mathbf{R}^r_Z f_* := H^r\mathbf{R}_Z f_*$ and $\mathrm{R}'^r_{\mathscr{X}} f_* := H^r\mathbf{R}\mathit{\Gamma}'_{\mathscr{X}} f_*$.

CHAPTER 2

The duality pseudofunctors over formal schemes

We mainly work with the category $\mathbb{G}$ of composites of open immersions and pseudo-proper maps between noetherian formal schemes. By Nagata's compactification theorem, every separated finite-type map of *ordinary* schemes lies in $\mathbb{G}$.

2.1. Grothendieck Duality on formal schemes

The results in [**AJL2**] and [**Nay**] extend the theory of $-^{\mathbf{!}}$ over ordinary schemes to that over $\mathbb{G}$. Thus, there is a contravariant pseudofunctor $(-)^!$ on $\mathbb{G}$ with values in $\mathbf{D}^+_{\mathrm{qct}}(\mathscr{X})$ for any formal scheme $\mathscr{X}$, such that if $f\colon \mathscr{X} \to \mathscr{Y}$ is pseudo-proper, there exists a functorial map $t_f\colon \mathbf{R}f_*f^! \to \mathbf{1}_{\mathbf{D}^+_{\mathrm{qct}}(\mathscr{Y})}$ such that $(f^!, t_f)$ is a right adjoint to $\mathbf{R}f_*\colon \mathbf{D}^+_{\mathrm{qct}}(\mathscr{X}) \to \mathbf{D}^+_{\mathrm{qct}}(\mathscr{Y})$ while if f is an open immersion or more generally, if f is adic étale, then $f^! = f^*$ pseudofunctorially. For a formally étale map f in $\mathbb{G}$, (e.g., a completion map $\mathscr{X} \to X$ where X is an ordinary scheme and $\mathscr{X}$ its completion along some coherent ideal in $\mathcal{O}_X$), we have $f^! \xrightarrow{\sim} \mathbf{R}\varGamma'_{\mathscr{X}} f^*$ (again pseudofunctorially), see [**Nay**, Theorem 7.1.6]. Note that $f^!$ does not preserve coherence of homology in general.

There is an extension of $(-)^!$ that we find convenient to use. For any $f\colon \mathscr{X} \to \mathscr{Y}$ in $\mathbb{G}$ the composite $f^!\mathbf{R}\varGamma'_{\mathscr{Y}}$ sends $\widetilde{\mathbf{D}}^+_{\mathrm{qc}}(\mathscr{Y})$ to $\mathbf{D}^+_{\mathrm{qct}}(\mathscr{X}) \subset \widetilde{\mathbf{D}}^+_{\mathrm{qc}}(\mathscr{X})$ and is isomorphic to $f^!$ when restricted to $\mathbf{D}^+_{\mathrm{qct}}(\mathscr{Y})$. The extended functor is also denoted as $f^!$. However, this extension $(-)^!$ only forms a pre-pseudofunctor (see beginning of Chapter 1). Thus, for $\mathscr{X} \xrightarrow{f} \mathscr{Y} \xrightarrow{g} \mathscr{Z}$ in $\mathbb{G}$, the comparison isomorphism $C^!_{f,g}\colon (gf)^! \xrightarrow{\sim} f^!g^!$ is induced by the usual one over $\mathbf{D}^+_{\mathrm{qct}}(-)$ and by the isomorphisms

$$(gf)^!\mathbf{R}\varGamma'_{\mathscr{Z}} \xrightarrow{\sim} f^!g^!\mathbf{R}\varGamma'_{\mathscr{Z}} \xrightarrow{\sim} f^!\mathbf{R}\varGamma'_{\mathscr{Y}}g^!\mathbf{R}\varGamma'_{\mathscr{Z}}.$$

The associativity condition for $C^!_{-,-}$ vis-á-vis composition of 3 maps easily results from the corresponding one over $\mathbf{D}^+_{\mathrm{qct}}(-)$. For the identity map $1_{\mathscr{X}}$ on $\mathscr{X}$ we have a natural map $1^!_{\mathscr{X}} = \mathbf{R}\varGamma'_{\mathscr{X}} \to \mathbf{1}$ and the comparison isomorphisms corresponding to composing f on the left or right by identity are the canonical ones $f^! \xrightarrow{\sim} f^!\mathbf{R}\varGamma'_{\mathscr{X}}$ and $f^! \xrightarrow{\sim} \mathbf{R}\varGamma'_{\mathscr{X}} f^!$.

This extended pre-pseudofunctorial version of $(-)^!$ is what we will use from now on. It has the following properties. For $f\colon \mathscr{X} \to \mathscr{Y}$ in $\mathbb{G}$, if f is pseudo-proper, then $f^!\colon \widetilde{\mathbf{D}}^+_{\mathrm{qc}}(\mathscr{Y}) \to \mathbf{D}^+_{\mathrm{qct}}(\mathscr{X})$ is right adjoint to $\mathbf{R}f_*\colon \mathbf{D}^+_{\mathrm{qct}}(\mathscr{X}) \to \widetilde{\mathbf{D}}^+_{\mathrm{qc}}(\mathscr{Y})$, while if f is formally étale, then $f^!$ is isomorphic to $\mathbf{R}\varGamma'_{\mathscr{X}} f^*\mathbf{R}\varGamma'_{\mathscr{Y}} \xrightarrow{\sim} \mathbf{R}\varGamma'_{\mathscr{X}} f^*$, (see Theorem 7.1.6 and §7.2 of [**Nay**]).

For $\mathbf{D}^+_{\mathrm{c}}$-related questions, it is useful to work with another generalization of $(-)^!$ from ordinary schemes. For $f\colon \mathscr{X} \to \mathscr{Y}$ in $\mathbb{G}$ we define $f^{\#}\colon \widetilde{\mathbf{D}}^+_{\mathrm{qc}}(\mathscr{Y}) \to$

$\widetilde{\mathbf{D}}^+_{\mathrm{qc}}(\mathscr{X})$ by the formula

$$f^{\#} := \mathbf{\Lambda}_{\mathscr{X}} f^{!}. \tag{2.1.1}$$

Since $f^! \xrightarrow{\sim} \mathbf{R}\mathit{\Gamma}'_{\mathscr{X}} f^{\#}$, therefore $(-)^!$ and $(-)^{\#}$ determine each other upto isomorphism.

Note that $(-)^{\#}$ is also a pre-pseudofunctor. For maps $\mathscr{X} \xrightarrow{f} \mathscr{Y} \xrightarrow{g} \mathscr{Z}$ in $\mathbb{G}$, the comparison isomorphism $C^{\#}_{f,g}\colon (gf)^{\#} \xrightarrow{\sim} f^{\#}g^{\#}$ is given by the composite

$$(gf)^{\#} = \mathbf{\Lambda}_{\mathscr{X}}(gf)^! \xrightarrow{\sim} \mathbf{\Lambda}_{\mathscr{X}} f^! g^! \xrightarrow{\sim} \mathbf{\Lambda}_{\mathscr{X}} f^! \mathbf{\Lambda}_{\mathscr{Y}} g^! = f^{\#}g^{\#}$$

where the last isomorphism is obtained from composite of the following sequence where we use $f^!\mathbf{R}\mathit{\Gamma}'_{\mathscr{Y}} \xrightarrow{\sim} f^!$ in the first and the last step:

$$f^! g^! \xleftarrow{\sim} f^!\mathbf{R}\mathit{\Gamma}'_{\mathscr{Y}} g^! \xrightarrow{\sim} f^!\mathbf{R}\mathit{\Gamma}'_{\mathscr{Y}}\mathbf{\Lambda}_{\mathscr{Y}} g^! \xrightarrow{\sim} f^!\mathbf{\Lambda}_{\mathscr{Y}} g^!.$$

The associativity condition for $C^{\#}_{-,-}$ vis-á-vis composition of 3 maps easily results from the corresponding one for $(-)^!$. For the identity map $1_{\mathscr{X}}$ on $\mathscr{X}$, there is a map $\mathbf{1} \to (1_{\mathscr{X}})^{\#} = \mathbf{\Lambda}_{\mathscr{X}}$ and the comparison isomorphisms corresponding to composing f on the left or right by identity are the canonical ones $f^{\#} \xrightarrow{\sim} \mathbf{\Lambda}_{\mathscr{X}} f^{\#}$ and $f^{\#} \xrightarrow{\sim} f^{\#}\mathbf{\Lambda}_{\mathscr{Y}}$.

If $f\colon \mathscr{X} \to \mathscr{Y}$ is a pseudo-proper map (whence a map in $\mathbb{G}$), then $f^{\#}$ is right adjoint to $\mathbf{R}f_*\mathbf{R}\mathit{\Gamma}'_{\mathscr{X}}$ so that we have a co-adjoint unit, the so-called *trace map*

$$\mathrm{Tr}_f\colon \mathbf{R}f_*\mathbf{R}\mathit{\Gamma}'_{\mathscr{X}} f^{\#} \to \mathbf{1}, \tag{2.1.2}$$

while if f is an open immersion (or more generally, if f is formally étale) then there is a natural isomorphism $f^{\#} \xrightarrow{\sim} \mathbf{\Lambda}_{\mathscr{X}} f^*$. Moreover these isomorphisms are pre-pseudofunctorial over the corresponding full subcategories of $\mathbb{G}$.

A cautionary remark. In [**AJL2**, Prop. 6.1.4], $f^{\#}$ is defined as $\mathbf{\Lambda}_{\mathscr{X}} f_{\mathrm{t}}^{\times}$ where $f_{\mathrm{t}}^{\times}$ is the right adjoint to the restriction of $\mathbf{R}f_*$ to $\mathbf{D}_{\mathrm{qct}}(\mathscr{X})$. The functor $\mathbf{\Lambda}_{\mathscr{X}} f_{\mathrm{t}}^{\times}$ is shown to be be a right adjoint to $\mathbf{R}f_*\mathbf{R}\mathit{\Gamma}'_{\mathscr{X}}$ for *every* f. Our definition of $f^{\#}$ agrees with that of [**AJL2**] when f is pseudo-proper, but not in general.

For any $f\colon \mathscr{X} \to \mathscr{Y}$ in $\mathbb{G}$, we have $f^{\#}(\mathbf{D}^+_{\mathrm{c}}(\mathscr{Y})) \subset \mathbf{D}^+_{\mathrm{c}}(\mathscr{X})$. This can be seen by reducing to the special cases when f is pseudoproper or f is an open immersion; in the latter case one uses that $\mathbf{\Lambda}_{\mathscr{X}}\big|_{\mathbf{D}_{\mathrm{c}}(\mathscr{X})}$ is isomorphic to the identity functor, so that $f^{\#} \xrightarrow{\sim} \mathbf{\Lambda}_{\mathscr{X}} f^* \xrightarrow{\sim} f^*$ on $\mathbf{D}_{\mathrm{c}}(\mathscr{X})$, while the former case is dealt with in [**AJL2**, p. 89, Proposition 8.3.2]. Thus $(-)^{\#}$ gives a $\mathbf{D}^+_{\mathrm{c}}$-valued pseudofunctor on $\mathbb{G}$. It also follows that if f is formally étale and $\mathscr{F} \in \mathbf{D}^+_{\mathrm{c}}(\mathscr{Y})$, then we have an isomorphism

$$f^*\mathscr{F} \xrightarrow{\sim} f^{\#}\mathscr{F} \tag{2.1.3}$$

which is pseudofunctorial over the category of formally étale maps. If $\mathscr{Y}$ is a formal scheme, $\mathscr{I} \subset \mathscr{O}_{\mathscr{Y}}$ an open coherent ideal, $\mathscr{W} := \widehat{\mathscr{Y}}$ the completion of $\mathscr{Y}$ by $\mathscr{I}$ and $\kappa\colon \mathscr{W} \to \mathscr{Y}$ the corresponding completion map, then κ is both pseudoproper and étale. For any $\mathscr{F} \in \mathbf{D}^+_{\mathrm{c}}(\mathscr{Y})$, the isomorphism of (2.1.3) is the same (see Lemma A.1.3 in Appendix below) as the map adjoint to the natural composite

$$\mathbf{R}\kappa_*\mathbf{R}\mathit{\Gamma}'_{\mathscr{W}}\kappa^* \xrightarrow{\sim} \mathbf{R}\mathit{\Gamma}_{\mathscr{I}} \to \mathbf{1}.$$

Nevertheless, while working with $(-)^{\#}$, it is also convenient to work with the larger category $\widetilde{\mathbf{D}}^+_{\mathrm{qc}}$ in addition to $\mathbf{D}^+_{\mathrm{c}}$ since, some of the functors, such as $\mathbf{R}f_*\mathbf{R}\mathit{\Gamma}'_{\mathscr{X}}$, the left adjoint to $f^{\#}$ when $f\colon \mathscr{X} \to \mathscr{Y}$ is pseudoproper, do not preserve coherence of homology in general.

Over ordinary schemes $(-)^{\#}$ and $(-)^{!}$ are canonically identified and so we use both interchangeably.

2.2. Flat base change

Suppose we have a cartesian square $\mathfrak{s}$ of noetherian formal schemes

$$\begin{tikzcd} \mathscr{V} \arrow[r, "v"] \arrow[d, "g"'] \arrow[dr, phantom, "\square"] & \mathscr{X} \arrow[d, "f"] \\ \mathscr{W} \arrow[r, "u"'] & \mathscr{Y} \end{tikzcd}$$

with f in $\mathbb{G}$ and u flat. The *flat-base-change theorem* for $(-)^{!}$ gives an isomorphism for $\mathscr{F} \in \widetilde{\mathbf{D}}^{+}_{\mathrm{qc}}(\mathscr{Y})$:

$$\beta^{!}_{\mathfrak{s}}(\mathscr{F}) \colon \mathbf{R}\varGamma'_{\mathscr{V}} v^{*} f^{!}\mathscr{F} \xrightarrow{\ \sim\ } g^{!}u^{*}\mathscr{F}$$

(see [**AJL2**, p. 77, Theorem 7.4] for the case when f is pseudoproper and [**Nay**, p. 261, Theorem 7.14] for the general case and also [**Nay**, §7.2]). For $\mathscr{F} \in \widetilde{\mathbf{D}}^{+}_{\mathrm{qc}}(\mathscr{Y})$ the corresponding flat-base-change isomorphism for $(-)^{\#}$

$$(2.2.1) \qquad \beta^{\#}_{\mathfrak{s}}(\mathscr{F}) \colon \mathbf{\Lambda}_{\mathscr{V}} v^{*} f^{\#}(\mathscr{F}) \xrightarrow{\ \sim\ } g^{\#}u^{*}(\mathscr{F})$$

is induced by the following sequence of natural isomorphisms of functors

$$\mathbf{\Lambda}_{\mathscr{V}} v^{*} f^{\#} = \mathbf{\Lambda}_{\mathscr{V}} v^{*} \mathbf{\Lambda}_{\mathscr{X}} f^{!} \xleftarrow{\ \sim\ } \mathbf{\Lambda}_{\mathscr{V}} v^{*} f^{!} \xleftarrow{\ \sim\ } \mathbf{\Lambda}_{\mathscr{V}} \mathbf{R}\varGamma'_{\mathscr{V}} v^{*} f^{!} \xrightarrow{\ \sim\ } \mathbf{\Lambda}_{\mathscr{V}} g^{!}u^{*} = g^{\#}u^{*}$$

where the last isomorphism is induced by $\beta^{!}_{\mathfrak{s}}$ (cf. [**AJL2**, p. 86, Theorem 8.1] for f, g pseudoproper).

If $\mathscr{F} \in \mathbf{D}^{+}_{\mathrm{c}}(\mathscr{Y})$, or if u is open or if $\mathscr{V}$ is an ordinary scheme, we have an isomorphism

$$(2.2.2) \qquad v^{*} f^{\#}\mathscr{F} \xrightarrow{\ \sim\ } g^{\#}u^{*}\mathscr{F},$$

(see [**AJL2**, Theorem 8.1, Corollary 8.3.3]).

Further properties of the base-change map are explored in Appendix A.1.

2.3. Traces with proper support

Let $f\colon X \to Y$ be a separated map of finite-type between ordinary schemes, and Z a closed subscheme of X which is proper over Y. The completion map $\kappa\colon \mathscr{X} \to X$ of X along Z, is formally étale and affine and the composition $\widehat{f} := f\kappa$ is pseudoproper. We define the trace map for f along Z

$$(2.3.1) \qquad \mathrm{Tr}_{f,Z} \colon \mathbf{R}_{Z} f_{*} f^{\#} \to \mathbf{1}$$

to be the composite

$$\mathbf{R}f_{*}\mathbf{R}\varGamma_{Z} f^{\#} \xrightarrow{\ \sim\ } \mathbf{R}f_{*}\kappa_{*}\mathbf{R}\varGamma'_{\mathscr{X}}\kappa^{*} f^{\#} \xrightarrow{\ \sim\ } \mathbf{R}\widehat{f}_{*}\mathbf{R}\varGamma'_{\mathscr{X}}\kappa^{\#} f^{\#} \xrightarrow{\ \sim\ } \mathbf{R}\widehat{f}_{*}\mathbf{R}\varGamma'_{\mathscr{X}}\widehat{f}^{\#} \xrightarrow{\mathrm{Tr}_{\widehat{f}}} \mathbf{1}$$

where the first isomorphism is from the canonical isomorphism $\mathbf{R}\varGamma_{Z} \xrightarrow{\ \sim\ } \kappa_{*}\mathbf{R}\varGamma'_{\mathscr{X}}\kappa^{*}$ while the remaining maps are the obvious natural ones.

There is an alternate description of $\mathrm{Tr}_{f,Z}$ involving compactifications. Let $(u, \bar{f})$ be a compactification of f, i.e., $u\colon X \to \overline{X}$ is an open immersion, $f\colon \overline{X} \to Y$ a *proper* map such that $f = \bar{f} \circ u$. A theorem of Nagata assures us that compactifications always exist (see [**N**], [**D3**], [**Lu**], and [**C2**]). According to Lemma A.3.5, $\mathrm{Tr}_{f,Z}$ can also be described as the composite

$$\mathbf{R}f_*\mathbf{R}\mathit{\Gamma}_Z f^{\#} \xrightarrow{\sim} \mathbf{R}\bar{f}_*\mathbf{R}\mathit{\Gamma}_{u(Z)}\bar{f}^{\#} \to \mathbf{R}\bar{f}_*\bar{f}^{\#} \xrightarrow{\mathrm{Tr}_{\bar{f}}} \mathbf{1}. \tag{2.3.2}$$

CHAPTER 3

Traces and Residues for Cohen-Macaulay maps

3.1. Cohen-Macaulay maps

Recall that a locally finite type map $f\colon X \to Y$ between ordinary schemes is said to be *Cohen-Macaulay of relative dimension* r if it is a flat map and all the non-empty fibres are Cohen-Macaulay and of pure dimension r. This is equivalent to saying that f is flat, $f^!\mathscr{O}_Y$ (which is defined locally if f is not separated) has homology concentrated in only degree $-r$, and the resulting $\mathscr{O}_X$-module $\omega_f^\#$ obtained by gluing the various local $\mathrm{H}^{-r}(f^!\mathscr{O}_Y)$ is coherent and flat over Y. We make the obvious generalization to formal schemes. First we need the following definition.

Definition 3.1.1. A map of formal schemes $f\colon \mathscr{X} \to \mathscr{Y}$ is said to be *locally in* $\mathbb{G}$ if for every point $x \in \mathscr{X}$, there exists an open neighbourhood U of x such that the restriction f_U of f to U is in $\mathbb{G}$.

Note that if $f\colon \mathscr{X} \to \mathscr{Y}$ is locally in $\mathbb{G}$ and $\mathscr{F} \in \mathbf{D}_c^+(\mathscr{Y})$, then locally on $\mathscr{X}$, $f^\#\mathscr{F}$ is defined, but it need not be defined globally, even though by pseudofunctoriality of $(-)^\#$ over $\mathbf{D}_c^+$, these local twisted inverse images are isomorphic on overlaps and these isomorphisms form a descent datum for the Zariski topology. However, in this case, for every integer n, and every $\mathscr{G} \in \mathbf{D}_c^+(\mathscr{Y})$ the sheaves $\mathrm{H}^n(f_U^\#(\mathscr{G}))$ do glue to give a coherent sheaf on $\mathscr{X}$ which we denote as $\mathrm{H}^n(f^\#\mathscr{G})$ (even though there might well be no $f^\#\mathscr{G}$). We are not aware of any example where $f^\#$ is defined locally but not globally.

Definition 3.1.2. A map of formal schemes $f\colon \mathscr{X} \to \mathscr{Y}$ is called *Cohen-Macaulay* (CM) *of relative dimension* r if it is flat, locally in $\mathbb{G}$ with $\mathrm{H}^i(f^\#\mathscr{O}_{\mathscr{Y}}) = 0$ for $i \neq -r$ and $\omega_f^\# := \mathrm{H}^{-r}(f^\#\mathscr{O}_{\mathscr{Y}})$ is flat over $\mathscr{Y}$. The coherent $\mathscr{O}_{\mathscr{X}}$-module $\omega_f^\#$ is called the *relative dualizing sheaf* for the CM map f. If such a map f is already in $\mathbb{G}$, we shall make the identification $f^\#\mathscr{O}_{\mathscr{Y}} = \omega_f^\#[r]$.

It is tempting to give alternate definitions for a map to be CM that are more local in nature. An extension of the theory of $(-)^!$ and $(-)^\#$ to a larger category containing maps that are essentially of pseudo-finite type (see [**LNS**, §2.1]) would provide a natural setting for proving equivalence between various possible alternate definitions. Since there is no such extension in literature and since we do not need such a result here, we do not pursue this matter further.

A map $f\colon \mathscr{X} \to \mathscr{Y}$ is said to be *smooth* if it is in $\mathbb{G}$ and is formally smooth. In this case the universally finite module of relative differential forms $\widehat{\Omega}^1_{\mathscr{X}/\mathscr{Y}}$ is a locally free module of finite rank and if it has constant rank r we say that f has relative dimension r, see [**LNS**, §2.6]. We use ω_f to denote the top exterior power of the universally finite module of differentials.

3.2. Abstract Trace for Cohen-Macaulay maps

Suppose $g\colon \mathscr{X} \to \mathscr{Y}$ is Cohen-Macaulay of relative dimension r and is *pseudo-proper*. Since $\omega_g^{\#}[r] = g^{\#}\mathscr{O}_{\mathscr{Y}}$, therefore, as in (2.1.2), we have a trace map

$$\mathrm{Tr}_g(\mathscr{O}_{\mathscr{Y}})\colon \mathbf{R}g_*\mathbf{R}\varGamma'_{\mathscr{X}}\omega_g^{\#}[r] \to \mathscr{O}_{\mathscr{Y}}.$$

DEFINITION 3.2.1. Let $g\colon \mathscr{X} \to \mathscr{Y}$ be as above (i.e., g is pseudo-proper and Cohen-Macaulay of relative dimension r). The *abstract trace map on* $\mathrm{R}'^{r}_{\mathscr{X}}g_*\omega_g^{\#}$ (or simply *the trace map on* $\mathrm{R}'^{r}_{\mathscr{X}}g_*\omega_g^{\#}$) is the map

$$\mathrm{tr}_g^{\#}\colon \mathrm{R}'^{r}_{\mathscr{X}}g_*\omega_g^{\#} \to \mathscr{O}_{\mathscr{Y}}. \tag{3.2.2}$$

given by $\mathrm{tr}_g^{\#} = \mathrm{H}^0(\mathrm{Tr}_g(\mathscr{O}_{\mathscr{Y}}))$.

THEOREM 3.2.3. *Let $g\colon \mathscr{X} \to \mathscr{Y}$ be pseudo-proper and Cohen-Macaulay of relative dimension r. Then for any quasi-coherent $\mathscr{O}_{\mathscr{X}}$-module $\mathscr{F}$ satisfying $\mathrm{R}'^{j}_{\mathscr{X}}g_*\mathscr{F} = 0$ for $j > r$, we have a functorial isomorphism*

$$\mathrm{Hom}_{\mathscr{X}}(\mathscr{F},\, \omega_g^{\#}) \xrightarrow{\ \sim\ } \mathrm{Hom}_{\mathscr{Y}}(\mathrm{R}'^{r}_{\mathscr{X}}g_*\mathscr{F},\, \mathscr{O}_{\mathscr{Y}}),$$

which is given by sending $\theta \in \mathrm{Hom}_{\mathscr{X}}(\mathscr{F},\, \omega_g^{\#})$ to the composite

$$\mathrm{R}'^{r}_{\mathscr{X}}g_*\mathscr{F} \xrightarrow{\mathrm{R}'^{r}_{\mathscr{X}}g_*(\theta)} \mathrm{R}'^{r}_{\mathscr{X}}g_*\omega_g^{\#} \xrightarrow{\mathrm{tr}_g^{\#}} \mathscr{O}_{\mathscr{Y}}.$$

PROOF. By adjointness we have a natural isomorphism

$$\mathbf{R}\mathrm{Hom}_{\mathscr{X}}(\mathscr{F},\, \omega_g^{\#}[r]) \xrightarrow{\ \sim\ } \mathbf{R}\mathrm{Hom}_{\mathscr{Y}}(\mathrm{R}'_{\mathscr{X}}g_*\mathscr{F},\, \mathscr{O}_{\mathscr{Y}}),$$

hence there are natural isomorphisms

$$\begin{aligned}
\mathrm{Hom}_{\mathscr{X}}(\mathscr{F},\, \omega_g^{\#}) &= \mathrm{Hom}_{\mathbf{D}(\mathscr{X})}(\mathscr{F},\, \omega_g^{\#})\\
&= H^{-r}\mathbf{R}\mathrm{Hom}_{\mathscr{X}}(\mathscr{F},\, \omega_g^{\#}[r])\\
&\cong H^{-r}\mathbf{R}\mathrm{Hom}_{\mathscr{Y}}(\mathrm{R}'_{\mathscr{X}}g_*\mathscr{F},\, \mathscr{O}_{\mathscr{Y}})\\
&= \mathrm{Hom}_{\mathbf{D}(\mathscr{Y})}(\mathrm{R}'_{\mathscr{X}}g_*\mathscr{F}[r],\, \mathscr{O}_{\mathscr{Y}})\\
&\cong \mathrm{Hom}_{\mathbf{D}(\mathscr{Y})}(\mathrm{R}'^{r}_{\mathscr{X}}g_*\mathscr{F},\, \mathscr{O}_{\mathscr{Y}}) \qquad \text{(see [\textbf{L4}, p. 37, Prop. 1.10.1])}\\
&= \mathrm{Hom}_{\mathscr{Y}}(\mathrm{R}'^{r}_{\mathscr{X}}g_*\mathscr{F},\, \mathscr{O}_{\mathscr{Y}}).
\end{aligned}$$

□

In the above proof, we don't know if $\omega_g^{\#}$ satisfies the hypotheses required of $\mathscr{F}$. We are interested in special cases when this is true. This leads to the following.

COROLLARY 3.2.4. *If $\mathrm{R}'^{j}_{\mathscr{X}}g_*(\mathscr{F}) = 0$ for every $j > r$ and every $\mathscr{F} \in \mathcal{A}_{\mathrm{c}}(\mathscr{X})$ (resp. $\mathscr{F} \in \mathcal{A}_{\vec{\mathrm{c}}}(\mathscr{X})$, resp. $\mathscr{F} \in \mathcal{A}_{\mathrm{qc}}(\mathscr{X})$), then $(\omega_g^{\#},\, \mathrm{tr}_g^{\#})$ represents the functor $\mathrm{Hom}_{\mathscr{Y}}(\mathrm{R}'^{r}_{\mathscr{X}}g_*(-),\, \mathscr{O}_{\mathscr{Y}})$ on $\mathcal{A}_{\mathrm{c}}(\mathscr{X})$ (resp. $\mathcal{A}_{\vec{\mathrm{c}}}(\mathscr{X})$, resp. $\mathcal{A}_{\mathrm{qc}}(\mathscr{X})$). In particular if $\mathscr{Y} = Y$ is an ordinary scheme, $f\colon X \to Y$ a Cohen-Macaulay map of ordinary schemes of relative dimension r, Z a closed subscheme of X such that the resulting map $Z \to Y$ is finite and flat, $\mathscr{X} = X_{/Z}$ the completion of X along Z, and $g\colon \mathscr{X} \to \mathscr{Y}$ the map induced by f, then $(\omega_g^{\#},\, \mathrm{tr}_g^{\#})$ represents the functor $\mathrm{Hom}_{\mathscr{Y}}(\mathrm{R}'^{r}_{\mathscr{X}}g_*(-),\, \mathscr{O}_{\mathscr{Y}})$ on $\mathcal{A}_{\vec{\mathrm{c}}}(\mathscr{X})$.*

PROOF. The first assertion holds since, if g is Cohen-Macaulay, then $\omega_g^{\#} \in \mathcal{A}_{\mathrm{c}}(\mathscr{X})$. The second assertion follows from the first since, if $\mathscr{I}$ is a coherent ideal defining Z in X, $\mathscr{F} \in \mathcal{A}_{\vec{c}}(\mathscr{X})$ and κ denotes the canonical map $\mathscr{X} \to X$, then $\mathscr{F} \xrightarrow{\sim} \kappa^*\mathscr{G}$ for some $\mathscr{G} \in \mathcal{A}_{\mathrm{qc}}(X)$ (see [**AJL2**, p. 31, Prop. 3.1.1]), and hence $\mathbf{R}g_*\mathbf{R}\mathit{\Gamma}'_{\mathscr{X}}\mathscr{F} \xrightarrow{\sim} \mathbf{R}f_*\kappa_*\mathbf{R}\mathit{\Gamma}'_{\mathscr{X}}\kappa^*\mathscr{G} \xrightarrow{\sim} \mathbf{R}f_*\mathbf{R}\mathit{\Gamma}_{\mathscr{I}}\mathscr{G}$ (see [**AJL2**, §5]). □

REMARK 3.2.5. In a slightly different direction, Lipman observed the following (private communication). First note that according to [**AJL2**, p. 39, Prop. 3.4.3], since all our schemes are noetherian, if $f\colon \mathscr{X} \to \mathscr{Y}$ is a map of schemes (possibly formal), the functor $\mathbf{R}f_*$ is bounded above on $\mathbf{D}_{\vec{c}}(\mathscr{X})$. In other words, there is an integer $e \geq 0$ such that if $\mathscr{H} \in \mathbf{D}_{\vec{c}}(\mathscr{X})$ and $H^i(\mathscr{H}) = 0$ for $i \geq i_0$, then $H^i(\mathbf{R}f_*\mathscr{H}) = 0$ for all $i \geq i_0 + e$. Next, by computing local cohomologies using stable Koszul complexes (see (C.5.2)) on affine open subschemes of $\mathscr{X}$ and using quasi-compactness of the noetherian scheme $\mathscr{X}$, we see that there is an integer t such that $H^j(\mathbf{R}\mathit{\Gamma}'_{\mathscr{X}}\mathscr{F}) = 0$ for $\mathscr{F} \in \mathcal{A}_{\vec{c}}(\mathscr{X})$ and $j > t$. It is then not hard to see that if $r = e + t$, and if $\mathscr{H} \in \mathbf{D}_{\vec{c}}(\mathscr{X})$ is such that $H^i(\mathscr{H}) = 0$ for $i > i_0$, then $H^j(\mathbf{R}f_*\mathbf{R}\mathit{\Gamma}'_{\mathscr{X}}(\mathscr{H})) = 0$ for $j > i_0 + r$. Now suppose f is *pseudo-proper*. By the argument given in [**L4**, p. 165, Lemma 4.1.8], we see that if $\mathscr{G} \in \mathcal{A}_{\vec{c}}(\mathscr{Y})$ is such that $f^{\#}\mathscr{G} \in \mathbf{D}_{\vec{c}}(\mathscr{X})$ then $H^j f^{\#}\mathscr{G} = 0$ for every any $j < -r$. Let $\omega_f^{\#} = H^{-r}(f^{\#}\mathscr{O}_{\mathscr{Y}})$. Then as we argued earlier, $\omega_f^{\#} \in \mathcal{A}_{\mathrm{c}}(\mathscr{X}) \subset \mathbf{D}_{\vec{c}}(\mathscr{X})$. The proof of Theorem 3.2.3 applies and we have a functorial isomorphism (without any Cohen-Macaulay hypotheses)

$$\mathrm{Hom}_{\mathscr{X}}(\mathscr{F},\, \omega_f^{\#}) \xrightarrow{\sim} \mathrm{Hom}_{\mathscr{Y}}(\mathrm{R}'^{r}_{\mathscr{X}}\mathscr{F},\, \mathscr{O}_{\mathscr{Y}})$$

for every $\mathscr{F} \in \mathcal{A}_{\vec{c}}(\mathscr{X})$. We point out that $\mathrm{R}'^{j}_{\mathscr{X}}|_{\mathcal{A}_{\vec{c}}(\mathscr{X})} = 0$ for $j > r$. In particular, in this argument, $\mathrm{R}'^{j}_{\mathscr{X}}\omega_f^{\#} = 0$ for $j > r$. We could not guarantee this for the r used in the Theorem.

3.3. Abstract Residue for Cohen-Macaulay maps

Throughout this subsection

$$f\colon X \to Y$$

is a *finite-type Cohen-Macaulay* map between ordinary schemes of relative dimension r. Suppose $Z \hookrightarrow X$ is a closed subscheme of X, *proper over* Y. Let $\mathscr{X} = X_{/Z}$ be the formal completion of X along Z, and $\kappa\colon \mathscr{X} \to X$ the completion map. Let $\widehat{f}\colon \mathscr{X} \to Y$ be the composite $f \circ \kappa$. We have $\widehat{f}^{\#} \xrightarrow{\sim} \kappa^{\#}f^{\#} \xrightarrow{\sim} \kappa^* f^{\#}$, whence $\mathrm{H}^j(\widehat{f}^{\#}) \xrightarrow{\sim} \kappa^*\mathrm{H}^j(f^{\#})$. Note that $\widehat{f}$ is pseudo-proper and Cohen-Macaulay of relative dimension r and therefore $\mathrm{tr}^{\#}_{\widehat{f}}$ is defined. In [**S2**, p. 742, (3.2)] a residue map along Z is defined using local compactifications. Here is a reformulation of that definition in terms of κ.

DEFINITION 3.3.1. Let Z and f be as above. The *abstract residue along* Z

$$\mathbf{res}^{\#}_Z \colon \mathrm{R}^r_Z f_*\omega_f^{\#} \to \mathscr{O}_Y \tag{3.3.2}$$

is the composite

$$\mathrm{R}^r_Z f_*\omega_f^{\#} \xrightarrow[\text{(A.3.1)}]{\sim} \mathrm{R}'^{r}_{\mathscr{X}}\widehat{f}_*\omega^{\#}_{\widehat{f}} \xrightarrow{\mathrm{tr}^{\#}_{\widehat{f}}} \mathscr{O}_Y.$$

It is worth unravelling the first isomorphism in the above composite a little more. The isomorphism $\kappa_*\mathbf{R}\varGamma'_{\mathscr{X}}\kappa^* \xrightarrow{\sim} \mathbf{R}\varGamma_Z$ gives rise to isomorphisms (one for every j)

$$\mathrm{R}^j_Z f_*\mathscr{F} \xrightarrow{\sim} \mathrm{H}^j(\mathbf{R}\widehat{f}_*\mathbf{R}\varGamma'_{\mathscr{X}}\kappa^*\mathscr{F}) = \mathrm{R}'^j_{\mathscr{X}}\widehat{f}_*\kappa^*\mathscr{F} \tag{3.3.3}$$

which are functorial in $\mathscr{F}$ varying over quasi-coherent $\mathscr{O}_X$-modules. In affine terms, if $X = \operatorname{Spec} R$, M an R-module, and Z is given by the ideal I, then writing $\widehat{R}$ for the I-adic completion of R, and $J = I\widehat{R}$, the above isomorphism is the well-known one

$$\mathrm{H}^j_I(M) \xrightarrow{\sim} \mathrm{H}^j_J(M \otimes_R \widehat{R}).$$

The isomorphism $\mathrm{R}^r_Z f_*\omega^\#_f \xrightarrow{\sim} \mathrm{R}'^r_{\mathscr{X}}\widehat{f}_*\omega^\#_{\widehat{f}}$ induced by (A.3.1) is the composite of the map (3.3.3), i.e., $\mathrm{R}^r_Z f_*\omega^\#_f \xrightarrow{\sim} \mathrm{R}'^r_{\mathscr{X}}\widehat{f}_*\kappa^*\omega^\#_f$, and the isomorphism induced by (2.1.3), i.e., $\mathrm{R}'^r_{\mathscr{X}}\widehat{f}_*\kappa^*\omega^\#_f \xrightarrow{\sim} \mathrm{R}'^r_{\mathscr{X}}\widehat{f}_*\kappa^\#\omega^\#_f \xrightarrow{\sim} \mathrm{R}'^r_{\mathscr{X}}\widehat{f}_*\omega^\#_{\widehat{f}}$.

In [**S2**, p.742, (3.2)] a different, but equivalent, definition is given of $\mathbf{res}^\#_Z$. In that situation f is *separated,* and therefore has a compactification by a result of Nagata, say $u\colon X \hookrightarrow \bar{X}$ of X over Y. Let $\bar{f}\colon \bar{X} \to Y$ be the structure morphism (by definition of a compactification, a proper map) of $\bar{X}$. In *loc. cit.*, the residue along Z is defined as the composite

$$\begin{aligned}
\mathrm{R}^r_Z f_*\omega^\#_f = \mathrm{H}^0(\mathbf{R}f_*\mathbf{R}\varGamma_Z\omega^\#_f[r]) = \mathrm{H}^0(\mathbf{R}f_*\mathbf{R}\varGamma_Z f^!\mathscr{O}_Y) &\xrightarrow{\quad\sim\quad} \mathrm{H}^0(\mathbf{R}\bar{f}_*\mathbf{R}\varGamma_{u(Z)}\bar{f}^!\mathscr{O}_Y)\\
&\xrightarrow{\qquad\quad} \mathrm{H}^0(\mathbf{R}\bar{f}_*\bar{f}^!\mathscr{O}_Y)\\
&\xrightarrow{\mathrm{H}^0(\mathrm{Tr}_{\bar{f}})} \mathscr{O}_Y.
\end{aligned}$$

By Lemma A.3.5 the two definitions coincide in the situation considered in [**S2**] and therefore the definition in [**S2**, p.742, (3.2)] is independent of the compactification $(u,\,\bar{f})$. This gives another proof of [**S2**, p.742, Proposition 3.1.1].

If f is *proper*, it follows that there is a commutative diagram:

$$\begin{tikzcd}
\mathrm{R}^r_Z f_*\omega^\#_f \arrow[r] \arrow[dr, "\mathbf{res}^\#_Z"'] & \mathrm{R}^r f_*\omega^\#_f \arrow[d, "\mathrm{tr}^\#_f"] \\
& \mathscr{O}_Y
\end{tikzcd} \tag{3.3.4}$$

REMARK 3.3.5. In [**ILN**, p. 746, Remark 2.3.4], Iyengar, Lipman, and Neeman give a generalization of the residue map in [**S2**]. Suppose $f\colon X \to Y$ is a separated essentially finite type map of ordinary schemes, W *a union of closed subsets* of X to each of which the restriction of f is proper. (Note that W need not be closed in X.) Then one has an integer d such that $\mathrm{H}^{-e}(f^!\mathscr{O}_Y) = 0$ for all $e > d$, while $\omega_f := \mathrm{H}^{-d}(f^!\mathscr{O}_Y) \neq 0$. Iyengar, Lipman, and Neeman then define a natural map

$$\mathrm{H}^d\mathbf{R}f_*\mathbf{R}\varGamma_W(\omega_f) \longrightarrow \mathscr{O}_Y \tag{3.3.6}$$

denote by them as $\int_W$, which generalizes the map denoted $\mathrm{res}_{\mathrm{W}}$ in [**S2**, §3.1]. In greater detail, if $\mathbf{D}_{\mathrm{qc}}(X)_W$ denotes the essential image of $\mathbf{R}\varGamma_W$ in $\mathbf{D}_{\mathrm{qc}}(X)$, then in [**ILN**, p. 746, Corollary 2.3.3] it is shown that for E in $\mathbf{D}_{\mathrm{qc}}(X)_W$ and G in $\mathbf{D}^+_{\mathrm{qc}}(Y)$, we have a functorial isomorphism

$$\mathrm{Hom}_{\mathbf{D}(Y)}(\mathbf{R}f_*E,\, G) \xrightarrow{\sim} \mathrm{Hom}_{\mathbf{D}(X)}(E,\, \mathbf{R}\varGamma_W f^!G).$$

In particular one has a counit

$$\mathbf{R}f_*\mathbf{R}\varGamma_W f^!\mathscr{O}_Y \longrightarrow \mathscr{O}_Y. \tag{3.3.7}$$

The map (3.3.6) is defined as the composite

$$\begin{aligned}\mathrm{H}^d\mathbf{R}f_*\mathbf{R}\varGamma_W(\omega_f) &= \mathrm{H}^0\mathbf{R}f_*\mathbf{R}\varGamma_W(\omega_f[d]) \\ &\longrightarrow \mathrm{H}^0\mathbf{R}f_*\mathbf{R}\varGamma_W(f^!\mathscr{O}_Y) \xrightarrow{\mathrm{H}^0(3.3.7)} \mathrm{H}^0\mathscr{O}_Y = \mathscr{O}_Y.\end{aligned}$$

3.4. Traces for finite Cohen-Macaulay maps.

We begin with a global construction. Suppose we have a commutative diagram of ordinary schemes

$$\begin{tikzcd} Z \arrow[r, hook, "i"] \arrow[dr, "h"'] & X \arrow[d, "f"] \\ & Y \end{tikzcd} \tag{3.4.1}$$

with f Cohen-Macaulay of relative dimension r, h a finite surjective map, i a closed immersion, the $\mathscr{O}_X$-ideal $\mathscr{I}$ of Z generated by $t_1, \dots, t_r \in \Gamma(X, \mathscr{O}_X)$ such that $\mathbf{t} = (t_1, \dots, t_r)$ is $\mathscr{O}_{X,z}$-regular for every $z \in Z \subset X$. Note that h is necessarily flat and is Cohen-Macaulay of relative dimension 0. Define

$$\boldsymbol{\tau}_h^{\#}(= \boldsymbol{\tau}_{h,f,i}^{\#})\colon h_*(i^*\omega_f^{\#} \otimes_{\mathscr{O}_Z} (\wedge^r_{\mathscr{O}_Z}\mathscr{I}/\mathscr{I}^2)^*) \longrightarrow \mathscr{O}_Y \tag{3.4.2}$$

as the unique map which fills the dotted arrow to make the diagram below commute where η_i' is induced by (C.2.13).

$$\begin{tikzcd} h_*(i^*\omega_f^{\#} \otimes_{\mathscr{O}_Z} (\wedge^r_{\mathscr{O}_Z}\mathscr{I}/\mathscr{I}^2)^*) \arrow[r, "\sim", "\eta_i'"'] \arrow[d, dotted, "\boldsymbol{\tau}_h^{\#}"'] & h_*\mathrm{H}^0(i^!f^!\mathscr{O}_Y) \arrow[r, "\sim"] & h_*\mathrm{H}^0(h^!\mathscr{O}_Y) \arrow[d, equal] \\ \mathscr{O}_Y & & \mathrm{H}^0(h_*h^!\mathscr{O}_Y) \arrow[ll, "\mathrm{tr}_h^{\#}"] \end{tikzcd}$$

We would like to show that $\boldsymbol{\tau}_h^{\#}$ factors through $\mathbf{res}_Z^{\#}\colon \mathrm{R}^r_Z f_*\omega_f^{\#} \to \mathscr{O}_Y$. To that end, we make the following definition. First, as in (C.2.10), let $i^{\blacktriangle} := \mathbf{L}i^*(-) \overset{\mathbf{L}}{\otimes}_{\mathscr{O}_Z} (\mathscr{N}_i^{\,r}[-r])$. Next, for a quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$, let

$$\psi = \psi(\mathscr{F})\colon h_*(i^*\mathscr{F} \otimes_{\mathscr{O}_Z} (\wedge^r_{\mathscr{O}_Z}\mathscr{I}/\mathscr{I}^2)^*) \longrightarrow \mathrm{R}^r_Z f_*\mathscr{F} \tag{3.4.3}$$

be defined by applying H^0 to the composite

$$h_*i^{\blacktriangle}\mathscr{F}[r] \xrightarrow{\sim} \mathbf{R}f_*i_*i^{\blacktriangle}\mathscr{F}[r] \xrightarrow[\text{(C.2.11)}]{\sim} \mathbf{R}f_*i_*i^{\flat}\mathscr{F}[r] \longrightarrow \mathbf{R}f_*\mathbf{R}\varGamma_Z\mathscr{F}[r]. \tag{3.4.4}$$

The map (3.4.3) is a sheafied version of (C.5.4.1), as (C.5.4.2) shows. Moreover it is functorial in $\mathscr{F}$.

The formal-scheme version is as follows. Let $\kappa\colon \mathscr{X} \to X$ be the completion of X along Z, $j\colon Z \to \mathscr{X}$ the natural closed immersion (so that $\kappa \circ j = i$) and $\widehat{f} = f \circ \kappa$. As before, let $j^{\blacktriangle}$ be as in (C.2.10). For $\mathscr{G} \in \mathcal{A}_{\vec{\mathrm{c}}}(\mathscr{X})$ we have a map

$$\widehat{\psi}\colon h_*(j^*\mathscr{G} \otimes_{\mathscr{O}_Z} \mathscr{N}_j^{\,r}) \longrightarrow \mathrm{R}'^{\,r}_{\mathscr{X}}\widehat{f}_*\mathscr{G} \tag{3.4.5}$$

defined by applying H^0 to the composite

$$h_* j^{\blacktriangle}\mathscr{G}[r] \xrightarrow{\sim} \mathbf{R}\widehat{f}_* j_* j^{\blacktriangle}\mathscr{G}[r] \xrightarrow{\sim} \mathbf{R}\widehat{f}_* j_* j^{\#}\mathscr{G}[r] \xrightarrow{\mathrm{Tr}_j} \mathbf{R}\widehat{f}_*\mathbf{R}\varGamma'_{\mathscr{X}}\mathscr{G}[r]. \tag{3.4.6}$$

Since the composite $i_*i^\flat \to \mathbf{R}\varGamma_Z \to \mathbf{1}$ is "evaluation at one", i.e., it is the trace map (if one identifies $i^\flat$ with $i^!$), it is easy to see that the diagram

$$\begin{tikzcd}
h_* i^{\blacktriangle}\mathscr{F}[r] \arrow[rr, "(3.4.4)"] \arrow[d, equal] & & \mathbf{R}f_*\mathbf{R}\varGamma_Z\mathscr{F}[r] \\
h_* j^{\blacktriangle}\kappa^*\mathscr{F}[r] \arrow[r, "(3.4.6)"] & \mathbf{R}\widehat{f}_*\mathbf{R}\varGamma'_{\mathscr{X}}\kappa^*\mathscr{F}[r] \arrow[r, "\sim"] & \mathbf{R}f_*\kappa_*\mathbf{R}\varGamma'_{\mathscr{X}}\kappa^*\mathscr{F}[r] \arrow[u, "\wr"']
\end{tikzcd} \tag{3.4.7}$$

commutes where the upward arrow on the right is induced by the isomorphism $\kappa_*\mathbf{R}\varGamma'_{\mathscr{X}}\kappa^* \xrightarrow{\sim} \mathbf{R}\varGamma_Z$.

THEOREM 3.4.8. *In the situation of* (3.4.1)*, the following diagram commutes.*

$$\begin{tikzcd}
h_*(i^*\omega_f^{\#} \otimes_{\mathscr{O}_Z} (\wedge^r_{\mathscr{O}_Z}\mathscr{I}/\mathscr{I}^2)^*) \arrow[r, "\boldsymbol{\tau}_h^{\#}"] \arrow[d, "\psi(\omega_f^{\#})"'] & \mathscr{O}_Y \arrow[d, equal] \\
\mathrm{R}^r_Z f_*\omega_f^{\#} \arrow[r, "\mathbf{res}_Z^{\#}"] & \mathscr{O}_Y
\end{tikzcd}$$

PROOF. The diagram in the statement of the theorem can be realized as the transpose of the border of the following one.

$$\begin{tikzcd}
h_*(i^*\omega_f^{\#} \otimes \mathscr{N}_i^r) \arrow[rrr, "\psi"] \arrow[dd] \arrow[dr] & & & \mathrm{R}^r_Z f_*\omega_f^{\#} \arrow[dl] \arrow[dd, "\mathbf{res}_Z^{\#}"] \\
& h_*(j^*\omega_{\widehat{f}}^{\#} \otimes \mathscr{N}_j^r) \arrow[r, "\widehat{\psi}"] \arrow[dl] & \mathrm{R}'^r_{\mathscr{X}}\widehat{f}_*\omega_{\widehat{f}}^{\#} \arrow[dr, "\mathrm{tr}_{\widehat{f}}^{\#}"] & \\
h_*\omega_h^{\#} \arrow[rrr, "\mathrm{tr}_h^{\#}"'] & & & \mathscr{O}_Y
\end{tikzcd}$$

We have to show the above diagram commutes. Applying H^0 to (3.4.7), with $\mathscr{F} = \omega_f^{\#}$, and using the isomorphism $\kappa^*\omega_f^{\#} \xrightarrow{\sim} \omega_{\widehat{f}}^{\#}$, we see that the upper trapezium commutes. The triangle on the right commutes by definition of $\mathbf{res}_Z^{\#}$ (see Definition 3.3.1), while the one on the left corresponds to the natural isomorphisms $i^{\#}f^{\#} \xrightarrow{\sim} j^{\#}\widehat{f}^{\#} \xrightarrow{\sim} h^{\#}$ (after applying H^0 and h_*). Finally, the lower trapezium corresponds to H^0 of the outer border of the following diagram.

$$\begin{tikzcd}
h_* j^{\blacktriangle}\widehat{f}^{\#}\mathscr{O}_Y \arrow[r, "\sim"] \arrow[d, "\wr"', "(\mathrm{C}.2.13)"] & \widehat{f}_* j_* j^{\blacktriangle}\widehat{f}^{\#}\mathscr{O}_Y \arrow[d, "\wr"', "(\mathrm{C}.2.13)"] & \\
h_* j^{\#}\widehat{f}^{\#}\mathscr{O}_Y \arrow[r, "\sim"] \arrow[d, "\wr"'] & \widehat{f}_* j_* j^{\#}\widehat{f}^{\#}\mathscr{O}_Y \arrow[r, "\mathrm{Tr}_j"'] & \widehat{f}_*\mathbf{R}\varGamma'_{\mathscr{X}}\widehat{f}^{\#}\mathscr{O}_Y \arrow[d, "\mathrm{Tr}_{\widehat{f}}"] \\
h_* h^{\#}\mathscr{O}_Y \arrow[rr, "\mathrm{Tr}_h"'] & & \mathscr{O}_Y
\end{tikzcd}$$

The upper rectangle commutes trivially while the lower one results from the identification of the adjoint $h^\#$ with the composition of the adjoints $j^\# \widehat{f}^\#$. $\square$

REMARK 3.4.9. The map ψ in (3.4.3) is compatible with open immersions in X containing Z. In greater detail, suppose i factors as

$$Z \xrightarrow{u} U \xrightarrow{x} X$$

with $x\colon U \to X$ an open immersion, and u (necessarily) a closed immersion. Then

$$\begin{tikzcd} h_* i^* \mathscr{F} \otimes \mathscr{N}_i^{\,r} \arrow[r, "(3.4.3)"] \arrow[d, equal] & \mathrm{R}^r_Z f_* \mathscr{F} \arrow[d, "\wr"] \\ h_* u^*(x^* \mathscr{F}) \otimes \mathscr{N}_u^{\,r} \arrow[r, "(3.4.3)"'] & \mathrm{R}^r_Z (fx)_* x^* \mathscr{F} \end{tikzcd}$$

commutes. We leave the verification to the reader, but point out that one method is to move to formal schemes, using (3.4.7), noting that the completion of X along Z is the same as the completion of U along Z. This means $\boldsymbol{\tau}_h^\#$ is unaffected if X is replaced by U.

3.5. A residue formula for Cohen-Macaulay maps

Consider again Diagram (3.4.1). Suppose now that X, Y, and Z are affine, say $X = \operatorname{Spec} R$, $Y = \operatorname{Spec} A$ and $Z = \operatorname{Spec} B$. In other words $A \to R$ is a finite-type map of rings which is Cohen-Macaulay of relative dimension r, we have an ideal I in R generated by a quasi-regular sequence $\mathbf{t} = (t_1, \ldots, t_r)$ in R, and $B = R/I$. Assume as before that h is a finite (and hence flat) surjective map.

Let us write $\omega^\#_{R/A} = \Gamma(X, \omega^\#_f)$, $\omega^\#_{B/A} = \Gamma(Z, \omega^\#_h)$, $\mathrm{tr}^\#_{B/A} = \Gamma(Y, \mathrm{tr}^\#_h)$. The global sections of $\boldsymbol{\tau}_h^\#$ give us an A-linear map

(3.5.1) $$\boldsymbol{\tau}^\#_{B/A} (= \boldsymbol{\tau}^\#_{B/A,R})\colon \omega^\#_{R/A} \otimes_R (\wedge^r_B I/I^2)^* \longrightarrow A$$

such that the following diagram commutes

$$\begin{tikzcd} \omega^\#_{R/A} \otimes_R (\wedge^r_B I/I^2)^* \arrow[r, "\sim"] \arrow[d, "\boldsymbol{\tau}^\#_{B/A}"'] & \omega^\#_{B/A} \arrow[d, "\mathrm{tr}^\#_{B/A}"] \\ A \arrow[r, equal] & A \end{tikzcd}$$

where the horizontal isomorphism on the top row is the global sections of the composite

$$h_*(i^*\omega^\#_f \otimes_{\mathscr{O}_Z} (\wedge^r_{\mathscr{O}_Z} \mathscr{I}/\mathscr{I}^2)^*) \underset{\eta'_i}{\xrightarrow{\sim}} h_* \mathrm{H}^0(i^! f^! \mathscr{O}_Y) = \mathrm{H}^0 h_*(i^! f^! \mathscr{O}_Y) \xrightarrow{\sim} \mathrm{H}^0(h_* h^! \mathscr{O}_Y).$$

3.5.2. Notation. In an obvious extension of our notational philosophy, we should use the symbol $\mathbf{res}^\#_I$ for the global sections of the residue map $\mathbf{res}^\#_Z$ in (3.3.2). However, for psychological reasons we will continue to use the symbol $\mathbf{res}^\#_Z$ to denote this map. Thus we have

$$\mathbf{res}^\#_Z \colon \mathrm{H}^r_I(\omega^\#_{R/A}) \to A.$$

In what follows, elements of $\mathrm{H}^r_I(\omega^\#_{R/A})$ are denoted by generalized fractions

$$\begin{bmatrix} \nu \\ t_1, \ldots, t_r \end{bmatrix}$$

as in §C.5 (see especially (C.5.2) and (C.5.3) and the discussions around them).

Finally, define

$$\frac{\mathbf{1}}{\mathbf{t}} \in (\wedge^r_B I/I^2)^* \tag{3.5.3}$$

as the element which sends $(t_1 + I^2) \wedge \cdots \wedge (t_r + I^2) \in \wedge^r_B I/I^2$ to 1.

PROPOSITION 3.5.4. *With the above notations, for any* $\nu \in \omega^{\#}_{R/A}$ *we have*

$$\mathbf{res}^{\#}_{Z} \begin{bmatrix} \nu \\ t_1, \ldots, t_r \end{bmatrix} = \boldsymbol{\tau}^{\#}_{B/A}\Big(\nu \otimes \frac{\mathbf{1}}{\mathbf{t}}\Big)$$

where $\frac{\mathbf{1}}{\mathbf{t}}$ *is as in* (3.5.3).

PROOF. According to Theorem 3.4.8, the following diagram commutes.

$$\begin{tikzcd} \omega^{\#}_{R/A} \otimes_R (\wedge^r_B I/I^2)^* \arrow[rr, "(C.5.4.1)"] \arrow[dr, "\boldsymbol{\tau}^{\#}_{B/A}"'] & & \mathrm{H}^r_I(\omega^{\#}_{R/A}) \arrow[dl, "\mathbf{res}^{\#}_Z"] \\ & A & \end{tikzcd}$$

The Proposition then follows from Lemma C.5.4. □

THEOREM 3.5.5. *Suppose* J *is another ideal in* R *such that* $I \subset J$ *and* J *is generated by a quasi-regular sequence* $\mathbf{g} = (g_1, \ldots, g_r)$. *Let* $t_i = \sum_j u_{ij} g_j$, $u_{ij} \in R$. *Let* $W = \operatorname{Spec} R/J$. *Then, for any* $\nu \in \omega^{\#}_{R/A}$

$$\mathbf{res}^{\#}_{Z} \begin{bmatrix} \det(u_{ij})\nu \\ t_1, \ldots, t_r \end{bmatrix} = \mathbf{res}^{\#}_{W} \begin{bmatrix} \nu \\ g_1, \ldots, g_r \end{bmatrix}$$

PROOF. This is an immediate consequence of Theorem C.7.2 and Proposition 3.5.4. □

CHAPTER 4

Base change for residues

4.1. Hypotheses

Throughout this chapter, we fix a commutative diagram of ordinary schemes

$$
\begin{tikzcd}
Z' \arrow[r, "w"] \arrow[d, "j"'] \arrow[dd, bend right=60, "h'"'] \arrow[dr, phantom, "\square"] & Z \arrow[d, "i"] \arrow[dd, bend left=60, "h"] \\
X' \arrow[r, "v"] \arrow[d, "g"'] \arrow[dr, phantom, "\square"] & X \arrow[d, "f"] \\
Y' \arrow[r, "u"'] & Y
\end{tikzcd}
\tag{4.1.1}
$$

with f separated Cohen-Macaulay of relative dimension r, the rectangles cartesian, $i\colon Z \to X$ a closed immersion such that $h = f \circ i\colon Z \to Y$ is finite and the quasi-coherent ideal sheaf $\mathscr{I}$ of Z is generated by global sections $t_1, \dots, t_r \in \Gamma(X, \mathscr{O}_X)$ with the property that $\mathbf{t} = (t_1, \dots, t_r)$ is $\mathscr{O}_{X,z}$-regular for every $z \in Z \subset X$. (Note that $Z \to Y$ is flat by [**EGA**, 0_{IV}, 15.1.16].) We also use the following additional notations: $\mathscr{J} = v^*\mathscr{I}$ is the ideal sheaf of Z', $\mathscr{N} = (\wedge^r_{\mathscr{O}_Z}\mathscr{I}/\mathscr{I}^2)^*$, and $\mathscr{N}' = (\wedge^r_{\mathscr{O}_{Z'}}\mathscr{J}/\mathscr{J}^2)^* = w^*\mathscr{N}$.

4.2. Base change for direct image with supports

Since f is *Cohen-Macaulay of relative dimension* r, therefore, according to [**S2**, p. 740, Theorem 2.3.5 (a)], we have a base-change isomorphism

$$\theta^f_u\colon v^*\omega^\#_f \xrightarrow{\ \sim\ } \omega^\#_g.$$

The principal aim of this section is to show that the composite

$$u^*h_*(i^*\omega^\#_f \otimes_{\mathscr{O}_Z} \mathscr{N}) \xrightarrow{\ \sim\ } h'_*(j^*v^*\omega^\#_f \otimes_{\mathscr{O}_{Z'}} \mathscr{N}') \xrightarrow[\theta^f_u]{\ \sim\ } h'_*(j^*\omega^\#_g \otimes_{\mathscr{O}_{Z'}} \mathscr{N}') \xrightarrow{\boldsymbol{\tau}^\#_{h'}} \mathscr{O}_{Y'}$$

is $u^*\boldsymbol{\tau}^\#_h$, i.e., speaking informally, $\boldsymbol{\tau}^\#_h$ is stable under base change (here the first isomorphism results from the fact that h is an affine map). We would also like to show that the result in Theorem 3.4.8 is stable under base change. Indeed, that is how we will prove that $\boldsymbol{\tau}^\#_h$ is stable under base change. To set things up, we now discuss, very briefly, base change for cohomology with supports, at least for the situation we are in.

In our situation, we have base-change maps (see, for example, [**S2**, p. 768, (A.5)]), one for each k

$$b(u, f) = b(u, f, k)\colon u^*\mathrm{R}^k_Z f_* \longrightarrow \mathrm{R}^k_{Z'} g_* v^*. \tag{4.2.1}$$

These are natural transformation of functors on quasi-coherent sheaves on X. In the event u is *flat*, $b(u,f)$ is an isomorphism. In fact, in this case, $b(u,f,k)$ is $H^k(-)$ applied to the composite of natural isomorphisms

$$u^*\mathbf{R}f_*\circ\mathbf{R}\varGamma_Z \xrightarrow{\ \sim\ } \mathbf{R}g_*v^*\circ\mathbf{R}\varGamma_Z \xrightarrow{\ \sim\ } \mathbf{R}g_*\circ\mathbf{R}\varGamma_{Z'}v^*.$$

It is useful for us to recast $b(u,f)$ in terms of the formal completions of X and X'. To that end, let $\kappa\colon \mathscr{X}\to X$ (resp. $\kappa'\colon \mathscr{X}'\to X'$) be the completion of X along Z (resp. of X' along Z') and let $\alpha\colon Z\to\mathscr{X}$, $\beta\colon Z'\to\mathscr{X}'$ be the natural closed immersions, so that $i=\kappa\circ\alpha$ and $j=\kappa'\circ\beta$. Let $\widehat{f}=f\circ\kappa$, $\widehat{g}=g\circ\kappa'$, and finally let $\widehat{v}\colon\mathscr{X}'\to\mathscr{X}$ be the natural map induced by v, so that the following diagram is cartesian:

(4.2.2)
$$\begin{tikzcd} \mathscr{X}' \arrow[r, "\widehat{v}"] \arrow[d, "\widehat{g}"'] \arrow[dr, phantom, "\square"] & \mathscr{X} \arrow[d, "\widehat{f}"] \\ Y' \arrow[r, "u"'] & Y \end{tikzcd}$$

The maps $b(u,f,k)$ in (4.2.1) give rise, in a natural way, maps

(4.2.3)
$$b(u,\widehat{f})=b(u,\widehat{f},k)\colon u^*\mathrm{R}'^{k}_{\mathscr{X}}\widehat{f}_* \longrightarrow \mathrm{R}'^{k}_{\mathscr{X}'}\widehat{g}_*\widehat{v}^*$$

induced by (A.3.1) applied to κ and to κ'. In the event u is flat, then as in the case of ordinary schemes, $b(u,\widehat{f},k)$ is an isomorphism and is, in fact, $H^k(-)$ applied to the natural composite

$$u^*\mathbf{R}\widehat{f}\circ\mathbf{R}\varGamma'_{\mathscr{X}} \xrightarrow{\ \sim\ } \mathbf{R}\widehat{g}\,\widehat{v}^*\circ\mathbf{R}\varGamma'_{\mathscr{X}} \xrightarrow{\ \sim\ } \mathbf{R}\widehat{g}\circ\mathbf{R}\varGamma'_{\mathscr{X}'}\widehat{v}^*.$$

We will in fact show that when $k=r$, the map $b(u,f,k)$ is an isomorphism *even when u is not flat.*

Proposition 4.2.4. *Suppose u is a flat map and $\mathscr{F}\in\mathbf{D}_{\mathrm{qc}}(X)$. Then the following diagram commutes, where the unlabelled arrows arise from the natural maps ("evaluation at 1") $i_*i^\flat\to\mathbf{R}\varGamma_Z$ and $j_*j^\flat\to\mathbf{R}\varGamma_{Z'}$:*

$$\begin{tikzcd} j_*j^{\blacktriangle}v^*\mathscr{F} \arrow[rr, "\sim", "(\mathrm{C.2.11})"'] \arrow[d, equal] & & j_*j^\flat v^*\mathscr{F} \arrow[r] & \mathbf{R}\varGamma_{Z'}v^*\mathscr{F} \\ j_*w^*i^{\blacktriangle}\mathscr{F} \arrow[r, equal] & v^*i_*i^{\blacktriangle}\mathscr{F} \arrow[r, "\sim", "(\mathrm{C.2.11})"'] & v^*i_*i^\flat\mathscr{F} \arrow[r] & v^*\mathbf{R}\varGamma_Z\mathscr{F} \arrow[u, "\wr"'] \end{tikzcd}$$

Remark: The maps (C.2.11) make sense for $\mathscr{F}\in\mathbf{D}_{\mathrm{qc}}(X)$ because X is an ordinary scheme (see discussion in §§C.2.15). Flat base change works in this case for all $\mathscr{F}\in\mathbf{D}_{\mathrm{qc}}(X)$ without boundedness hypotheses because i_* takes perfect complexes to perfect complexes (see [**L4**, p. 197, Thm. 4.7.4]).

Proof. Let $\mathrm{Tr}^\flat_i\colon i_*i^\flat\to\mathbf{1}$ be as in (B.1.1), i.e., $\mathrm{Tr}^\flat_i$ is the composite $i_*i^\flat\to\mathbf{R}\varGamma_Z\to\mathbf{1}$ (see the discussion in Subsection B.1 with $\mathscr{X}=Z$ and $\mathscr{X}=X$). Then $\mathrm{Tr}^\flat_i$ is simply evaluation at 1, and hence equals the composite

$$i_*i^\flat \xrightarrow{\ \sim\ } i_*i^! \xrightarrow{\mathrm{Tr}_i} \mathbf{1}.$$

The two maps, $i_*i^\flat \to \mathbf{R}\varGamma_Z$ and $\mathrm{Tr}_i^\flat$, determine each other and hence we have show that the diagram

$$(\dagger)\qquad \begin{tikzcd} j_*j^{\blacktriangle}v^* \arrow[r, "\sim", "\text{(C.2.11)}"'] \arrow[d, equal] & j_*j^\flat v^* \arrow[r, "\mathrm{Tr}_j^\flat"] & v^* \arrow[d, equal] \\ v^*i_*i^{\blacktriangle} \arrow[r, "\sim", "\text{(C.2.11)}"'] & v^*i_*i^\flat \arrow[r, "v^*(\mathrm{Tr}_i^\flat)"] & v^* \end{tikzcd}$$

commutes.

The composite $i_*i^{\blacktriangle} \xrightarrow{\text{(C.2.11)}} i_*i^\flat \xrightarrow{\mathrm{Tr}_i^\flat} \mathbf{1}$ is clearly the same as the composite $i_*i^{\blacktriangle} \xrightarrow{\text{(C.2.13)}} i_*i^! \xrightarrow{\mathrm{Tr}_i} \mathbf{1}$. We will denote the common value by

$$\mathrm{Tr}_i^{\blacktriangle}\colon i_*i^{\blacktriangle} \longrightarrow \mathbf{1}.$$

We have to show that

$$v^* \circ \mathrm{Tr}_i^{\blacktriangle} = \mathrm{Tr}_j^{\blacktriangle} \circ v^*.$$

The question in local on X and X' and hence we assume that $X = \operatorname{Spec} R$, $Z = \operatorname{Spec} A$, $X' = \operatorname{Spec} R'$, $Z' = \operatorname{Spec} A'$ where $A' = A \otimes_R R'$. We write I for the ideal in R generated by $\mathbf{t}$, J for its extension to R', N for the A-module $(\wedge_A^r I/I^2)^*$, and N' for $N \otimes_A A' = (\wedge_{A'}^r J/J^2)^*$. Finally, let $\mathrm{Tr}_{A/R}^{\blacktriangle}$ and $\mathrm{Tr}_{A'/R'}^{\blacktriangle}$ be the maps in $\mathbf{D}(\mathrm{Mod}_R)$ and $\mathbf{D}(\mathrm{Mod}_{R'})$ whose "sheafified" versions are $\mathrm{Tr}_i^{\blacktriangle}$ and $\mathrm{Tr}_j^{\blacktriangle}$ respectively. The discussions in Remark C.2.14 and in §§C.2.15 apply. In particular, from the commutative diagram (C.2.14.1), we only have to show:

$$(*)\qquad \mathrm{Tr}_{A/R}^{\blacktriangle}(R) \otimes_R R' = \mathrm{Tr}_{A'/R'}^{\blacktriangle}(R').$$

This follows from the explicit description of $\mathrm{Tr}_{A/R}(R)$ in (C.2.15.1), for the maps $\varphi_{\mathbf{t}}$ and $\pi_{\mathbf{t}}$ occuring in *loc.cit.* are compatible with base change. In greater detail, if t_i' are the images in R' of t_i and $\mathbf{t}' = (t_1', \dots, t_r')$, then $\varphi_{\mathbf{t}} \otimes_R R' = \varphi_{\mathbf{t}'}$ and $\pi_{\mathbf{t}} \otimes_R R' = \pi_{\mathbf{t}'}$. Since $\mathrm{Tr}_{A/R}(R) = \pi_{\mathbf{t}} \circ \varphi_{\mathbf{t}}^{-1}$ and $\mathrm{Tr}_{A'/R'}(R') = \pi_{\mathbf{t}'} \circ \varphi_{\mathbf{t}'}^{-1}$, the relation asserted in $(*)$ is true. $\square$

Remark 4.2.5. Formula $(*)$ in the above proof is true in greater generality. Suppose $Z = \operatorname{Spec} A$, $X = \operatorname{Spec} R$, and we have a regular immersion $i\colon Z \hookrightarrow X$ given by an R-sequence $\mathbf{t} = (t_1, \dots, t_r)$. Then $(*)$ remains valid without the assumption that X be Cohen-Macaulay over another scheme. In fact, by checking locally one easily deduces that if $i\colon Z \hookrightarrow X$ is a regular immersion (not necessarily of affine schemes, and not necessarily given globally by the vanishing of a sequence of the form $\mathbf{t}$) and we have a cartesian diagram

$$\begin{tikzcd} Z' \arrow[r, "w"] \arrow[d, "j"'] \arrow[dr, phantom, "\square"] & Z \arrow[d, "i"] \\ X' \arrow[r, "v"'] & X \end{tikzcd}$$

with u flat and if $\Theta\colon w^*i^!\mathscr{O}_X \xrightarrow{\sim} j^!\mathscr{O}_{X'}$ is the flat base change isomorphism, then the following diagram commutes;

$$\begin{tikzcd}
w^*\mathscr{N}_i^r[-r] \arrow[r, equal] \arrow[d, "\eta'_j"'] & \mathscr{N}_j^r[-r] \arrow[d, "\eta'_i"] \\
w^*i^!\mathscr{O}_X \arrow[r, "\sim", "\Theta"'] & j^!\mathscr{O}_{X'}
\end{tikzcd}$$

The flatness hypothesis on v can be relaxed, since $(*)$ works even when R' is not flat over R, but for now, we leave matters as they are.

4.3. Base-change theorems

We now prove that $\boldsymbol{\tau}_h^\#$ is stable under arbitrary base change. We embed that result in a larger set of base-change results, namely in Theorem 4.3.1.

THEOREM 4.3.1. *With the hypotheses as in §4.1 we have:*

(a) *For a coherent $\mathscr{O}_X$-module $\mathscr{F}$, the diagram*

$$\begin{tikzcd}
h'_*(j^*v^*\mathscr{F}\otimes_{\mathscr{O}_{Z'}}\mathscr{N}') \arrow[r, equal] \arrow[d, "(3.4.3)"'] & u^*h_*(i^*\mathscr{F}\otimes_{\mathscr{O}_Z}\mathscr{N}) \arrow[d, "u^*(3.4.3)"] \\
\mathrm{R}^r_{Z'}v^*\mathscr{F} & u^*\mathrm{R}^r_Z\mathscr{F} \arrow[l, "(4.2.1)"]
\end{tikzcd}$$

commutes.

(b) *The map*

$$b(u,f)\colon u^*\mathrm{R}^r_Z f_* \longrightarrow \mathrm{R}^r_{Z'}g_*v^*$$

is an isomorphism.

(c) *The diagram*

$$\begin{tikzcd}
h'_*(j^*(\omega_g^\#)\otimes_{\mathscr{O}_{Z'}}\mathscr{N}') \arrow[dd, "\boldsymbol{\tau}_{h'}^\#"'] & h'_*(j^*v^*(\omega_f^\#)\otimes_{\mathscr{O}_{Z'}}w^*\mathscr{N}) \arrow[l, "\sim"', "h'_*(\theta_u^f\otimes\mathbf{1})"] \arrow[d, equal] \\
 & h'_*w^*(i^*(\omega_f^\#)\otimes_{\mathscr{O}_{Z'}}w^*\mathscr{N}) \arrow[d, equal] \\
\mathscr{O}_{Y'} & u^*h_*(i^*(\omega_f^\#)\otimes_{\mathscr{O}_Z}\mathscr{N}) \arrow[l, "u^*(\boldsymbol{\tau}_h^\#)"]
\end{tikzcd}$$

commutes, where $\theta_u^f\colon v^*\omega_f^\# \xrightarrow{\sim} \omega_g^\#$ is the base-change isomorphism of [**S2**, p. 740, Theorem 2.3.5 (a)].

(d) *The diagram*

$$\begin{tikzcd}
u^*\mathrm{R}^r_Z f_*\omega_f^\# \arrow[r, "\sim", "b(u,f)"'] \arrow[d, "u^*(\mathbf{res}_Z^\#)"'] & \mathrm{R}^r_{Z'}g_*v^*\omega_f^\# \arrow[d, "\mathrm{R}^r_{Z'}g_*(\theta_u^f)"] \\
\mathscr{O}_{Y'} & \mathrm{R}^r_{Z'}g_*\omega_g^\# \arrow[l, "\mathbf{res}_{Z'}^\#"]
\end{tikzcd}$$

commutes.

PROOF. From the definitions, we may, without loss of generality, assume that $Y = \operatorname{Spec} A$ and $Y' = \operatorname{Spec} A'$. Consider the composite natural transform of functors of quasi-coherent $\mathscr{O}_X$-modules:

$$\operatorname{Ext}^r_A(\mathscr{O}_Z,\, -) \longrightarrow \operatorname{Ext}^r_A(\mathscr{O}_Z,\, v_*v^*(-)) \longrightarrow \operatorname{Ext}^r_{A'}(\mathscr{O}_{Z'},\, v^*(-))$$

giving a base change map

$$A' \otimes_A \operatorname{Ext}^r_A(\mathscr{O}_Z,\, -) \longrightarrow \operatorname{Ext}^r_{A'}(\mathscr{O}_{Z'},\, v^*(-)) \tag{4.3.1.1}$$

Fron the definition of (4.2.1) it is easy to see that

$$\begin{tikzcd}
A' \otimes_A \operatorname{Ext}^r_A(\mathscr{O}_Z,\, -) \arrow[r, "(4.3.1.1)"] \arrow[d] & \operatorname{Ext}^r_{A'}(\mathscr{O}_{Z'},\, v^*(-)) \arrow[d] \\
A' \otimes_A \mathrm{H}^r_Z(X,\, -) \arrow[r, "(4.2.1)"'] & \mathrm{H}^r_{Z'}(X',\, v^*(-))
\end{tikzcd} \tag{4.3.1.2}$$

commutes.

Let $\mathscr{E}xt^i_f(\mathscr{O}_Z,\, -)$ be the i^{th} right derived functor of $f_*\mathscr{H}om_X(\mathscr{O}_Z,\, -)$. Since Z is affine and $\mathscr{E}xt^i_f(\mathscr{O}_Z,\, -)$ is supported on Z, this is simply $h_*\operatorname{Ext}^i_X(\mathscr{O}_Z,\, -)$. Similarly, one defines $\mathscr{E}xt^i_g(\mathscr{O}_{Z'}, -)$. Using this, and computing $\mathscr{E}xt^r_X(\mathscr{O}_Z, -)$ and $\mathscr{E}xt^r_{X'}(\mathscr{O}_{Z'}, -)$ via the Koszul resolutions on $\mathbf{t}$ of $\mathscr{O}_Z$ and $\mathscr{O}_{Z'}$, we get see easily that the fundamental local isomorphisms (C.2.7) is compatible with (4.3.1.1). In other words, the following diagram of functors of coherent $\mathscr{O}_X$-modules commutes:

$$\begin{tikzcd}
h'_*(j^*v^*(-) \otimes_{\mathscr{O}_{Z'}} \mathscr{N}') \arrow[r, equal] \arrow[d, "(C.2.7)"'] & u^*h_*(i^*(-) \otimes_{\mathscr{O}_Z} \mathscr{N}) \arrow[d, "(C.2.7)"] \\
\mathscr{E}xt^r_g(\mathscr{O}_{Z'},\, v^*(-)) & u^*\, \mathscr{E}xt^r_f(\mathscr{O}_Z,\, -) \arrow[l, "(4.3.1.1)"]
\end{tikzcd} \tag{4.3.1.3}$$

This together with (4.3.1.2) gives part (a). In particular, applied to coherent $\mathscr{O}_X$-modules, (4.3.1.1) is an isomorphism.

Applying the fact that (4.3.1.1) is an isomorphism to the closed schemes Z_n of X defined by $t_1^n, \dots, t_r^n$, and taking the direct limit as $n \to \infty$ we get (b) from (4.3.1.2).

According to [**S2**, pp. 755–756, Prop. 6.2.2 (b) and (c)], part (d) is true when either u is flat or when $Z \hookrightarrow X$ is a *good immersion* for f, i.e., it satisfies:

- There is an affine open covering $\mathscr{U} = \{U_\alpha = \operatorname{Spec} A_\alpha\}$ of Y, and for each index α an affine open scheme $V_\alpha = \operatorname{Spec} R_\alpha$ of $f^{-1}(U_\alpha)$ such that $Z \cap f^{-1}(U_\alpha)$ is contained in V_α.
- The closed immersion i is given in V_α by a quasi-regular R_α-sequence.
- $Z \to Y$ is finite.

(See also [**S2**, p. 744, Def. 3.1.4] and [**HK1**, pp. 77–78, Assumptions 4.3].)

Let $\mathfrak{p}$ be a prime ideal of A, y the point in Y corresponding to $\mathfrak{p}$, $\widetilde{Y}$ the completion of the local ring $A_{\mathfrak{p}}$, and $\widetilde{Y}'$ the completion of $A'_{\mathfrak{p}}$ with respect to the ideal $\mathfrak{p}A'_{\mathfrak{p}}$. We then have a commutative diagram

$$\begin{tikzcd}
& X' \arrow[rr, "v"] \arrow[dd, "g"] & & X \arrow[dd, "f"] \\
\widetilde{X}' \arrow[ur, "\vartheta"] \arrow[rr, "\widetilde{v}"] \arrow[dd, "\widetilde{g}"'] & & \widetilde{X} \arrow[ur, "t"] \arrow[dd, "\widetilde{f}"] & \\
& Y' \arrow[rr, "u"] & & Y \\
\widetilde{Y}' \arrow[ur, "\sigma"] \arrow[rr, "\widetilde{u}"'] & & \widetilde{Y} \arrow[ur, "s"'] &
\end{tikzcd} \tag{4.3.1.4}$$

All the lateral faces are cartesian, however the top and bottom faces need not be. We set $\widetilde{Z} = t^{-1}(Z)$ and $\widetilde{Z}' = \vartheta^{-1}(Z')$.

One checks easily that

$$b(\sigma, g) \circ \sigma^* b(u, f) = b(u \circ \sigma, f) = b(\widetilde{u}, \widetilde{f}) \circ \widetilde{u}^* b(s, f) \tag{$*$}$$

and according to [**S2**, p. 747, Remark 3.3.2], we have

$$\theta^{\widetilde{f}}_{\widetilde{u}} \circ \widetilde{v}^* \theta^f_s = \theta^f_{u\sigma} = \theta^g_\sigma \circ \vartheta^* \theta^f_u. \tag{$\dagger$}$$

We remark that Cohen-Macaulay maps of relative dimension r are, in the terminology of *ibid*, locally r-compactifiable.

From our observations about the compatibility of residues with certain base changes, (d) is true for the left, right and front faces of (4.3.1.4). Indeed, s and σ are flat, whereas $\widetilde{Z}$ is a good immersion for $\widetilde{f}$. We therefore have:

$$\begin{aligned}
\mathbf{res}^{\#}_{\widetilde{Z}} \circ \mathrm{R}^r_{\widetilde{Z}} \widetilde{f}_*(\theta^f_s) \circ b(s, f) &= s^*(\mathbf{res}^{\#}_Z) \\
\mathbf{res}^{\#}_{\widetilde{Z}'} \circ \mathrm{R}^r_{\widetilde{Z}'} \widetilde{g}_*(\theta^{\widetilde{f}}_{\widetilde{u}}) \circ b(\widetilde{u}, \widetilde{f}) &= \widetilde{u}^*(\mathbf{res}^{\#}_{\widetilde{Z}}) \\
\mathbf{res}^{\#}_{\widetilde{Z}'} \circ \mathrm{R}^r_{\widetilde{Z}'} \widetilde{g}_*(\theta^g_\sigma) \circ b(\sigma, g) &= \sigma^*(\mathbf{res}^{\#}_{Z'})
\end{aligned} \tag{$\ddagger$}$$

The formulas ($*$), ($\dagger$), and ($\ddagger$) say that the diagram in part (d) of the statement of the theorem commutes after applying σ^*. Now use part (a), which we have proven, to see that the diagram in (c) commutes after applying σ^*. Since all the sheaves involved in the diagram are cohenrent, this means the diagram in (c) commutes in a Zariski open neighbourhood of $u^{-1}(y)$. This proves (c) since $y \in Y$ is arbitrary.

Part (d) now follows by replacing Z by Z_n as before, where Z_n is defined by $t_1^n, \ldots, t_r^n$, applying (c) to Z_n, and taking direct limits. $\square$

CHAPTER 5

Iterated traces

An important formula concerning residues is a Fubini like statement for iterated residues (see [**RD**, p.198, (R4)]). To establish this via our approach to residues, i.e., via Verdier's isomorphism, we have to understand iterated traces (for a composite of pseudo-proper maps) in various ways. That is the main thrust of this section. The circle of ideas is sometimes referred to as "transitivity" (cf. [**LS**]). In somewhat greater detail suppose

$$\mathscr{X} \xrightarrow{f} \mathscr{Y} \xrightarrow{g} \mathscr{Z}$$

is a pair of pseudo-proper maps. Recall that $\mathrm{Tr}_f\colon \mathbf{R}f_*\mathbf{R}\varGamma'_{\mathscr{X}}f^{\#} \to \mathbf{1}_{\mathbf{D}(\mathscr{Y})}$ factors through the natural map $\mathbf{R}\varGamma'_{\mathscr{Y}} \to \mathbf{1}_{\mathbf{D}(\mathscr{Y})}$. Moreover, we abuse notation and write $\mathrm{Tr}_f\colon \mathbf{R}f_*\mathbf{R}\varGamma'_{\mathscr{X}}f^{\#} \to \mathbf{R}\varGamma'_{\mathscr{Y}}$ for the missing factor in the just mentioned factorization of $\mathrm{Tr}_f\colon \mathbf{R}f_*\mathbf{R}\varGamma'_{\mathscr{X}}f^{\#} \to \mathbf{1}_{\mathbf{D}(\mathscr{Y})}$. Given $\mathscr{F},\mathscr{G} \in \widetilde{\mathbf{D}}^{+}_{\mathrm{qc}}(\mathscr{Y})$ the torsion version of the projection isomorphism, which we shall denote as $p^t_f(\mathscr{F},\mathscr{G})$, is the following composition

$$\mathscr{F} \overset{\mathbf{L}}{\otimes}_{\mathscr{O}_{\mathscr{X}}} \mathbf{R}f_*\mathbf{R}\varGamma'_{\mathscr{X}}\mathscr{G} \xrightarrow{\ \sim\ } \mathbf{R}f_*(\mathbf{L}f^*\mathscr{F} \overset{\mathbf{L}}{\otimes}_{\mathscr{O}_{\mathscr{X}}} \mathbf{R}\varGamma'_{\mathscr{X}}\mathscr{G}) \xrightarrow{\ \sim\ } \mathbf{R}f_*\mathbf{R}\varGamma'_{\mathscr{X}}(\mathbf{L}f^*\mathscr{F} \overset{\mathbf{L}}{\otimes}_{\mathscr{O}_{\mathscr{X}}} \mathscr{G})$$

where the first isomorphism is induced by projection. In this situation, we have the following iterated trace on $\mathbf{R}(gf)_*\mathbf{R}\varGamma'_{\mathscr{X}}(\mathbf{L}f^*g^{\#}\mathscr{O}_{\mathscr{Z}} \overset{\mathbf{L}}{\otimes} f^{\#}\mathscr{O}_{\mathscr{Y}})$ where the map labelled p is the natural one induced by $(p^t_f)^{-1}$ while the one labelled T is induced by Tr_f:

$$\begin{aligned}
\mathbf{R}(gf)_*\mathbf{R}\varGamma'_{\mathscr{X}}(\mathbf{L}f^*g^{\#}\mathscr{O}_{\mathscr{Z}} \overset{\mathbf{L}}{\otimes} f^{\#}\mathscr{O}_{\mathscr{Y}}) &\xrightarrow{\ \sim\ } \mathbf{R}g_*\mathbf{R}f_*\mathbf{R}\varGamma'_{\mathscr{X}}(\mathbf{L}f^*g^{\#}\mathscr{O}_{\mathscr{Z}} \overset{\mathbf{L}}{\otimes} f^{\#}\mathscr{O}_{\mathscr{Y}})\\
&\xrightarrow{\ p\ } \mathbf{R}g_*(g^{\#}\mathscr{O}_{\mathscr{Z}} \overset{\mathbf{L}}{\otimes} \mathbf{R}f_*\mathbf{R}\varGamma'_{\mathscr{X}}f^{\#}\mathscr{O}_{\mathscr{Y}})\\
&\xrightarrow{\ T\ } \mathbf{R}g_*(g^{\#}\mathscr{O}_{\mathscr{Z}} \overset{\mathbf{L}}{\otimes} \mathbf{R}\varGamma'_{\mathscr{Y}}\mathscr{O}_{\mathscr{Y}})\\
&\xrightarrow{\ \sim\ } \mathbf{R}g_*\mathbf{R}\varGamma'_{\mathscr{Y}}g^{\#}\mathscr{O}_{\mathscr{Z}}\\
&\xrightarrow{\ \mathrm{Tr}_g\ } \mathscr{O}_{\mathscr{Z}}.
\end{aligned}$$

By adjointness, this gives us a map

$$\chi_{[g,f]}\colon \mathbf{L}f^*(g^{\#}\mathscr{O}_{\mathscr{Z}}) \overset{\mathbf{L}}{\otimes} f^{\#}\mathscr{O}_{\mathscr{Y}} \to (gf)^{\#}\mathscr{O}_{\mathscr{Z}}.$$

In fact one does not need f and g to be pseudo-proper to define $\chi_{[g,f]}$. Our definition below works under the assumption that each of them is a composite of compactifiable maps.

Part of the theme of transitivity is to work out a concrete formula for $\chi_{[g,f]}$ when f and g are smooth, and when $g^{\#}\mathscr{O}_{\mathscr{Z}}$, $f^{\#}\mathscr{O}_{\mathscr{Y}}$, and $(gf)^{\#}\mathscr{O}_{\mathscr{Z}}$ are substituted with suitable differential forms (placed in the appropriate degree) via Verdier's isomorphism [**V**]. That is done in Part 2, based on the work done here.

5.1. Traces in affine terms

If $A \to B$ is a pseudo-finite-type map of adic rings, $I \subset A$ and $J \subset B$ defining ideals for the adic rings A and B respectively, and $f\colon \mathrm{Spf}\,(B,\,J) \to \mathrm{Spf}\,(A,\,I)$ the resulting map of formal schemes, then the complex $f^{\#}\mathscr{O}_{\mathrm{Spf}B}$ can be represented by a bounded-below complex

$$\omega^{\#\bullet}{}_{(B,J)/(A,I)} = \omega^{\#\bullet}{}_{B/A} \in \mathbf{D}^{+}(\mathrm{Mod}_B)$$

which has finitely generated cohomology modules, where the more elaborate notation $\omega^{\#\bullet}{}_{(B,J)/(A,I)}$ is used only when the role of the adic structures needs to be emphasised. To simplify notation further, we shall use $\omega^{\bullet}_{B/A}$ in place of $\omega^{\#\bullet}{}_{B/A}$ from now on.

It then follows that if f is *Cohen-Macaulay* then $\omega^{\bullet}_{B/A} = \omega^{\#}_{B/A}[d]$.

Regarding the affine version of traces there are two related situations which we wish to discuss.

A. Suppose $A \to B/J$ is finite. Recall that the trace map

$$\mathrm{Tr}_f\colon \mathbf{R}\varGamma'_{\mathrm{Spf}(B,J)}f^{\#}\mathscr{O}_{\mathrm{Spf}(A,I)} \to \mathscr{O}_{\mathrm{Spf}(A,I)}[0]$$

factors through the natural map $\mathbf{R}\varGamma'_{\mathrm{Spf}(A,I)}\mathscr{O}_{\mathrm{Spf}(A,I)}[0] \to \mathscr{O}_{\mathrm{Spf}(A,I)}[0]$ and that the map $\mathbf{R}\varGamma'_{\mathrm{Spf}(B,J)}f^{\#}\mathscr{O}_{\mathrm{Spf}(A,I)} \to \mathbf{R}\varGamma'_{\mathrm{Spf}(A,I)}\mathscr{O}_{\mathrm{Spf}(A,I)}[0]$ inducing this trace map is also called the trace map, and is also denoted Tr_f. Corresponding to these maps Tr_f we have, at the affine level, two maps, again denoted by the same symbol $\mathrm{Tr}_{B/A}(= \mathrm{Tr}_{(B,J)/(A,I)})$

(5.1.1) $$\mathrm{Tr}_{B/A} = \mathrm{Tr}_{(B,J)/(A,I)}\colon \mathbf{R}\Gamma_J\omega^{\bullet}_{B/A} \to \mathbf{R}\Gamma_I\, A[0].$$

and

(5.1.2) $$\mathrm{Tr}_{B/A} = \mathrm{Tr}_{(B,J)/(A,I)}\colon \mathbf{R}\Gamma_J\omega^{\bullet}_{B/A} \to A[0].$$

Note that the two uses of the symbol $\mathrm{Tr}_{B/A}$ occur in the following commutative diagram:

$$\begin{tikzcd} \mathbf{R}\Gamma_J\omega^{\bullet}_{B/A} \arrow[d, "\mathrm{Tr}_{B/A}"'] \arrow[dr, "\mathrm{Tr}_{B/A}"] & \\ \mathbf{R}\Gamma_I\, A[0] \arrow[r] & A[0] \end{tikzcd}$$

B. Next suppose A and B both have discrete topology, and we have a finite-type map $A \to B$. Suppose J is an ideal in B such that $A \to B/J$ is finite. Let $\widehat{B}$ be the completion of B with respect to J. Note that if $\kappa\colon \mathrm{Spf}(\widehat{B}, J\widehat{B}) \to \mathrm{Spec}\,B$ is the completion map, then the canonical isomorphism $\kappa^*f^{\#} \xrightarrow{\sim} (f\kappa)^{\#}$ results in a canonical isomorphism $\omega^{\bullet}_{B/A}\otimes_B \widehat{B} \xrightarrow{\sim} \omega^{\bullet}_{\widehat{B}/A}$. Define

(5.1.3) $$\mathrm{Tr}_J\colon \mathbf{R}\Gamma_J\omega^{\bullet}_{B/A} \to A[0]$$

as the composite

$$\begin{aligned} \mathbf{R}\Gamma_J\omega^{\bullet}_{B/A} &\xrightarrow{\ \sim\ } \mathbf{R}\Gamma_{J\widehat{B}}(\omega^{\bullet}_{B/A}\otimes_B \widehat{B}) \\ &\xrightarrow{\ \sim\ } \mathbf{R}\Gamma_{J\widehat{B}}\omega^{\bullet}_{\widehat{B}/A} \\ &\xrightarrow[\mathrm{Tr}_{\widehat{B}/A}]{} A[0] \end{aligned}$$

5.1.4. There is potential for confusion over the symbol $\omega^{\bullet}_{B/A}$ in a situation we will be in and we would like to clarify the issues here. Let (A, I) and (B, J) be adic rings. Let $L = IB + J \subset B$, and assume further that B is also L-adically complete. Suppose there is a ring homomorphism $A \to B$ such that the induced map $A \to B/J$ is finite. Then $A \to B/L$ is also finite and the formal-scheme maps $\mathrm{Spf}(B,\, J) \xrightarrow{p} \mathrm{Spec}\, A$ and $\mathrm{Spf}(B,\, L) \xrightarrow{f} \mathrm{Spf}(A,\, I)$ are both pseudo-finite. Moreover we have a cartesian square as follows.

$$\begin{tikzcd}
\mathrm{Spf}(B,\, L) \arrow[r, "\kappa_L"] \arrow[d, "f"'] \arrow[dr, phantom, "\square"] & \mathrm{Spf}(B,\, J) \arrow[d, "p"] \\
\mathrm{Spf}(A,\, I) \arrow[r, "\kappa_I"'] & \mathrm{Spec}\, A
\end{tikzcd}$$

Since κ_I is flat, we have $\kappa_L^* p^{\#}\mathscr{O}_{\mathrm{Spec}\, A} \xrightarrow{\ \sim\ } f^{\#}\kappa_I^*\mathscr{O}_{\mathrm{Spec}\, A} = f^{\#}\mathscr{O}_{\mathrm{Spf}(A,\, I)}$. This means we can, and *we will*, identify $\omega^{\bullet}_{(B,L)/(A,I)}$ and $\omega^{\bullet}_{(B,J/)(A,0)}$. Therefore, denoting the common complex $\omega^{\bullet}_{B/A}$ in this situation causes no confusion. Thus,

$$\omega^{\bullet}_{B/A} = \omega^{\bullet}_{(B,L)/(A,I)} = \omega^{\bullet}_{(B,J)/(A,0)}.$$

We have two maps $\mathrm{Tr}_L \colon \mathbf{R}\Gamma_L\omega^{\bullet}_{B/A} \to \mathbf{R}\Gamma_I(A[0])$ and $\mathrm{Tr}_J \colon \mathbf{R}\Gamma_J\omega^{\bullet}_{B/A} \to A[0]$ corresponding to Tr_f (cf. (5.1.1)) and Tr_p (cf. (5.1.2)) respectively. In these circumstances, according to Proposition A.2.1 in the appendix, the following diagram commutes:

(5.1.4.1)
$$\begin{tikzcd}
\mathbf{R}\Gamma_I\mathbf{R}\Gamma_J(\omega^{\bullet}_{B/A}) \arrow[r, "\sim"] \arrow[d, "\mathbf{R}\Gamma_I(\mathrm{Tr}_J)"'] & \mathbf{R}\Gamma_L(\omega^{\bullet}_{B/A}) \arrow[d, "\mathrm{Tr}_J"] \\
\mathbf{R}\Gamma_I A[0] \arrow[r, equal] & \mathbf{R}\Gamma_I A[0]
\end{tikzcd}$$

5.2. Abstract Transitivity

This section is a digression on setting up a suitable bifunctor for every morphism in $\mathbb{G}$ which will then be used to define an abstract transitivity relation.

For a morphism $f \colon \mathscr{X} \to \mathscr{Y}$ in $\mathbb{G}$, and complexes $\mathscr{F}, \mathscr{G} \in \widetilde{\mathbf{D}}^+_{\mathrm{qc}}(\mathscr{Y})$, we shall now define a bifunctorial map

$$\chi^f(\mathscr{F}, \mathscr{G}) \colon \mathbf{L}f^*\mathscr{F} \overset{\mathbf{L}}{\otimes}_{\mathscr{O}_{\mathscr{X}}} f^{\#}\mathscr{G} \longrightarrow f^{\#}(\mathscr{F} \overset{\mathbf{L}}{\otimes}_{\mathscr{O}_{\mathscr{Y}}} \mathscr{G}) \tag{5.2.1}$$

which, a-priori, will depend on the choice of a factorization $f = f_n f_{n-1} \cdots f_1$ where each f_i is either an open immersion or a pseudoproper map. In these two special cases, there is a simple version of this bifunctorial map and the general case is handled by putting together these special ones. In Proposition 5.2.4 below we prove that $\chi^f(-,-)$ is independent of the choice of the factorization.

Since $(-)^{\#}$ is only a pre-pseudofunctor, even for f any identity map, say $f = 1_{\mathscr{X}}$, some non-trivial considerations arise. For $\mathscr{F}, \mathscr{G} \in \widetilde{\mathbf{D}}^+_{\mathrm{qc}}(\mathscr{X})$, we define

$$q_{\mathscr{X}}(\mathscr{F}, \mathscr{G}) \colon \mathscr{F} \overset{\mathbf{L}}{\otimes}_{\mathscr{O}_{\mathscr{X}}} \mathbf{\Lambda}_{\mathscr{X}}\mathscr{G} \to \mathbf{\Lambda}_{\mathscr{X}}(\mathscr{F} \overset{\mathbf{L}}{\otimes}_{\mathscr{O}_{\mathscr{X}}} \mathscr{G})$$

to be the map, which, via right adjointness of $\mathbf{\Lambda}_{\mathscr{X}}$ to $\mathbf{R}\varGamma'_{\mathscr{X}}$, corresponds to the composite of natural maps

$$\mathbf{R}\varGamma'_{\mathscr{X}}(\mathscr{F} \overset{\mathbf{L}}{\otimes}_{\mathscr{O}_{\mathscr{X}}} \mathbf{\Lambda}_{\mathscr{X}}\mathscr{G}) \xrightarrow{\ \sim\ } \mathscr{F} \overset{\mathbf{L}}{\otimes}_{\mathscr{O}_{\mathscr{X}}} \mathbf{R}\varGamma'_{\mathscr{X}}\mathbf{\Lambda}_{\mathscr{X}}\mathscr{G} \to \mathscr{F} \overset{\mathbf{L}}{\otimes}_{\mathscr{O}_{\mathscr{X}}} \mathscr{G}.$$

Below we shall define $\chi^{1_{\mathscr{X}}}$ to be $q_{\mathscr{X}}$. For now, we collect a few properties of $q_{\mathscr{X}}$ that we shall use.

The natural map $1 \to \mathbf{\Lambda}_{\mathscr{X}}$ on $\mathscr{F} \overset{\mathbf{L}}{\otimes}_{\mathscr{O}_{\mathscr{X}}} \mathscr{G}$ factors through $q_{\mathscr{X}}(\mathscr{F}, \mathscr{G})$:

$$\mathscr{F} \overset{\mathbf{L}}{\otimes}_{\mathscr{O}_{\mathscr{X}}} \mathscr{G} \to \mathscr{F} \overset{\mathbf{L}}{\otimes}_{\mathscr{O}_{\mathscr{X}}} \mathbf{\Lambda}_{\mathscr{X}}\mathscr{G} \xrightarrow{q_{\mathscr{X}}(\mathscr{F},\mathscr{G})} \mathbf{\Lambda}_{\mathscr{X}}(\mathscr{F} \overset{\mathbf{L}}{\otimes}_{\mathscr{O}_{\mathscr{X}}} \mathscr{G}).$$

Note that $q_{\mathscr{X}}(\mathscr{F}, \mathscr{G})$ is an isomorphism if both $\mathscr{G}$ and $\mathscr{F} \overset{\mathbf{L}}{\otimes}_{\mathscr{O}_{\mathscr{X}}} \mathscr{G}$ have coherent homology or if $\mathscr{F}$ is perfect, i.e., locally isomorphic to bounded complex of finite-rank locally free modules. Also note that $\mathbf{R}\varGamma'_{\mathscr{X}} q_{\mathscr{X}}$ is an isomorphism and hence $\mathbf{\Lambda}_{\mathscr{X}} q_{\mathscr{X}}$, which is isomorphic to $\mathbf{\Lambda}_{\mathscr{X}}\mathbf{R}\varGamma'_{\mathscr{X}} q_{\mathscr{X}}$, is also an isomorphism, i.e., the natural map is an isomorphism

$$\mathbf{\Lambda}_{\mathscr{X}} q_{\mathscr{X}}(\mathscr{F}, \mathscr{G})\colon \mathbf{\Lambda}_{\mathscr{X}}(\mathscr{F} \overset{\mathbf{L}}{\otimes}_{\mathscr{O}_{\mathscr{X}}} \mathbf{\Lambda}_{\mathscr{X}}\mathscr{G}) \xrightarrow{\ \sim\ } \mathbf{\Lambda}_{\mathscr{X}}(\mathscr{F} \overset{\mathbf{L}}{\otimes}_{\mathscr{O}_{\mathscr{X}}} \mathscr{G}).$$

Via natural identifications, $q_{\mathscr{X}}(\mathscr{O}_{\mathscr{X}}, \mathscr{G})$ identifies with the identity map on $\mathbf{\Lambda}_{\mathscr{X}}\mathscr{G}$. Finally, to get a more explicit description of $q_{\mathscr{X}}$, if we choose the adjoint pair $(\mathbf{\Lambda}_{\mathscr{X}}, \varepsilon)$ to $\mathbf{R}\varGamma'_{\mathscr{X}}$ to be $\mathbf{\Lambda}_{\mathscr{X}}\mathscr{M} = \mathbf{R}\mathscr{H}om_{\mathscr{X}}(\mathbf{R}\varGamma'_{\mathscr{X}}\mathcal{O}_{\mathscr{X}}, \mathscr{M})$ with ε being the following composition on canonical maps

$$\mathbf{R}\varGamma'_{\mathscr{X}}\mathbf{R}\mathscr{H}om_{\mathscr{X}}(\mathbf{R}\varGamma'_{\mathscr{X}}\mathcal{O}_{\mathscr{X}}, \mathscr{M}) \xrightarrow{\ \sim\ } \mathbf{R}\mathscr{H}om_{\mathscr{X}}(\mathbf{R}\varGamma'_{\mathscr{X}}\mathcal{O}_{\mathscr{X}}, \mathscr{M}) \overset{\mathbf{L}}{\otimes}_{\mathscr{X}} \mathbf{R}\varGamma'_{\mathscr{X}}\mathcal{O}_{\mathscr{X}} \xrightarrow{\text{eval}} \mathscr{M},$$

then $q_{\mathscr{X}}(\mathscr{F}, \mathscr{G})$ can be described as the canonical map

$$\mathscr{F} \overset{\mathbf{L}}{\otimes}_{\mathscr{O}_{\mathscr{X}}} \mathbf{R}\mathscr{H}om_{\mathscr{X}}(\mathbf{R}\varGamma'_{\mathscr{X}}\mathcal{O}_{\mathscr{X}}, \mathscr{G}) \to \mathbf{R}\mathscr{H}om_{\mathscr{X}}(\mathbf{R}\varGamma'_{\mathscr{X}}\mathcal{O}_{\mathscr{X}}, \mathscr{F} \overset{\mathbf{L}}{\otimes}_{\mathscr{O}_{\mathscr{X}}} \mathscr{G}).$$

We will now set up some notation that will be useful for keeping track of the numerous issues that arise out handling factorizations of maps into open immersions and pseudoproper maps.

Let $f\colon \mathscr{X} \to \mathscr{Y}$ be a morphism in $\mathbb{G}$ and $f = f_n f_{n-1} \cdots f_1$ a factorization where each $f_i\colon \mathscr{X}_i \to \mathscr{X}_{i+1}$ is an open immersion or a pseudoproper map with $\mathscr{X}_1 := \mathscr{X}$ and $\mathscr{X}_{n+1} := \mathscr{Y}$. Let us assign to each f_i a label λ_i, with λ_i being one of either O or P (where O = open immersions and P = pseudoproper maps), together with the requirement that each f_i lies in the subcategory corresponding to λ_i. We shall denote the labelled map as $f_i^{\lambda_i}$ and the above factorization together with the assigned labels will be called a labelled factorization (of f). The corresponding sequence $F = (f_1^{\lambda_1}, \ldots, f_n^{\lambda_n})$ will be called a labelled sequence of length n and $|F|$ shall denote the composite f. To ease notation, the labels shall often be suppressed and we shall spell them out only when it is necessary. Thus we shall often denote a labelled map f^{λ} by the underlying map f itself. If $F = (f_1, \ldots, f_n)$ and $G = (g_1, \ldots, g_m)$ are labelled sequences, and if $g_1 f_n$ makes sense, then we denote the composite labelled sequence $(f_1, \ldots, f_n, g_1, \ldots, g_m)$ as (F, G), which is evidently a labelled factorization of $|(F,G)| = |G||F|$.

For a labelled sequence $F = (f_1, \ldots, f_n)$, set

$$F^* := \mathbf{L}f_1^*\mathbf{L}f_2^* \cdots \mathbf{L}f_n^*, \qquad\qquad F^{\#} := f_1^{\#} f_2^{\#} \cdots f_n^{\#}.$$

With $|F| = f$, there are canonical pseudofunctorial isomorphisms $F^* \xrightarrow{\ \sim\ } \mathbf{L}f^*$ and $F^{\#} \xrightarrow{\ \sim\ } f^{\#}$. If F, G are labelled sequences such that the composite (F,G) exists, then $(F,G)^{\#} = F^{\#}G^{\#}$ and $(F,G)^* = F^*G^*$.

For a labelled sequence $F = (f_1, \dots, f_n)$ factoring $f\colon \mathscr{X} \to \mathscr{Y}$ and for complexes $\mathscr{F}, \mathscr{G} \in \widetilde{\mathbf{D}}^+_{\mathrm{qc}}(\mathscr{Y})$, we recursively define

$$\chi_F(\mathscr{F}, \mathscr{G})\colon F^*\mathscr{F} \overset{\mathbf{L}}{\otimes}_{\mathscr{O}_\mathscr{X}} F^\#\mathscr{G} \longrightarrow F^\#(\mathscr{F} \overset{\mathbf{L}}{\otimes}_{\mathscr{O}_\mathscr{Y}} \mathscr{G})$$

as follows. If $n = 1$, then $F = f^{\mathsf{O}}$ or $F = f^{\mathsf{P}}$ and moreover $F^* = \mathbf{L}f^* = f^*$, $F^\# = f^\#$. If $F = f^{\mathsf{O}}$, so that f is an open immersion, then using $f^\# = \mathbf{\Lambda}_\mathscr{X} f^* \xrightarrow{\sim} f^*\mathbf{\Lambda}_\mathscr{Y}$, we take $\chi_F(\mathscr{F}, \mathscr{G})$ to be the composite along the top row of the following commutative diagram

$$\begin{tikzcd}
f^*\mathscr{F} \overset{\mathbf{L}}{\otimes}_{\mathscr{O}_\mathscr{X}} \mathbf{\Lambda}_\mathscr{X} f^*\mathscr{G} \arrow[r, "q_\mathscr{X}"] \arrow[d, "\wr"] & \mathbf{\Lambda}_\mathscr{X}(f^*\mathscr{F} \overset{\mathbf{L}}{\otimes}_{\mathscr{O}_\mathscr{X}} f^*\mathscr{G}) \arrow[r, "\sim"] & \mathbf{\Lambda}_\mathscr{X} f^*(\mathscr{F} \overset{\mathbf{L}}{\otimes}_{\mathscr{O}_\mathscr{Y}} \mathscr{G}) \arrow[d, "\wr"] \\
f^*\mathscr{F} \overset{\mathbf{L}}{\otimes}_{\mathscr{O}_\mathscr{X}} f^*\mathbf{\Lambda}_\mathscr{Y}\mathscr{G} \arrow[r, "\sim"] & f^*(\mathscr{F} \overset{\mathbf{L}}{\otimes}_{\mathscr{O}_\mathscr{X}} \mathbf{\Lambda}_\mathscr{Y}\mathscr{G}) \arrow[r, "f^*q_\mathscr{Y}"] & f^*\mathbf{\Lambda}_\mathscr{Y}(\mathscr{F} \overset{\mathbf{L}}{\otimes}_{\mathscr{O}_\mathscr{Y}} \mathscr{G})
\end{tikzcd} \tag{5.2.2}$$

where q_- is defined above. (The commutativity of this diagram, which will only be used later, follows easily from the explicit description of q_- above.) If $F = f^{\mathsf{P}}$, so that f is pseudoproper, we set $\chi_F(\mathscr{F}, \mathscr{G})$ to be the map adjoint to the composite

$$\mathbf{R}f_*\mathbf{R}\varGamma'_\mathscr{X}(\mathbf{L}f^*\mathscr{F} \overset{\mathbf{L}}{\otimes}_{\mathscr{O}_\mathscr{X}} f^\#\mathscr{G}) \underset{\cong}{\xrightarrow{\text{via } (p^t_f)^{-1}}} \mathscr{F} \overset{\mathbf{L}}{\otimes}_{\mathscr{O}_\mathscr{Y}} \mathbf{R}f_*\mathbf{R}\varGamma'_\mathscr{X} f^\#\mathscr{G} \xrightarrow{\text{via } \mathrm{Tr}_f} \mathscr{F} \overset{\mathbf{L}}{\otimes}_{\mathscr{O}_\mathscr{Y}} \mathscr{G}$$

where p^t_- is the torsion projection isomorphism as defined in the beginning of this chapter. In general, if $n > 1$, we decompose F as $\mathscr{X} \xrightarrow{f_1} \mathscr{X}_2 \xrightarrow{g} \mathscr{Y}$ where $G = (f_2, \dots, f_n)$ gives a labelled factorization of g while f_1 is naturally a labelled sequence of length 1. Assuming $\chi_G(\mathscr{F}, \mathscr{G})$ is already defined, we define $\chi_F(\mathscr{F}, \mathscr{G})$ to be the composite (with $\otimes_\mathscr{X} = \otimes_{\mathcal{O}_\mathscr{X}}$, $\otimes_\mathscr{Y} = \otimes_{\mathcal{O}_\mathscr{Y}}$)

$$\begin{aligned}
F^*\mathscr{F} \overset{\mathbf{L}}{\otimes}_\mathscr{X} F^\#\mathscr{G} = f_1^*G^*\mathscr{F} \overset{\mathbf{L}}{\otimes}_\mathscr{X} f_1^\# G^\#\mathscr{G} &\xrightarrow{\chi_{f_1}(G^*\mathscr{F}, G^\#\mathscr{G}))} f_1^\#(G^*\mathscr{F} \overset{\mathbf{L}}{\otimes}_{\mathscr{X}_2} G^\#\mathscr{G}) \\
&\xrightarrow{f_1^\#\chi_G(\mathscr{F},\mathscr{G})} f_1^\# G^\#(\mathscr{F} \overset{\mathbf{L}}{\otimes}_\mathscr{Y} \mathscr{G}) = F^\#(\mathscr{F} \overset{\mathbf{L}}{\otimes}_\mathscr{Y} \mathscr{G}).
\end{aligned}$$

In more concise terms, we may write that if $F = (f_1, G)$, then $\chi_F = (f_1^\#\chi_G) \circ \chi_{f_1}$. It follows from the recursive nature of the definition that for any decomposition $F = (F_1, F_2)$ we have $\chi_F = \chi_{(F_1, F_2)} = (F_1^\#\chi_{F_2}) \circ \chi_{F_1}$.

For $\mathscr{F} = \mathscr{O}_\mathscr{X}$, via the obvious natural identifications $F^*\mathscr{O}_\mathscr{Y} \overset{\mathbf{L}}{\otimes}_\mathscr{X} F^\#\mathscr{G} \xrightarrow{\sim} F^\#\mathscr{G}$ and $F^\#(\mathscr{O}_\mathscr{Y} \overset{\mathbf{L}}{\otimes}_\mathscr{Y} \mathscr{G}) \xrightarrow{\sim} F^\#\mathscr{G}$ we see that $\chi_F(\mathscr{O}_\mathscr{X}, \mathscr{G})$ identifies with the identity map on $F^\#\mathscr{G}$.

The identity map $1_\mathscr{X}$ for any formal scheme $\mathscr{X}$, being in both O and P, forms a factorization of length 1 of itself for any of the two labels. With either label, we see that $\chi_{1_\mathscr{X}}$ equals $q_\mathscr{X}$ defined above.

More generally, $f\colon \mathscr{X} \to \mathscr{Y}$ is in O and P iff f is an isomorphism of $\mathscr{X}$ onto a connected component of $\mathscr{Y}$ and so any such f is a length-one factorization of itself with either label.

LEMMA 5.2.3. *If $f\colon \mathscr{X} \to \mathscr{Y}$ is an isomorphism, then $\chi_{f^{\mathsf{P}}} = \chi_{f^{\mathsf{O}}}$.*

PROOF. In this case, $f^* = \mathbf{L}f^*$ and $f_* = \mathbf{R}f_*$ are both left and right adjoint to each other and the unit/counit maps for either adjoint pair is given by the canonical isomorphisms $f_*f^* \cong 1$, $f^*f_* \cong 1$. Since $f^\# = \mathbf{\Lambda}f^*$, the result follows from the commutativity of the outer border of the following diagram for $\mathscr{F},\mathscr{G} \in \widetilde{\mathbf{D}}^{+}_{\mathrm{qc}}(\mathscr{Y})$, where to reduce clutter, the derived functors are denoted by their non-derived counterparts and moreover $\mathbf{\Lambda} = \mathbf{\Lambda}_{\mathscr{X}}$, $\varGamma' = \mathbf{R}\varGamma'_{\mathscr{X}}$.

$$\begin{tikzcd}
 & (f^*\mathscr{F} \otimes \mathbf{\Lambda} f^*\mathscr{G}) \arrow[dl] \arrow[d] & \\
\mathbf{\Lambda}\varGamma'(f^*\mathscr{F} \otimes \mathbf{\Lambda} f^*\mathscr{G}) \arrow[r] \arrow[d] & \mathbf{\Lambda} f^* f_*\varGamma'(f^*\mathscr{F} \otimes \mathbf{\Lambda} f^*\mathscr{G}) \arrow[d] & \\
\mathbf{\Lambda}(f^*\mathscr{F} \otimes \varGamma'\mathbf{\Lambda} f^*\mathscr{G}) \arrow[r] \arrow[d] & \mathbf{\Lambda} f^* f_*(f^*\mathscr{F} \otimes \varGamma'\mathbf{\Lambda} f^*\mathscr{G}) \arrow[r] \arrow[d] & \mathbf{\Lambda} f^*(\mathscr{F} \otimes f_*\varGamma'\mathbf{\Lambda} f^*\mathscr{G}) \arrow[d] \\
\mathbf{\Lambda}(f^*\mathscr{F} \otimes f^*\mathscr{G}) \arrow[r] \arrow[dr] & \mathbf{\Lambda} f^* f_*(f^*\mathscr{F} \otimes f^*\mathscr{G}) \arrow[r] & \mathbf{\Lambda} f^*(\mathscr{F} \otimes f_* f^*\mathscr{G}) \arrow[dl] \\
 & \mathbf{\Lambda} f^*(\mathscr{F} \otimes \mathscr{G}) &
\end{tikzcd}$$

The bottom triangle is seen to commute by unravelling the definition of the projection isomorphism. The remaining parts commute trivially. $\square$

If $f\colon \mathscr{X} \to \mathscr{Y}$ is in $\mathbb{G}$ and F is a labelled factorization of f, for $\mathscr{F},\mathscr{G} \in \widetilde{\mathbf{D}}^{+}_{\mathrm{qc}}(\mathscr{Y})$ we set $\chi^f_F(\mathscr{F},\mathscr{G})$ to be the composite

$$\mathbf{L}f^*\mathscr{F} \overset{\mathbf{L}}{\otimes}_{\mathscr{O}_{\mathscr{X}}} f^{\#}\mathscr{G} \xrightarrow{\ \sim\ } F^*\mathscr{F} \overset{\mathbf{L}}{\otimes}_{\mathscr{O}_{\mathscr{X}}} F^{\#}\mathscr{G} \xrightarrow{\chi_F} F^{\#}(\mathscr{F} \overset{\mathbf{L}}{\otimes}_{\mathscr{O}_{\mathscr{Y}}} \mathscr{G}) \xrightarrow{\ \sim\ } f^{\#}(\mathscr{F} \overset{\mathbf{L}}{\otimes}_{\mathscr{O}_{\mathscr{Y}}} \mathscr{G}).$$

If $f = 1_{\mathscr{X}}$, then $\chi^f_f = q_{\mathscr{X}}$ for either label as mentioned before.

Proposition-Definition 5.2.4.

(i) *If F_1 and F_2 are two labelled factorizations of a map $f\colon \mathscr{X} \to \mathscr{Y}$ in $\mathbb{G}$, then $\chi^f_{F_1}(-,-) = \chi^f_{F_2}(-,-)$. We thus define $\chi^f(-,-)$ in (5.2.1) to be $\chi^f_F(-,-)$ for any choice of a labelled factorization F of f.*

(ii) *If $\mathscr{X} \xrightarrow{f} \mathscr{Y} \xrightarrow{g} \mathscr{Z}$ are maps in $\mathbb{G}$, then for any complexes $\mathscr{F},\mathscr{G} \in \widetilde{\mathbf{D}}^{+}_{\mathrm{qc}}(\mathscr{Z})$, the following diagram commutes.*

$$\begin{tikzcd}
\mathbf{L}f^*\mathbf{L}g^*\mathscr{F} \overset{\mathbf{L}}{\otimes}_{\mathscr{O}_{\mathscr{X}}} f^{\#}g^{\#}\mathscr{G} \arrow[r, "\chi^f"] \arrow[d, "\cong"] & f^{\#}(\mathbf{L}g^*\mathscr{F} \overset{\mathbf{L}}{\otimes}_{\mathscr{O}_{\mathscr{Y}}} g^{\#}\mathscr{G}) \arrow[r, "f^{\#}\chi^g"] & f^{\#}g^{\#}(\mathscr{F} \overset{\mathbf{L}}{\otimes}_{\mathscr{O}_{\mathscr{Z}}} \mathscr{G}) \arrow[d, "\cong"'] \\
\mathbf{L}(gf)^*\mathscr{F} \overset{\mathbf{L}}{\otimes}_{\mathscr{O}_{\mathscr{X}}} (gf)^{\#}\mathscr{G} \arrow[rr, "\chi^{gf}"] & & (gf)^{\#}(\mathscr{F} \overset{\mathbf{L}}{\otimes}_{\mathscr{O}_{\mathscr{Z}}} \mathscr{G})
\end{tikzcd}$$

Part (ii) of Proposition 5.2.4 is the transitivity property for χ. It is an easy consequence of part (i) and the relation $\chi_{(F,G)} = (F^{\#}\chi_G) \circ \chi_F$ where F, G are labelled factorizations of f, g respectively so that (F,G) can be chosen as a labelled factorization of fg.

The proof of Proposition 5.2.4(i) is somewhat long and proceeds via several special cases of both parts (i) and (ii) first. We tackle these in the next few lemmas. In all these proofs, to reduce clutter in numerous diagrams, we shall use the following shorthand notation where f is a generic name for a map and $\mathscr{X}$ for a formal scheme.

$$f^* = \mathbf{L}f^*, \qquad \otimes = \overset{\mathbf{L}}{\otimes}, \qquad \varGamma'_{\mathscr{X}} = \mathbf{R}\varGamma'_{\mathscr{X}}, \qquad f^t_* = \mathbf{R}f_*\mathbf{R}\varGamma'_{\mathscr{X}}$$

LEMMA 5.2.5. *Let $\mathscr{X} \xrightarrow{f} \mathscr{Y}$ be a map in $\mathbb{G}$.*

(i) *If F_1, F_2 are labelled factorizations of f such that $\chi^f_{F_1} = \chi^f_{F_2}$, then for any maps $\mathscr{W} \xrightarrow{h} \mathscr{X}$ and $\mathscr{Y} \xrightarrow{g} \mathscr{Z}$ in $\mathbb{G}$ and labelled factorizations G, H of g, h respectively, we have $\chi^{gf}_{(F_1,G)} = \chi^{gf}_{(F_2,G)}$ and $\chi^{fh}_{(H,F_1)} = \chi^{fh}_{(H,F_2)}$.*

(ii) *For any labelled factorization F of f we have $\chi^f_{(1_{\mathscr{X}},F)} = \chi^f_F = \chi^f_{(F,1_{\mathscr{Y}})}$.*

PROOF. (i) To prove that $\chi^{gf}_{(F_1,G)} = \chi^{gf}_{(F_2,G)}$ it suffices to check that for any complexes $\mathscr{F}, \mathscr{G} \in \widetilde{\mathbf{D}}^+_{\mathrm{qc}}(\mathscr{Z})$ the outer border of the following diagram commutes.

$$\begin{tikzcd}
F_1^*G^*\mathscr{F} \otimes F_1^\#G^\#\mathscr{G} \arrow[r] \arrow[d, "\cong"] & F_1^\#(G^*\mathscr{F} \otimes G^\#\mathscr{G}) \arrow[r] \arrow[d, "\cong"] & F_1^\#G^\#(\mathscr{F} \otimes \mathscr{G}) \arrow[d, "\cong"] \\
f^*g^*\mathscr{F} \otimes f^\#g^\#\mathscr{G} \arrow[d, "\cong"] & f^\#(G^*\mathscr{F} \otimes G^\#\mathscr{G}) \arrow[d, "\cong"] & f^\#g^\#(\mathscr{F} \otimes \mathscr{G}) \arrow[d, "\cong"] \\
F_2^*G^*\mathscr{F} \otimes F_2^\#G^\#\mathscr{G} \arrow[r] & F_2^\#(G^*\mathscr{F} \otimes G^\#\mathscr{G}) \arrow[r] & F_2^\#G^\#(\mathscr{F} \otimes \mathscr{G})
\end{tikzcd}$$

Along the leftmost and the rightmost columns, the composite of maps remains unchanged if in the objects of the middle row, $g^*, g^\#$ are replaced by $G^*, G^\#$ respectively. Thus the left half commutes if $\chi^f_{F_1} = \chi^f_{F_2}$ while the right one commutes for functorial reasons. The proof of the other relation is similar.

(ii) For $\mathscr{F}, \mathscr{G} \in \widetilde{\mathbf{D}}^+_{\mathrm{qc}}(\mathscr{Y})$, the following diagram, where $1 = 1_{\mathscr{X}}$, is easily seen to commute keeping in mind the isomorphisms $F^\# \xrightarrow{\sim} \mathbf{\Lambda}_{\mathscr{X}}F^\# = 1^\#F^\#$.

$$\begin{tikzcd}
f^*\mathscr{F} \otimes f^\#\mathscr{G} \arrow[r, "\cong"] \arrow[d, "\cong"'] & F^*\mathscr{F} \otimes F^\#\mathscr{G} \arrow[r] \arrow[dl, "\cong"'] \arrow[d] & F^\#(\mathscr{F} \otimes \mathscr{G}) \arrow[r, "\cong"] \arrow[d, "\cong"'] & f^\#(\mathscr{F} \otimes \mathscr{G}) \arrow[dl, "\cong"] \\
1^*F^*\mathscr{F} \otimes 1^\#F^\#\mathscr{G} \arrow[r] & 1^\#(F^*\mathscr{F} \otimes F^\#\mathscr{G}) \arrow[r] & 1^\#F^\#(\mathscr{F} \otimes \mathscr{G}) &
\end{tikzcd}$$

From the outer border we get that $\chi^f_{(1_{\mathscr{X}},F)} = \chi^f_F$. The other relation is proved similarly. $\square$

LEMMA 5.2.6. *Let $f\colon \mathscr{X} \to \mathscr{Y}$ be a map in $\mathbb{G}$ and $F = (f_1, \dots, f_n)$ a labelled sequence factoring f such that all the f_i's have the same label, say λ, so that f is also in λ. Then $\chi^f_F = \chi^f_{f^\lambda}$.*

PROOF. It suffices to prove the case $n = 2$ for then, by Lemma 5.2.5(i), in the general case we have $\chi^f_{(f_1,\dots,f_n)} = \chi^f_{(f_2f_1,\dots,f_n)}$, whence the result follows by induction.

In effect, we have reduced to proving Proposition 5.2.4 (ii), with $\chi^f, \chi^g, \chi^{gf}$ replaced by $\chi_f, \chi_g, \chi_{gf}$ respectively. For the rest of the proof, we use the notation from there. Let $\mathscr{F}, \mathscr{G} \in \widetilde{\mathbf{D}}^+_{\mathrm{qc}}(\mathscr{Z})$.

If $\lambda = \mathsf{O}$, then the result follows from the outer border of the following commutative diagram where (‡) commutes by (5.2.2) and the remaining parts commute for trivial reasons.

$$\begin{tikzcd}
f^*g^*\mathscr{F} \otimes \mathbf{\Lambda}_{\mathscr{X}} f^*\mathbf{\Lambda}_{\mathscr{Y}} g^*\mathscr{G} \arrow[r] \arrow[d] & \mathbf{\Lambda}_{\mathscr{X}}(f^*g^*\mathscr{F} \otimes f^*\mathbf{\Lambda}_{\mathscr{Y}} g^*\mathscr{G}) \arrow[r] \arrow[d] \arrow[dr, phantom, "(\ddagger)"] & \mathbf{\Lambda}_{\mathscr{X}} f^*(g^*\mathscr{F} \otimes \mathbf{\Lambda}_{\mathscr{Y}} g^*\mathscr{G}) \arrow[d] \\
f^*g^*\mathscr{F} \otimes \mathbf{\Lambda}_{\mathscr{X}} f^*g^*\mathscr{G} \arrow[r] \arrow[d] \arrow[dr] & \mathbf{\Lambda}_{\mathscr{X}}(f^*g^*\mathscr{F} \otimes \mathbf{\Lambda}_{\mathscr{X}} f^*g^*\mathscr{G}) \arrow[d] & \mathbf{\Lambda}_{\mathscr{X}} f^*\mathbf{\Lambda}_{\mathscr{Y}}(g^*\mathscr{F} \otimes g^*\mathscr{G}) \arrow[d] \arrow[ddl] \\
(gf)^*\mathscr{F} \otimes \mathbf{\Lambda}_{\mathscr{X}}(gf)^*\mathscr{G} \arrow[d] & \mathbf{\Lambda}_{\mathscr{X}}(f^*g^*\mathscr{F} \otimes f^*g^*\mathscr{G}) \arrow[dl] \arrow[d] & \mathbf{\Lambda}_{\mathscr{X}} f^*\mathbf{\Lambda}_{\mathscr{Y}} g^*(\mathscr{F} \otimes \mathscr{G}) \arrow[dd] \\
\mathbf{\Lambda}_{\mathscr{X}}((gf)^*\mathscr{F} \otimes (gf)^*\mathscr{G}) \arrow[d] & \mathbf{\Lambda}_{\mathscr{X}} f^*(g^*\mathscr{F} \otimes g^*\mathscr{G}) \arrow[dr] & \\
\mathbf{\Lambda}_{\mathscr{X}}(gf)^*(\mathscr{F} \otimes \mathscr{G}) \arrow[rr] & & \mathbf{\Lambda}_{\mathscr{X}} f^*g^*(\mathscr{F} \otimes \mathscr{G})
\end{tikzcd}$$

If $\lambda = \mathsf{P}$, then by adjointness it suffices to check that the outer border of the following diagram commutes where p^t_{-} is the torsion projection isomorphism as defined in the beginning of this chapter.

$$\begin{tikzcd}
g^t_*f^t_*(f^*g^*\mathscr{F} \otimes f^\#g^\#\mathscr{G}) \arrow[r, "(p^t_f)^{-1}"] \arrow[d] & g^t_*(g^*\mathscr{F} \otimes f^t_*f^\#g^\#\mathscr{G}) \arrow[r, "\mathrm{Tr}_f"] \arrow[d, "(p^t_g)^{-1}"] & g^t_*(g^*\mathscr{F} \otimes g^\#\mathscr{G}) \arrow[d, "(p^t_g)^{-1}"] \\
g_*f^t_*(f^*g^*\mathscr{F} \otimes f^\#g^\#\mathscr{G}) \arrow[d, "\cong"] & \mathscr{F} \otimes g^t_*f^t_*f^\#g^\#\mathscr{G} \arrow[r, "\mathrm{Tr}_f"] \arrow[d, "\cong"] & \mathscr{F} \otimes g^t_*g^\#\mathscr{G} \arrow[d, "\mathrm{Tr}_g"] \\
(gf)^t_*((gf)^*\mathscr{F} \otimes (gf)^\#\mathscr{G}) \arrow[r, "(p^t_{gf})^{-1}"] & \mathscr{F} \otimes (gf)^t_*(gf)^\#\mathscr{G} \arrow[r, "\mathrm{Tr}_{gf}"] & \mathscr{F} \otimes \mathscr{G}
\end{tikzcd}$$

The upper rectangle on the right commutes trivially, while the lower one on the right commutes because of the way the composite of adjoints is identified as an adjoint pseudofunctorially. Commutativity of the diagram on the left follows easily from the outer border of the following one with obvious natural maps where we use $\mathscr{E} = f^\#g^\#\mathscr{G}$.

$$\begin{tikzcd}
g^t_*f^t_*(f^*g^*\mathscr{F} \otimes \mathscr{E}) \arrow[r] \arrow[d] & g^t_*f_*(f^*g^*\mathscr{F} \otimes \varGamma'_{\mathscr{X}}\mathscr{E}) \arrow[r] \arrow[ddl] & g^t_*(g^*\mathscr{F} \otimes f_*\varGamma'_{\mathscr{X}}\mathscr{E}) \arrow[dl] \arrow[d] \\
g_*f^t_*(f^*g^*\mathscr{F} \otimes \mathscr{E}) \arrow[d] \arrow[dd, bend right] & g_*(g^*\mathscr{F} \otimes f_*\varGamma'_{\mathscr{X}}\mathscr{E}) \arrow[dl] \arrow[d] & g_*(g^*\mathscr{F} \otimes \varGamma'_{\mathscr{Y}}f_*\varGamma'_{\mathscr{X}}\mathscr{E}) \arrow[l] \arrow[d] \\
g_*f_*(f^*g^*\mathscr{F} \otimes \varGamma'_{\mathscr{X}}\mathscr{E}) \arrow[dr] \arrow[r, phantom, "(\dagger)"] & \mathscr{F} \otimes g_*f_*\varGamma'_{\mathscr{X}}\mathscr{E} \arrow[dr] & \mathscr{F} \otimes g^t_*f_*\varGamma'_{\mathscr{X}}\mathscr{E} \arrow[l] \arrow[d] \\
(gf)^t_*((gf)^*\mathscr{F} \otimes \mathscr{E}) \arrow[r] & (gf)_*((gf)^*\mathscr{F} \otimes \varGamma'_{\mathscr{X}}\mathscr{E}) \arrow[r] & \mathscr{F} \otimes (gf)_*\varGamma'_{\mathscr{X}}\mathscr{E}
\end{tikzcd}$$

Here (†) commutes by [**L4**, p. 125, Prop. 3.7.1]. Commutativity of the remaining parts is obvious. □

Consider a cartesian diagram in $\mathbb{G}$ as follows.

(5.2.7)
$$\begin{tikzcd} \mathscr{W} \arrow[r, "v"] \arrow[d, "g"'] & \mathscr{X} \arrow[d, "f"] \\ \mathscr{Z} \arrow[r, "u"'] & \mathscr{Y} \end{tikzcd}$$

Pick labelled factorizations $F = (f_1, \dots, f_n)$ and $U = (u_1, \dots, u_m)$ of f, u respectively so that these in turn, induce corresponding ones G, V of g, v by base change in the obvious manner. Thus the composite map $h = fv = ug$ admits two labelled factorizations, namely, (V, F) and (G, U).

LEMMA 5.2.8. *In the above setup,* $\chi^h_{(V,F)} = \chi^h_{(G,U)}$.

PROOF. By decomposing the factorizations F and U, let us first reduce to the case $m = n = 1$. For instance, a decomposition $U = (U', U'')$, induces a corresponding one $V = (V', V'')$ and we have a horizontally decomposed cartesian diagram as follows.

$$\begin{tikzcd} \mathscr{W} \arrow[r, "v'"] \arrow[d, "g"'] & \mathscr{W}' \arrow[r, "v''"] \arrow[d, "g'"] & \mathscr{X} \arrow[d, "f"] \\ \mathscr{Z} \arrow[r, "u'"'] & \mathscr{Z}' \arrow[r, "u''"'] & \mathscr{Y} \end{tikzcd}$$

Set $h' := u'g = g'v'$ and $h'' := u''g' = fv''$. If G' is the induced factorization of g', then by Lemma 5.2.5(i), it suffices to prove that $\chi^{h'}_{(G,U')} = \chi^{h'}_{(V',G')}$ and $\chi^{h''}_{(G',U'')} = \chi^{h''}_{(V'',F)}$. Thus we inductively reduce to the case $m = 1$. A similar argument further reduces it to $n = 1$. Moreover, after assuming $m = n = 1$, by Lemma 5.2.6 it suffices to consider the case when f, g have label P while u, v have label O.

For this special case, using the identifications $u^\# = \mathbf{\Lambda}_{\mathscr{Z}} u^*, v^\# = \mathbf{\Lambda}_{\mathscr{W}} v^*$, proving the relation $\chi^h_{(v,f)} = \chi^h_{(g,u)}$ amounts to proving that the following diagram commutes for $\mathscr{F}, \mathscr{G} \in \widetilde{\mathbf{D}}^+_{\mathrm{qc}}(\mathscr{Y})$.

(5.2.8.1)
$$\begin{tikzcd} v^*f^*\mathscr{F} \otimes \mathbf{\Lambda}_{\mathscr{W}} v^* f^\# \mathscr{G} \arrow[r, "\sim"] \arrow[d] & h^*\mathscr{F} \otimes h^\#\mathscr{G} \arrow[r, "\sim"] & g^*u^*\mathscr{F} \otimes g^\# \mathbf{\Lambda}_{\mathscr{Z}} u^*\mathscr{G} \arrow[d] \\ \mathbf{\Lambda}_{\mathscr{W}} v^*(f^*\mathscr{F} \otimes f^\#\mathscr{G}) \arrow[d] & & g^\#(u^*\mathscr{F} \otimes \mathbf{\Lambda}_{\mathscr{Z}} u^*\mathscr{G}) \arrow[d] \\ \mathbf{\Lambda}_{\mathscr{W}} v^* f^\#(\mathscr{F} \otimes \mathscr{G}) \arrow[r, "\sim"] & h^\#(\mathscr{F} \otimes \mathscr{G}) \arrow[r, "\sim"] & g^\# \mathbf{\Lambda}_{\mathscr{Z}} u^*(\mathscr{F} \otimes \mathscr{G}) \end{tikzcd}$$

The composite along each of the two rows is induced by the composite isomorphism $v^\# f^\# \xrightarrow{\sim} h^\# \xrightarrow{\sim} g^\# u^\#$ which in fact, identifies with the base-change isomorphism $\beta^\# \colon \mathbf{\Lambda}_{\mathscr{W}} v^* f^\# \xrightarrow{\sim} g^\# \mathbf{\Lambda}_{\mathscr{Z}} u^*$. Using the adjointness of $g^\#$ to $g^t_* = g_* \varGamma'_{\mathscr{W}}$, we consider the adjoint of (5.2.8.1). The adjoint diagram is expanded below, which, for convenience, is broken into two parts. Thus the rightmost column of (5.2.8.2) is the same as the left column of (5.2.8.3) and the outer border of the conjoined diagram is the adjoint of (5.2.8.1). The maps are natural ones induced by isomorphisms

$$\varGamma'_?(\mathcal{M}\otimes\mathcal{N}) \xrightarrow{\ \sim\ } \varGamma'_?\mathcal{M}\otimes\mathcal{N}, \quad \varGamma'_?\mathbf{\Lambda} \xrightarrow{\ \sim\ } \varGamma'_?, \quad \varGamma'_{\mathscr{Z}}u^* \xrightarrow{\ \sim\ } u^*\varGamma'_{\mathscr{Y}}, \quad \varGamma'_{\mathscr{W}}v^* \xrightarrow{\ \sim\ } v^*\varGamma'_{\mathscr{X}},$$

and also the isomorphisms $\varGamma'_{\mathscr{X}}f^{\#} \xrightarrow{\ \sim\ } f^!\varGamma'_{\mathscr{Y}}$, $\varGamma'_{\mathscr{W}}g^{\#} \xrightarrow{\ \sim\ } g^!\varGamma'_{\mathscr{Z}}$.

(5.2.8.2)

$$\begin{tikzcd}
g^t_*(v^*f^*\mathscr{F}\otimes\mathbf{\Lambda}_{\mathscr{W}}v^*f^{\#}\mathscr{G}) \arrow[rr] \arrow[dr] \arrow[dd] & & g^t_*(g^*u^*\mathscr{F}\otimes g^{\#}\mathbf{\Lambda}_{\mathscr{Z}}u^*\mathscr{G}) \arrow[d] \\
 & g_*(v^*f^*\mathscr{F}\otimes v^*f^!\varGamma'_{\mathscr{Y}}\mathscr{G}) \arrow[r] \arrow[d] & g_*(g^*u^*\mathscr{F}\otimes g^!u^*\varGamma'_{\mathscr{Y}}\mathscr{G}) \arrow[d] \\
g^t_*\mathbf{\Lambda}_{\mathscr{W}}v^*(f^*\mathscr{F}\otimes f^{\#}\mathscr{G}) \arrow[r] \arrow[d] & g_*v^*(f^*\mathscr{F}\otimes f^!\varGamma'_{\mathscr{Y}}\mathscr{G}) \arrow[d] \arrow[dr, phantom, "\blacksquare"] & u^*\mathscr{F}\otimes g_*g^!u^*\varGamma'_{\mathscr{Y}}\mathscr{G} \arrow[d] \\
g^t_*\mathbf{\Lambda}_{\mathscr{W}}v^*f^{\#}(\mathscr{F}\otimes\mathscr{G}) \arrow[d] & g_*v^*f^!(\mathscr{F}\otimes\varGamma'_{\mathscr{Y}}\mathscr{G}) \arrow[d] & u^*\mathscr{F}\otimes u^*\varGamma'_{\mathscr{Y}}\mathscr{G} \arrow[d] \\
g_*v^*f^!\varGamma'_{\mathscr{Y}}(\mathscr{F}\otimes\mathscr{G}) \arrow[ur] & u^*f_*f^!(\mathscr{F}\otimes\varGamma'_{\mathscr{Y}}\mathscr{G}) \arrow[r] & u^*(\mathscr{F}\otimes\varGamma'_{\mathscr{Y}}\mathscr{G})
\end{tikzcd}$$

(5.2.8.3)

$$\begin{tikzcd}
g^t_*(g^*u^*\mathscr{F}\otimes g^{\#}\mathbf{\Lambda}_{\mathscr{Z}}u^*\mathscr{G}) \arrow[r, equal] \arrow[d] & g^t_*(g^*u^*\mathscr{F}\otimes g^{\#}\mathbf{\Lambda}_{\mathscr{Z}}u^*\mathscr{G}) \arrow[d] \\
g_*(g^*u^*\mathscr{F}\otimes g^!u^*\varGamma'_{\mathscr{Y}}\mathscr{G}) \arrow[r] \arrow[d] & g_*(g^*u^*\mathscr{F}\otimes \varGamma'_{\mathscr{W}}g^{\#}\mathbf{\Lambda}_{\mathscr{Z}}u^*\mathscr{G}) \arrow[d] \\
u^*\mathscr{F}\otimes g_*g^!u^*\varGamma'_{\mathscr{Y}}\mathscr{G} \arrow[r] \arrow[d] & u^*\mathscr{F}\otimes g_*\varGamma'_{\mathscr{W}}g^{\#}\mathbf{\Lambda}_{\mathscr{Z}}u^*\mathscr{G} \arrow[d] \\
u^*\mathscr{F}\otimes u^*\varGamma'_{\mathscr{Y}}\mathscr{G} \arrow[r] \arrow[d] & u^*\mathscr{F}\otimes u^*\mathbf{\Lambda}_{\mathscr{Y}}\mathscr{G} \arrow[d] \\
u^*(\mathscr{F}\otimes\varGamma'_{\mathscr{Y}}\mathscr{G}) \arrow[r] & \mathbf{\Lambda}_{\mathscr{Z}}(u^*\mathscr{F}\otimes u^*\mathscr{G})
\end{tikzcd}$$

For the commutativity of the diagram labelled as ■ in (5.2.8.2) we refer to the diagram at the bottom of [**L4**, p. 196]. The commutativity of the remaining parts is straightforward. □

Lemma 5.2.9. *For any labelled factorization F of the identity map $1_{\mathscr{X}}$, we have $\chi_F^{1_{\mathscr{X}}} = \chi_{1_{\mathscr{X}}}^{1_{\mathscr{X}}} = q_{\mathscr{X}}$.*

Proof. First we consider the special case where the length of F is two, say $F = (f_1, f_2)$, and where the label of f_1 is P. If the label of f_2 is also P then the result follows from Lemma 5.2.6 while if the label is O, then f_2, which is necessarily surjective (as $f_2f_1 = 1_{\mathscr{X}}$), is an isomorphism. By Lemma 5.2.3, $\chi_F^{1_{\mathscr{X}}}$ does not change if we replace the label of f_2 by P, and upon doing so, the result follows from Lemma 5.2.6.

In general, fix an integer $n \geq 2$ and let $F = (f_1, \ldots, f_n)$ be a factorization of $1_{\mathscr{X}}$ so that with $f_i \colon \mathscr{X}_i \to \mathscr{X}_{i+1}$ we have $\mathscr{X}_1 = \mathscr{X} = \mathscr{X}_{n+1}$. Let $r(F)$ be the largest integer between 1 and n such that if $1 \leq i \leq r$ then f_i has label P. We prove the result by descending induction on $r(F)$. If $r(F) = n$, then the result follows by Lemma 5.2.6. Let $r(F) = k$ and assume that the result is true for any complex for which $r > k$. Consider the following diagram containing a fibered square

$$\begin{tikzcd} \mathscr{X} \arrow[r, "\Delta"] & \mathscr{X} \times_{\mathscr{X}_n} \mathscr{X} \arrow[r, "p_2"] \arrow[d, "p_1"] & \mathscr{X} \arrow[d, "g"] & \\ & \mathscr{X} \arrow[r, "g"] & \mathscr{X}_n \arrow[r, "f_n"] & \mathscr{X} \end{tikzcd}$$

where $g = f_{n-1} \cdots f_1$ is pseudoproper, Δ is the diagonal map and p_i are the usual projections. For the map g which is drawn in parallel to p_2 we choose the factorization $G = (f_1, \ldots, f_{n-1})$ while for the other one we choose the length 1 factorization g itself with label P. The parallel edges pick up corresponding labelled factorizations by base change: for p_2 we denote it as $G' = (f'_1, \ldots, f'_{n-1})$ while for p_1 it is p_1 itself with label P. Finally, we assign to Δ, the label P.

By the special case considered in the first para, $\chi^{1_{\mathscr{X}}}_{(\Delta, p_1)} = \chi^{1_{\mathscr{X}}}_{1_{\mathscr{X}}} = \chi^{1_{\mathscr{X}}}_{(g, f_n)}$. Since (Δ, G') has length n and $r(\Delta, G') = k + 1$, hence by induction, $\chi^{1_{\mathscr{X}}}_{(\Delta, G')} = \chi^{1_{\mathscr{X}}}_{1_{\mathscr{X}}}$. Therefore, by Lemma 5.2.5 and Lemma 5.2.8,

$$\begin{aligned} \chi^{1_{\mathscr{X}}}_{F} = \chi^{1_{\mathscr{X}}}_{(G, f_n)} = \chi^{1_{\mathscr{X}}}_{(1_{\mathscr{X}}, G, f_n)} &= \chi^{1_{\mathscr{X}}}_{(\Delta, p_1, G, f_n)} \\ &= \chi^{1_{\mathscr{X}}}_{(\Delta, G', g, f_n)} = \chi^{1_{\mathscr{X}}}_{(1_{\mathscr{X}}, g, f_n)} = \chi^{1_{\mathscr{X}}}_{(g, f_n)} = \chi^{1_{\mathscr{X}}}_{1_{\mathscr{X}}}. \end{aligned}$$

□

Using the above lemmas, Proposition 5.2.4 is proved as follows.

Proof of 5.2.4(i). Consider the following diagram with a fibered square.

$$\begin{tikzcd} \mathscr{X} \arrow[r, "\Delta"] & \mathscr{X} \times_{\mathscr{Y}} \mathscr{X} \arrow[r, "p_2"] \arrow[d, "p_1"'] & \mathscr{X} \arrow[d, "f"] \\ & \mathscr{X} \arrow[r, "f"'] & \mathscr{Y} \end{tikzcd}$$

We choose for the map f which is drawn parallel to p_1, the factorization F_1, and denote the induced factorization of p_1 by F'_1, while for the other f we choose F_2 as a factorization and denote the induced one on p_2 by F'_2. Assigning to Δ, the label P, by Lemma 5.2.5, Lemma 5.2.8 and Lemma 5.2.9 we have

$$\chi^{f}_{F_2} = \chi^{f}_{(1_{\mathscr{X}}, F_2)} = \chi^{f}_{(\Delta, F'_1, F_2)} = \chi^{f}_{(\Delta, F'_2, F_1)} = \chi^{f}_{(1_{\mathscr{X}}, F_1)} = \chi^{f}_{F_1}.$$

□

Proof of 5.2.4(ii). Let us pick labelled factorizations F, G of f, g respectively so that (F, G) is a factorization for gf. It suffices to prove that the outer border of the following diagram of obvious natural maps commutes.

$$\begin{tikzcd}
f^*g^*\mathscr{F}\otimes f^\#g^\#\mathscr{G} \arrow[r] \arrow[d] \arrow[dr, phantom, "\Box_f"] & f^\#(g^*\mathscr{F}\otimes g^\#\mathscr{G}) \arrow[r] \arrow[d] & f^\#g^\#(\mathscr{F}\otimes\mathscr{G}) \arrow[d] \\
F^*g^*\mathscr{F}\otimes F^\#g^\#\mathscr{G} \arrow[r] \arrow[d] & F^\#(g^*\mathscr{F}\otimes g^\#\mathscr{G}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\Box_g"] & F^\#g^\#(\mathscr{F}\otimes\mathscr{G}) \arrow[d] \\
F^*G^*\mathscr{F}\otimes F^\#G^\#\mathscr{G} \arrow[r] \arrow[d] & F^\#(G^*\mathscr{F}\otimes G^\#\mathscr{G}) \arrow[r] \arrow[d, phantom, "\Box_{gf}"] & F^\#G^\#(\mathscr{F}\otimes\mathscr{G}) \arrow[d] \\
(gf)^*\mathscr{F}\otimes(gf)^\#\mathscr{G} \arrow[rr] & & (gf)^\#(\mathscr{F}\otimes\mathscr{G})
\end{tikzcd}$$

Here $\Box_f, \Box_g, \Box_{gf}$ commute by definition of $\chi^f_F, \chi^g_G, \chi^{gf}_{(F,G)}$ respectively while the rest of the diagram commutes trivially. $\Box$

Here are some additional properties of χ that we need below. We begin with compatibility with flat base change. For simplicity we shall assume that the complexes have coherent homology.

PROPOSITION 5.2.10. *Suppose σ is a cartesian square as follows*

$$\begin{tikzcd}
\mathscr{V} \arrow[r, "v"] \arrow[d, "g"'] \arrow[dr, phantom, "\Box"] & \mathscr{X} \arrow[d, "f"] \\
\mathscr{U} \arrow[r, "u"'] & \mathscr{Y}
\end{tikzcd}$$

with f and g in $\mathbb{G}$ and u flat. *Then for any $\mathscr{F},\mathscr{G}\in\mathbf{D}^+_{\mathrm{c}}(\mathscr{Y})$ the diagram D_σ given as follows, commutes:*

$$\begin{tikzcd}
v^*(f^*\mathscr{F}\overset{\mathbf{L}}{\otimes} f^\#\mathscr{G}) \arrow[r, "v^*\chi^f"] \arrow[d, equal] & v^*f^\#(\mathscr{F}\overset{\mathbf{L}}{\otimes}\mathscr{G}) \arrow[d, "\wr"] \\
g^*u^*\mathscr{F}\overset{\mathbf{L}}{\otimes} v^*f^\#\mathscr{G} \arrow[d, "\wr"'] & g^\#u^*(\mathscr{F}\overset{\mathbf{L}}{\otimes}\mathscr{G}) \arrow[d, equal] \\
g^*u^*\mathscr{F}\overset{\mathbf{L}}{\otimes} g^\#u^*\mathscr{G} \arrow[r, "\chi^g"'] & g^\#(u^*\mathscr{F}\overset{\mathbf{L}}{\otimes} u^*\mathscr{G})
\end{tikzcd}$$

where the two isomorphisms displayed are the ones arising from the flat base change isomorphism $\beta^\#_\sigma\colon v^*f^\# \xrightarrow{\sim} g^\#u^*$.

PROOF. If f is an open immersion, the base-change isomorphism $\beta^\#_\sigma$ is induced, via canonical identifications, by the pseudofunctoriality of $(-)^*$ and the same is true of χ^f, χ^g, hence the result is obvious in this case. If f is pseudoproper, the result follows from the proof of commutativity of (5.2.8.1). In general, suppose $f = f_2 f_1$

where $f_i \in \mathbb{G}$, so that σ can be correspondingly expanded into a diagram as follows.

$$
\begin{tikzcd}
\mathscr{V} \arrow[r, "v"] \arrow[d, "g_1"'] \arrow[dr, phantom, "\sigma_1"] & \mathscr{X} \arrow[d, "f_1"] \\
\mathscr{W} \arrow[r, "w"] \arrow[d, "g_2"'] \arrow[dr, phantom, "\sigma_2"] & \mathscr{Z} \arrow[d, "f_2"] \\
\mathscr{U} \arrow[r, "u"'] & \mathscr{Y}
\end{tikzcd}
$$

Then checking commutativity of D_σ reduces to checking that of the outer border of the following diagram of obvious natural maps.

$$
\begin{tikzcd}
v^*f_1^*f_2^*\mathscr{F} \overset{\mathbf{L}}{\otimes} v^*f_1^{\#}f_2^{\#}\mathscr{G} \arrow[r] \arrow[d] & v^*(f_1^*f_2^*\mathscr{F} \overset{\mathbf{L}}{\otimes} f_1^{\#}f_2^{\#}\mathscr{G}) \arrow[r] \arrow[d, phantom, "D_{\sigma_1}"] & v^*f_1^{\#}(f_2^*\mathscr{F} \overset{\mathbf{L}}{\otimes} f_2^{\#}\mathscr{G}) \arrow[d] \arrow[ddl] \\
g_1^*w^*f_2^*\mathscr{F} \overset{\mathbf{L}}{\otimes} g_1^{\#}w^*f_2^{\#}\mathscr{G} \arrow[r] \arrow[d] & g_1^{\#}(w^*f_2^*\mathscr{F} \overset{\mathbf{L}}{\otimes} w^*f_2^{\#}\mathscr{G}) \arrow[d] \arrow[ddl] & v^*f_1^{\#}f_2^{\#}(\mathscr{F} \overset{\mathbf{L}}{\otimes} \mathscr{G}) \arrow[d] \\
g_1^*g_2^*u^*\mathscr{F} \overset{\mathbf{L}}{\otimes} g_1^{\#}g_2^{\#}u^*\mathscr{G} \arrow[d] & g_1^{\#}w^*(f_2^*\mathscr{F} \overset{\mathbf{L}}{\otimes} f_2^{\#}\mathscr{G}) \arrow[r] \arrow[d, phantom, "D_{\sigma_2}"] & g_1^{\#}w^*f_2^{\#}(\mathscr{F} \overset{\mathbf{L}}{\otimes} \mathscr{G}) \arrow[d] \\
g_1^{\#}(g_2^*u^*\mathscr{F} \overset{\mathbf{L}}{\otimes} g_2^{\#}u^*\mathscr{G}) \arrow[r] & g_1^{\#}g_2^{\#}(u^*\mathscr{F} \overset{\mathbf{L}}{\otimes} u^*\mathscr{G}) \arrow[r] & g_1^{\#}g_2^{\#}u^*(\mathscr{F} \overset{\mathbf{L}}{\otimes} \mathscr{G})
\end{tikzcd}
$$

Since the unlabelled parts commute trivially, we reduce to checking commutativity of $D_{\sigma_1}, D_{\sigma_2}$. Thus if we fix a labelled factorization of f then proceeding inductively we reduce to the already-resolved case of when the length of the factorization is 1. $\square$

LEMMA 5.2.11. *Let $\mathscr{Y}$ be a formal scheme and $\mathscr{I}$ a coherent open $\mathscr{O}_{\mathscr{Y}}$-ideal. Let $\kappa\colon \mathscr{X} \to \mathscr{Y}$ be the completion of $\mathscr{Y}$ with respect to $\mathscr{I}$. Let $\mathscr{F}, \mathscr{G} \in \mathbf{D}_{\mathrm{c}}^+(\mathscr{Y})$. Then the following diagram commutes where the vertically drawn maps are induced by the natural isomorphisms $\kappa^* \xrightarrow{\sim} \kappa^{\#}$ on $\mathbf{D}_{\mathrm{c}}^+(\mathscr{Y})$ while the map in the top row is the obvious isomorphism.*

$$
\begin{tikzcd}
\kappa^*\mathscr{F} \overset{\mathbf{L}}{\otimes}_{\mathscr{O}_{\mathscr{X}}} \kappa^*\mathscr{G} \arrow[r] \arrow[d] & \kappa^*(\mathscr{F} \overset{\mathbf{L}}{\otimes}_{\mathscr{O}_{\mathscr{Y}}} \mathscr{G}) \arrow[d] \\
\kappa^*\mathscr{F} \overset{\mathbf{L}}{\otimes}_{\mathscr{O}_{\mathscr{X}}} \kappa^{\#}\mathscr{G} \arrow[r, "\chi^{\kappa}"] & \kappa^{\#}(\mathscr{F} \overset{\mathbf{L}}{\otimes}_{\mathscr{O}_{\mathscr{Y}}} \mathscr{G})
\end{tikzcd}
$$

PROOF. By adjointness of $\kappa^{\#}$ to $\kappa_*\mathbf{R}\varGamma'_{\mathscr{X}}$, the assertion reduces to checking commutativity of the corresponding adjoint diagram which appears as the outer border of the following one.

$$\begin{tikzcd}
\kappa_*\mathbf{R}\varGamma'_{\mathscr{X}}(\kappa^*\mathscr{F}\overset{\mathbf{L}}{\otimes}_{\mathscr{O}_{\mathscr{X}}}\kappa^*\mathscr{G}) \arrow[r] \arrow[dd] \arrow[dr] & \kappa_*\mathbf{R}\varGamma'_{\mathscr{X}}\kappa^*(\mathscr{F}\overset{\mathbf{L}}{\otimes}_{\mathscr{O}_{\mathscr{Y}}}\mathscr{G}) \arrow[r] \arrow[dr, phantom, "\blacksquare"] & \mathbf{R}\varGamma_{\mathscr{I}}(\mathscr{F}\overset{\mathbf{L}}{\otimes}_{\mathscr{O}_{\mathscr{Y}}}\mathscr{G}) \arrow[d] \\
 & \mathscr{F}\overset{\mathbf{L}}{\otimes}_{\mathscr{O}_{\mathscr{Y}}}\kappa_*\mathbf{R}\varGamma'_{\mathscr{X}}\kappa^*\mathscr{G} \arrow[r] \arrow[d] & \mathscr{F}\overset{\mathbf{L}}{\otimes}_{\mathscr{O}_{\mathscr{Y}}}\mathbf{R}\varGamma_{\mathscr{I}}\mathscr{G} \arrow[d] \\
\kappa_*\mathbf{R}\varGamma'_{\mathscr{X}}(\kappa^*\mathscr{F}\overset{\mathbf{L}}{\otimes}_{\mathscr{O}_{\mathscr{X}}}\kappa^{\#}\mathscr{G}) \arrow[r] & \mathscr{F}\overset{\mathbf{L}}{\otimes}_{\mathscr{O}_{\mathscr{Y}}}\kappa_*\mathbf{R}\varGamma'_{\mathscr{X}}\kappa^{\#}\mathscr{G} \arrow[r] & \mathscr{F}\overset{\mathbf{L}}{\otimes}_{\mathscr{O}_{\mathscr{Y}}}\mathscr{G}
\end{tikzcd}$$

The unlabelled parts of the above diagram commute trivially, while commutativity of ■ follows from that of the outer border of the following commutative diagram of obvious natural isomorphisms, where for convenience, $\mathbf{R}\varGamma'_{-}$ is replaced by $\varGamma'_{-}$.

$$\begin{tikzcd}
\mathscr{F}\overset{\mathbf{L}}{\otimes}_{\mathscr{Y}}\kappa_*\varGamma'_{\mathscr{X}}\kappa^*\mathscr{G} \arrow[r] \arrow[d] & \kappa_*\varGamma'_{\mathscr{X}}(\kappa^*\mathscr{F}\overset{\mathbf{L}}{\otimes}_{\mathscr{X}}\kappa^*\mathscr{G}) \arrow[r] \arrow[d] & \kappa_*\varGamma'_{\mathscr{X}}\kappa^*(\mathscr{F}\overset{\mathbf{L}}{\otimes}_{\mathscr{Y}}\mathscr{G}) \arrow[d] \\
\mathscr{F}\overset{\mathbf{L}}{\otimes}_{\mathscr{Y}}\kappa_*\varGamma'_{\mathscr{X}}\kappa^*\varGamma'_{\mathscr{Y}}\mathscr{G} \arrow[r] \arrow[d] & \kappa_*\varGamma'_{\mathscr{X}}(\kappa^*\mathscr{F}\overset{\mathbf{L}}{\otimes}_{\mathscr{X}}\kappa^*\varGamma'_{\mathscr{Y}}\mathscr{G}) \arrow[r] \arrow[d] & \kappa_*\varGamma'_{\mathscr{X}}\kappa^*(\mathscr{F}\overset{\mathbf{L}}{\otimes}_{\mathscr{Y}}\varGamma'_{\mathscr{Y}}\mathscr{G}) \arrow[d] \\
\mathscr{F}\overset{\mathbf{L}}{\otimes}_{\mathscr{Y}}\kappa_*\kappa^*\varGamma'_{\mathscr{Y}}\mathscr{G} \arrow[r] \arrow[drr] & \kappa_*(\kappa^*\mathscr{F}\overset{\mathbf{L}}{\otimes}_{\mathscr{X}}\kappa^*\varGamma'_{\mathscr{Y}}\mathscr{G}) \arrow[r] & \kappa_*\kappa^*(\mathscr{F}\overset{\mathbf{L}}{\otimes}_{\mathscr{Y}}\varGamma'_{\mathscr{Y}}\mathscr{G}) \arrow[d] \\
 & & \mathscr{F}\overset{\mathbf{L}}{\otimes}_{\mathscr{Y}}\varGamma'_{\mathscr{Y}}\mathscr{G}
\end{tikzcd}$$

□

For a map $f\colon \mathscr{X}\to\mathscr{Y}$ in $\mathbb{G}$, and $\mathscr{F},\mathscr{G}\in\widetilde{\mathbf{D}}^+_{\mathrm{qc}}(\mathscr{Y})$, we define a conjugate version of $\chi^f(\mathscr{F},\mathscr{G})$, denoted as $\overline{\chi}^f(\mathscr{F},\mathscr{G})$, to be the following composite of obvious natural maps

$$f^{\#}\mathscr{F}\overset{\mathbf{L}}{\otimes}_{\mathscr{X}}\mathbf{L}f^*\mathscr{G}\ \xrightarrow{\ \sim\ }\ \mathbf{L}f^*\mathscr{G}\overset{\mathbf{L}}{\otimes}_{\mathscr{X}}f^{\#}\mathscr{F}\xrightarrow{\chi^f}f^{\#}(\mathscr{G}\overset{\mathbf{L}}{\otimes}_{\mathscr{Y}}\mathscr{F})\ \xrightarrow{\ \sim\ }\ f^{\#}(\mathscr{F}\overset{\mathbf{L}}{\otimes}_{\mathscr{Y}}\mathscr{G}). \tag{5.2.12}$$

Transitivity, completions and traces. We apply the abstract results of the previous subsection to relative dualizing modules.

LEMMA 5.2.13. *Suppose*

$$\mathscr{X}\xrightarrow{f}\mathscr{Y}_1\xrightarrow{\kappa}\mathscr{Y}_2\xrightarrow{g}\mathscr{Z}$$

are maps in $\mathbb{G}$ *with* κ *a completion map with respect to an ideal* $\mathscr{I}$ *of* $\mathscr{O}_{\mathscr{Y}_2}$. *Then the following diagram commutes where the unlabelled arrows are the obvious natural* *isomorphisms.*

$$
\begin{tikzcd}
\mathbf{L}f^*\kappa^\# g^\#\mathscr{O}_{\mathscr{Z}} \overset{\mathbf{L}}{\otimes}_{\mathscr{O}_{\mathscr{X}}} f^\#\kappa^*\mathscr{O}_{\mathscr{Y}_2} \arrow[r] \arrow[d, "\kappa^*\cong\,\kappa^\#"'] & \mathbf{L}f^*(g\kappa)^\#\mathscr{O}_{\mathscr{Z}} \overset{\mathbf{L}}{\otimes}_{\mathscr{O}_{\mathscr{X}}} f^\#\mathscr{O}_{\mathscr{Y}_1} \arrow[r, "\chi^f"] & f^\#(g\kappa)^\#\mathscr{O}_{\mathscr{Z}} \arrow[d] \\
\mathbf{L}f^*\kappa^* g^\#\mathscr{O}_{\mathscr{Z}} \overset{\mathbf{L}}{\otimes}_{\mathscr{O}_{\mathscr{X}}} f^\#\kappa^\#\mathscr{O}_{\mathscr{Y}_2} \arrow[r] & \mathbf{L}(\kappa f)^* g^\#\mathscr{O}_{\mathscr{Z}} \overset{\mathbf{L}}{\otimes}_{\mathscr{O}_{\mathscr{X}}} (\kappa f)^\#\mathscr{O}_{\mathscr{Y}_2} \arrow[r, "\chi^{\kappa f}"] & (\kappa f)^\# g^\#\mathscr{O}_{\mathscr{Z}}
\end{tikzcd}
$$

PROOF. It suffices to prove that the following diagram commutes since the outer border gives us the required commutativity. As before, to simplify notation, we use f^* instead of $\mathbf{L}f^*$ and drop the subscripts to $\otimes$. For the definition of $\overline{\chi}$ we refer to (5.2.12).

$$
\begin{tikzcd}
f^*(g\kappa)^\#\mathscr{O}_{\mathscr{Z}} \otimes f^\#\mathscr{O}_{\mathscr{Y}_1} \arrow[r, "\chi^f"] \arrow[d] & f^\#((g\kappa)^\#\mathscr{O}_{\mathscr{Z}} \otimes \mathscr{O}_{\mathscr{Y}_1}) \arrow[r, "f^\#\overline{\chi}^{g\kappa}"] \arrow[d] \arrow[dr, phantom, "\overline{\Diamond}"] & f^\#(g\kappa)^\#\mathscr{O}_{\mathscr{Z}} \arrow[d] \\
f^*\kappa^\# g^\#\mathscr{O}_{\mathscr{Z}} \otimes f^\#\kappa^*\mathscr{O}_{\mathscr{Y}_2} \arrow[r, "\chi^f"] \arrow[d, "\kappa^*\cong\,\kappa^\#"'] & f^\#(\kappa^\# g^\#\mathscr{O}_{\mathscr{Z}} \otimes \kappa^*\mathscr{O}_{\mathscr{Y}_2}) \arrow[d, "\kappa^*\cong\,\kappa^\#"'] \arrow[dr, "f^\#\overline{\chi}^{\kappa}"] \arrow[dr, phantom, "\triangle"'] & f^\#\kappa^\# g^\#\mathscr{O}_{\mathscr{Z}} \arrow[dd, bend left] \\
f^*\kappa^* g^\#\mathscr{O}_{\mathscr{Z}} \otimes_{\mathscr{O}_{\mathscr{X}}} f^\#\kappa^\#\mathscr{O}_{\mathscr{Y}_2} \arrow[r, "\chi^f"] \arrow[d] & f^\#(\kappa^* g^\#\mathscr{O}_{\mathscr{Z}} \otimes \kappa^\#\mathscr{O}_{\mathscr{Y}_2}) \arrow[r, "f^\#\chi^{\kappa}"] \arrow[d, phantom, "\Diamond"] & f^\#\kappa^\#(g^\#\mathscr{O}_{\mathscr{Z}} \otimes \mathscr{O}_{\mathscr{Y}_2}) \arrow[u, "f^\#\kappa^\#\overline{\chi}^{g}"'] \\
(\kappa f)^* g^\#\mathscr{O}_{\mathscr{Z}} \otimes (\kappa f)^\#\mathscr{O}_{\mathscr{Y}_2} \arrow[r, "\chi^{\kappa f}"] & (\kappa f)^\#(g^\#\mathscr{O}_{\mathscr{Z}} \otimes \mathscr{O}_{\mathscr{Y}_2}) \arrow[r, "(\kappa f)^\#\overline{\chi}^{g}"] \arrow[ur] & (\kappa f)^\# g^\#\mathscr{O}_{\mathscr{Z}}
\end{tikzcd}
$$

The unlabelled parts commute for functorial reasons. Both $\Diamond, \overline{\Diamond}$ commute by Proposition 5.2.4 (ii), namely transitivity of χ (which also implies transitivity of $\overline{\chi}$). Finally for $\triangle$ we use the outer border of the following diagram where $\mathscr{G} = g^\#\mathscr{O}_{\mathscr{Z}}$ and where θ denotes the canonical isomorphism $\mathscr{M} \otimes \mathscr{N} \xrightarrow{\sim} \mathscr{N} \otimes \mathscr{M}$.

$$
\begin{tikzcd}
\kappa^\#\mathscr{G} \otimes \kappa^*\mathscr{O}_{\mathscr{Y}_2} \arrow[r, "\theta"] \arrow[d] & \kappa^*\mathscr{O}_{\mathscr{Y}_2} \otimes \kappa^\#\mathscr{G} \arrow[r] \arrow[d] \arrow[dr, phantom, "5.2.11"] & \kappa^\#(\mathscr{O}_{\mathscr{Y}_2} \otimes \mathscr{G}) \arrow[d] \\
\kappa^*\mathscr{G} \otimes \kappa^*\mathscr{O}_{\mathscr{Y}_2} \arrow[r, "\theta"] \arrow[d] \arrow[drr] \arrow[dr, phantom, "5.2.11"] & \kappa^*\mathscr{O}_{\mathscr{Y}_2} \otimes \kappa^*\mathscr{G} \arrow[r] & \kappa^*(\mathscr{O}_{\mathscr{Y}_2} \otimes \mathscr{G}) \arrow[d] \\
\kappa^*\mathscr{G} \otimes \kappa^\#\mathscr{O}_{\mathscr{Y}_2} \arrow[r] & \kappa^\#(\mathscr{G} \otimes \mathscr{O}_{\mathscr{Y}_2}) \arrow[r] & \kappa^*(\mathscr{G} \otimes \mathscr{O}_{\mathscr{Y}_2})
\end{tikzcd}
$$

$\square$

PROPOSITION 5.2.14. *Suppose*

$$
\begin{tikzcd}
\widehat{\mathscr{X}} \arrow[r, "\kappa_2"] \arrow[d, "\widehat{f}"'] & \mathscr{X} \arrow[d, "f"] \\
\widehat{\mathscr{Y}} \arrow[r, "\kappa_1"] \arrow[dr, "\widehat{g}"'] & \mathscr{Y} \arrow[d, "g"] \\
 & \mathscr{Z}
\end{tikzcd}
$$

is a commutative diagram of formal schemes with κ_1 and κ_2 being completions with respect to open coherent ideals of $\mathscr{O}_{\mathscr{Y}}$ and $\mathscr{O}_{\mathscr{X}}$ respectively. Then, making the identifications $\kappa_i^{\#} = \kappa_i^*$, $i = 1, 2$, the following diagram commutes, with the map labelled α being the isomorphism arising from $\kappa_2^* f^{\#} = \kappa_2^{\#} f^{\#} \xrightarrow{\ \sim\ } \widehat{f}^{\#}\kappa_1^{\#} = \widehat{f}^{\#}\kappa_1^*$.

$$\begin{tikzcd}
\kappa_2^*(\mathbf{L}f^*g^{\#}\mathscr{O}_{\mathscr{Z}} \overset{\mathbf{L}}{\otimes} f^{\#}\mathscr{O}_{\mathscr{Y}}) \arrow[r, "\kappa_2^*\chi^f"] \arrow[d, equal] & \kappa_2^* f^{\#} g^{\#}\mathscr{O}_{\mathscr{Z}} \arrow[r] & \kappa_2^*(gf)^{\#}\mathscr{O}_{\mathscr{Z}} \arrow[d, equal] \\
\kappa_2^*\mathbf{L}f^*g^{\#}\mathscr{O}_{\mathscr{Z}} \overset{\mathbf{L}}{\otimes} \kappa_2^* f^{\#}\mathscr{O}_{\mathscr{Y}}) \arrow[d, "\alpha", "\wr"'] & & \kappa_2^{\#}(gf)^{\#}\mathscr{O}_{\mathscr{Z}} \arrow[d, "\wr"] \\
\mathbf{L}\widehat{f}^*\kappa_1^* g^{\#}\mathscr{O}_{\mathscr{Z}} \overset{\mathbf{L}}{\otimes} \widehat{f}^{\#}\kappa_1^*\mathscr{O}_{\mathscr{Y}} \arrow[d, "\wr"'] & & (gf\kappa_2)^{\#}\mathscr{O}_{\mathscr{Z}} \arrow[d, equal] \\
\mathbf{L}\widehat{f}^*\widehat{g}^{\#}\mathscr{O}_{\mathscr{Z}} \overset{\mathbf{L}}{\otimes} \widehat{f}^{\#}\mathscr{O}_{\widehat{\mathscr{Y}}} \arrow[r, "\chi^{\widehat{g},\widehat{f}}"] & \widehat{f}^{\#}\widehat{g}^{\#}\mathscr{O}_{\mathscr{Z}} \arrow[r] & (\widehat{g}\widehat{f})^{\#}\mathscr{O}_{\mathscr{Z}}
\end{tikzcd}$$

Proof. This follows from the outer border of the following diagram where the unlabelled parts commute trivially.

$$\begin{tikzcd}
 & & \kappa_2^*(f^*g^{\#}\mathscr{O}_{\mathscr{Z}} \otimes f^{\#}\mathscr{O}_{\mathscr{Y}}) \arrow[r, "\kappa_2^*\chi^f"] \arrow[d, "\kappa_2^*\cong\kappa_2^{\#}"'] & \kappa_2^* f^{\#} g^{\#}\mathscr{O}_{\mathscr{Z}} \arrow[d, "\kappa_2^*\cong\kappa_2^{\#}"] \\
\kappa_2^* f^* g^{\#}\mathscr{O}_{\mathscr{Z}} \otimes \kappa_2^* f^{\#}\mathscr{O}_{\mathscr{Y}} \arrow[urr] \arrow[drr] \arrow[d, "\alpha"'] & \text{5.2.11} & \kappa_2^{\#}(f^*g^{\#}\mathscr{O}_{\mathscr{Z}} \otimes f^{\#}\mathscr{O}_{\mathscr{Y}}) \arrow[r, "\kappa_2^{\#}\chi^f"] \arrow[dr, phantom, "\text{5.2.4(ii)}"] & \kappa_2^{\#} f^{\#} g^{\#}\mathscr{O}_{\mathscr{Z}} \arrow[d] \\
\widehat{f}^*\kappa_1^* g^{\#}\mathscr{O}_{\mathscr{Z}} \otimes \widehat{f}^{\#}\kappa_1^*\mathscr{O}_{\mathscr{Y}} \arrow[drr] \arrow[d, "\kappa_1^*\cong\kappa_1^{\#}"'] & & \kappa_2^* f^* g^{\#}\mathscr{O}_{\mathscr{Z}} \otimes \kappa_2^{\#} f^{\#}\mathscr{O}_{\mathscr{Y}} \arrow[u, "\chi^{\kappa_2}"] \arrow[r, "\chi^{f\kappa_2}"'] \arrow[d] & (f\kappa_2)^{\#} g^{\#}\mathscr{O}_{\mathscr{Z}} \arrow[d] \\
\widehat{f}^*\kappa_1^{\#} g^{\#}\mathscr{O}_{\mathscr{Z}} \otimes \widehat{f}^{\#}\kappa_1^*\mathscr{O}_{\mathscr{Y}} \arrow[rr, "\kappa_1^*\cong\kappa_1^{\#}"'] \arrow[d] & & \widehat{f}^*\kappa_1^* g^{\#}\mathscr{O}_{\mathscr{Z}} \otimes \widehat{f}^{\#}\kappa_1^{\#}\mathscr{O}_{\mathscr{Y}} \arrow[r, "\chi^{\kappa_1\widehat{f}}"'] \arrow[d, phantom, "\text{5.2.13}"] & (\kappa_1\widehat{f})^{\#} g^{\#}\mathscr{O}_{\mathscr{Z}} \arrow[d] \\
\widehat{f}^*\widehat{g}^{\#}\mathscr{O}_{\mathscr{Z}} \otimes \widehat{f}^{\#}\mathscr{O}_{\widehat{\mathscr{Y}}} \arrow[rr] & & \widehat{f}^{\#}\widehat{g}^{\#}\mathscr{O}_{\mathscr{Z}} \arrow[r] & \widehat{f}^{\#}\kappa_1^{\#} g^{\#}\mathscr{O}_{\mathscr{Z}}
\end{tikzcd}$$

$\square$

Definition 5.2.15. Let $\mathscr{X} \xrightarrow{f} \mathscr{Y} \xrightarrow{g} \mathscr{Z}$ be maps in $\mathbb{G}$. We define $\chi_{[g,f]}$ to the composite of the following natural maps:

$$\mathbf{L}f^*(g^{\#}\mathscr{O}_{\mathscr{Z}}) \overset{\mathbf{L}}{\otimes}_{\mathscr{O}_{\mathscr{X}}} f^{\#}\mathscr{O}_{\mathscr{Y}} \xrightarrow{\chi^f} f^{\#}(g^{\#}\mathscr{O}_{\mathscr{Z}} \overset{\mathbf{L}}{\otimes}_{\mathscr{O}_{\mathscr{Y}}} \mathscr{O}_{\mathscr{Y}}) \xrightarrow{\ \sim\ } f^{\#}g^{\#}\mathscr{O}_{\mathscr{Y}} \xrightarrow{\ \sim\ } (gf)^{\#}\mathscr{O}_{\mathscr{Y}}.$$

When, f, g are both pseudoproper, this definition agrees with the one given in the beginning of this chapter.

Definition 5.2.16. $A \to R \to S$ be pseudo-finite maps between adic rings. Let $f\colon \mathrm{Spf}(S) \to \mathrm{Spf}(R)$ and $g\colon \mathrm{Spf}(R) \to \mathrm{Spf}(A)$ denote the resulting maps of formal schemes. We define

$$\chi_{[S/R/A]}\colon \omega^{\bullet}_{R/A} \overset{\mathbf{L}}{\otimes}_R \omega^{\bullet}_{S/R} \to \omega^{\bullet}_{S/A}$$

to be the map corresponding to $\chi_{[g,f]}$ of 5.2.15, where $\omega^{\bullet}$ is defined as in the beginning of §5.1.

Proposition 5.2.17. *Let $A \to R \to S$ be a pair of maps of rings, $I \subset R$, $J \subset S$ ideals, such that $A \to R/I$ and $R \to S/J$ are finite. Let $L = IS + J$.*

(a) *Suppose R is complete in the I-adic topology and S is complete in the L-adic topology (so that S is then complete in the J-adic topology too). The following diagram commutes, with $\chi = \chi_{[S/R/A]}$:*

$$\begin{tikzcd}
\mathbf{R}\Gamma_L(\omega^{\bullet}_{R/A} \overset{\mathbf{L}}{\otimes}_R \omega^{\bullet}_{S/R}) \arrow[rr, "\mathbf{R}\Gamma_L(\chi)"] & & \mathbf{R}\Gamma_L\omega^{\bullet}_{S/A} \arrow[dr, "\mathrm{Tr}_{S/A}"] & \\
\mathbf{R}\Gamma_I\,\mathbf{R}\Gamma_J(\omega^{\bullet}_{R/A} \overset{\mathbf{L}}{\otimes}_R \omega^{\bullet}_{S/R}) \arrow[u, "\wr"] & & & A[0] \\
\mathbf{R}\Gamma_I\,(\omega^{\bullet}_{R/A} \overset{\mathbf{L}}{\otimes}_R \mathbf{R}\Gamma_J(\omega^{\bullet}_{S/R})) \arrow[u, "\wr"] \arrow[rr, "\mathbf{R}\Gamma_I\,(\mathbf{1}\otimes\mathrm{Tr}_{S/R})"] & & \mathbf{R}\Gamma_I\,(\omega^{\bullet}_{R/A}) \arrow[ur, "\mathrm{Tr}_{R/A}"'] &
\end{tikzcd}$$

(b) *Suppose the topology on A, R, and S are discrete, and $A \to R$, $R \to S$ are of finite type. Then the following diagram commutes with $\chi = \chi_{[S/R/A]}$:*

$$\begin{tikzcd}
\mathbf{R}\Gamma_L(\omega^{\bullet}_{R/A} \overset{\mathbf{L}}{\otimes}_R \omega^{\bullet}_{S/R}) \arrow[rr, "\mathbf{R}\Gamma_L(\chi)"] & & \mathbf{R}\Gamma_L\omega^{\bullet}_{S/A} \arrow[dr, "\mathrm{Tr}_L"] & \\
\mathbf{R}\Gamma_I\,\mathbf{R}\Gamma_J(\omega^{\bullet}_{R/A} \overset{\mathbf{L}}{\otimes}_R \omega^{\bullet}_{S/R}) \arrow[u, "\wr"] & & & A[0] \\
\mathbf{R}\Gamma_I\,(\omega^{\bullet}_{R/A} \overset{\mathbf{L}}{\otimes}_R \mathbf{R}\Gamma_J(\omega^{\bullet}_{S/R})) \arrow[u, "\wr"] \arrow[rr, "\mathbf{R}\Gamma_I\,(\mathbf{1}\otimes\mathrm{Tr}_J)"] & & \mathbf{R}\Gamma_I\,(\omega^{\bullet}_{R/A}) \arrow[ur, "\mathrm{Tr}_I"'] &
\end{tikzcd}$$

Proof. For part (a), first note that the diagram below commutes:

$$\begin{tikzcd}
\mathbf{R}\Gamma_L(\omega^{\bullet}_{R/A} \overset{\mathbf{L}}{\otimes}_R \omega^{\bullet}_{S/R}) \arrow[r, "\sim"] & \omega^{\bullet}_{R/A} \overset{\mathbf{L}}{\otimes}_R \mathbf{R}\Gamma_L(\omega^{\bullet}_{S/R}) \\
\mathbf{R}\Gamma_I\,\mathbf{R}\Gamma_J(\omega^{\bullet}_{R/A} \overset{\mathbf{L}}{\otimes}_R \omega^{\bullet}_{S/R}) \arrow[u, "\wr"] \arrow[r, "\sim"] & \omega^{\bullet}_{R/A} \overset{\mathbf{L}}{\otimes}_R \mathbf{R}\Gamma_I\,\mathbf{R}\Gamma_J(\omega^{\bullet}_{S/R}) \arrow[u, "\wr"'] \arrow[d, "\wr"] \\
& \mathbf{R}\Gamma_I\,(\omega^{\bullet}_{R/A} \overset{\mathbf{L}}{\otimes}_R \mathbf{R}\Gamma_J(\omega^{\bullet}_{S/R})) \arrow[ul, "\sim"']
\end{tikzcd}$$

The assertion now follows from the commutativity of (5.1.4.1) and the definition of $\chi_{[S/R/A]}$.

For part (b), let $\widehat{R}$ be the completion of R with respect to I, S' the completion of S with respect to J and $\widehat{S}$ the completion of S with respect to L. Let $\widehat{I} = I\widehat{R}$, $J' = JS'$, $L' = LS'$, $\widehat{L} = L\widehat{S}$. Let $\mathscr{X} = \mathrm{Spf}(\widehat{S}, \widehat{L})$, $X' = \mathrm{Spf}(S', J')$, $X = \operatorname{Spec} S$, $\mathscr{Y} = \mathrm{Spf}(\widehat{R}, \widehat{I})$, $Y = \operatorname{Spec} R$, and $Z = \operatorname{Spec} A$. The various natural relations between the adic rings can be represented by a commutative diagram of formal schemes:

$$\begin{tikzcd}
\mathscr{X} \arrow[r, "\kappa_1"] \arrow[d, "\widehat{f}"'] & X' \arrow[r, "\kappa_2"] \arrow[d, "f'"'] & X \arrow[dl, "f"] \\
\mathscr{Y} \arrow[r, "\kappa_3"] \arrow[dr, "\widehat{g}"'] & Y \arrow[d, "g"] & \\
& Z &
\end{tikzcd}$$

We have:

$$\kappa_1^* {f'}^{\#}\mathscr{O}_Y \xrightarrow{\ \sim\ } (\kappa_1 f')^{\#}\mathscr{O}_Y \xrightarrow{\ \sim\ } \widehat{f}^{\#}\kappa_3^*\mathscr{O}_Y = \widehat{f}^{\#}\mathscr{O}_{\mathscr{Y}}.$$

Moreover, $\kappa_2^* f^{\#}\mathscr{O}_Y \xrightarrow{\ \sim\ } {f'}^{\#}\mathscr{O}_Y$. We may thus make the following identifications: $\omega^{\bullet}_{S/R} \otimes_S \widehat{S} = \omega^{\bullet}_{S/R} \otimes_S \widehat{S} = \omega^{\bullet}_{\widehat{S}/\widehat{R}}$, and $\omega^{\bullet}_{S'/R} = \omega^{\bullet}_{S/R} \otimes_S S'$. The natural isomorphism $\kappa_3^* g^{\#}\mathscr{O}_Z \xrightarrow{\ \sim\ } \widehat{g}^{\#}\mathscr{O}_Z$ allows us to make the identification $\omega^{\bullet}_{\widehat{R}/A} = \omega^{\bullet}_{R/A} \otimes_R \widehat{R}$. Let us write $\chi = \chi_{[S/R/A]}$ and $\widehat{\chi} = \chi_{[\widehat{S}/\widehat{R}/A]}$. The above identifications and Proposition 5.2.14 gives $\chi \otimes \widehat{S} = \widehat{\chi}$, whence the following diagram commutes:

$$\begin{tikzcd}
\mathbf{R}\Gamma_L(\omega^{\bullet}_{R/A} \overset{\mathbf{L}}{\otimes}_R \omega^{\bullet}_{S/R}) \arrow[r, "\chi"] \arrow[d, "\wr"'] & \mathbf{R}\Gamma_L \omega^{\bullet}_{S/A} \arrow[d, "\wr"] \\
\mathbf{R}\Gamma_{\widehat{L}}(\omega^{\bullet}_{\widehat{R}/A} \overset{\mathbf{L}}{\otimes}_{\widehat{R}} \omega^{\bullet}_{\widehat{S}/\widehat{R}}) \arrow[r, "\widehat{\chi}"'] & \mathbf{R}\Gamma_{\widehat{L}} \omega^{\bullet}_{\widehat{S}/A}
\end{tikzcd}$$

By part (a), it is therefore enough to prove that the diagram below commutes:

$$(*)\qquad \begin{tikzcd}
\mathbf{R}\Gamma_L(\omega^{\bullet}_{R/A} \overset{\mathbf{L}}{\otimes}_R \omega^{\bullet}_{S/R}) \arrow[r, "\sim"] & \mathbf{R}\Gamma_{\widehat{L}}(\omega^{\bullet}_{\widehat{R}/A} \overset{\mathbf{L}}{\otimes}_{\widehat{R}} \omega^{\bullet}_{\widehat{S}/\widehat{R}}) \\
\mathbf{R}\Gamma_I \mathbf{R}\Gamma_J(\omega^{\bullet}_{R/A} \overset{\mathbf{L}}{\otimes}_R \omega^{\bullet}_{S/R}) \arrow[r, "\sim"] \arrow[u, "\wr"] & \mathbf{R}\Gamma_{\widehat{I}} \mathbf{R}\Gamma_{\widehat{J}}(\omega^{\bullet}_{\widehat{R}/A} \overset{\mathbf{L}}{\otimes}_{\widehat{R}} \omega^{\bullet}_{\widehat{S}/\widehat{R}}) \arrow[u, "\wr"'] \\
\mathbf{R}\Gamma_I(\omega^{\bullet}_{R/A} \overset{\mathbf{L}}{\otimes}_R \mathbf{R}\Gamma_J(\omega^{\bullet}_{S/R})) \arrow[r, "\sim"] \arrow[u, "\wr"] \arrow[d, "\mathrm{Tr}_J"'] & \mathbf{R}\Gamma_{\widehat{I}}(\omega^{\bullet}_{\widehat{R}/A} \overset{\mathbf{L}}{\otimes}_{\widehat{R}} \mathbf{R}\Gamma_{\widehat{J}}(\omega^{\bullet}_{\widehat{S}/\widehat{R}})) \arrow[u, "\wr"'] \arrow[d, "\mathrm{Tr}_{\widehat{S}/\widehat{R}}"] \\
\mathbf{R}\Gamma_I(\omega^{\bullet}_{R/A}) \arrow[r, "\sim"] \arrow[d, "\mathrm{Tr}_I"'] & R\Gamma_{\widehat{I}}(\omega^{\bullet}_{\widehat{R}/A}) \arrow[d, "\mathrm{Tr}_{\widehat{R}/A}"'] \\
A[0] \arrow[r, equal] & A[0]
\end{tikzcd}$$

The proof of the commutativity of $(*)$ is as follows. Suppose F is a bounded-below complex of R-modules with finitely generated cohomology and G is a bounded-below complex of finitely generated S-modules, we have a bifunctorial commutative diagram (with $\widehat{F} = F \otimes_R \widehat{R}$, $G' = G \otimes_S S'$, and $\widehat{G} = G \otimes_S \widehat{S}$):

$$
\begin{tikzcd}
\mathbf{R}\Gamma_L(F \overset{\mathbf{L}}{\otimes}_R G) \arrow[r, "\sim"] & \mathbf{R}\Gamma_{L'}(F \overset{\mathbf{L}}{\otimes}_R G') \arrow[r, "\sim"] & \mathbf{R}\Gamma_{\widehat{L}}(\widehat{F} \overset{\mathbf{L}}{\otimes}_{\widehat{R}} \widehat{G}) \\
\mathbf{R}\Gamma_I\,\mathbf{R}\Gamma_J(F \overset{\mathbf{L}}{\otimes}_R G) \arrow[r, "\sim"] \arrow[u, "\wr"] & \mathbf{R}\Gamma_I\,\mathbf{R}\Gamma_{J'}(F \overset{\mathbf{L}}{\otimes}_R G') \arrow[r, "\sim"] \arrow[u, "\wr"'] & \mathbf{R}\Gamma_{\widehat{I}}\mathbf{R}\Gamma_{\widehat{J}}(\widehat{F} \overset{\mathbf{L}}{\otimes}_{\widehat{R}} \widehat{G}) \arrow[u, "\wr"'] \\
\mathbf{R}\Gamma_I(F \overset{\mathbf{L}}{\otimes}_R \mathbf{R}\Gamma_J G) \arrow[r, "\sim"] \arrow[u, "\wr"] & \mathbf{R}\Gamma_I(F \overset{\mathbf{L}}{\otimes}_R \mathbf{R}\Gamma_{J'} G') \arrow[r, "\sim"] \arrow[u, "\wr"'] & \mathbf{R}\Gamma_{\widehat{I}}(\widehat{F} \overset{\mathbf{L}}{\otimes}_{\widehat{R}} \mathbf{R}\Gamma_{\widehat{J}} \widehat{G}) \arrow[u, "\wr"']
\end{tikzcd}
$$

This shows that the top two rectangles in $(*)$ commute. For the rest of $(*)$ it is enough to show that the following diagram commutes:

$$
\begin{tikzcd}
\mathbf{R}\Gamma_I(\omega^\bullet_{R/A} \overset{\mathbf{L}}{\otimes}_R \mathbf{R}\Gamma_J(\omega^\bullet_{S/R})) \arrow[r, "\mathrm{Tr}_J"] \arrow[d, "\wr"'] & \mathbf{R}\Gamma_I(\omega^\bullet_{R/A}) \arrow[d, equal] \arrow[dr, "\mathrm{Tr}_I"] & \\
\mathbf{R}\Gamma_I(\omega^\bullet_{R/A} \overset{\mathbf{L}}{\otimes}_R \mathbf{R}\Gamma_{J'}(\omega^\bullet_{S'/R})) \arrow[r, "\mathrm{Tr}_{S'/R}"] \arrow[d, "\wr"'] & \mathbf{R}\Gamma_I(\omega^\bullet_{R/A}) \arrow[d, "\wr"'] & A[0] \\
\mathbf{R}\Gamma_{\widehat{I}}(\omega^\bullet_{R/A} \overset{\mathbf{L}}{\otimes}_{\widehat{R}} \mathbf{R}\Gamma_{\widehat{J}}(\omega^\bullet_{\widehat{S}/\widehat{R}})) \arrow[r, "\mathrm{Tr}_{\widehat{S}/\widehat{R}}"'] & \mathbf{R}\Gamma_{\widehat{I}}(\omega^\bullet_{\widehat{R}/A}) \arrow[ur, "\mathrm{Tr}_{\widehat{R}/A}"'] &
\end{tikzcd}
$$

The rectangle on the top commutes by definition of Tr_J. The triangle on the right end of the diagram commutes by definition of Tr_I. The rectangle at the bottom commutes by flat base change, since the following diagram is cartesian:

$$
\begin{tikzcd}
\mathrm{Spf}(\widehat{S},\, \widehat{J}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathrm{Spf}(S',\, J') \arrow[d] \\
\mathrm{Spf}(\widehat{R},\, \widehat{I}) \arrow[r] & \mathrm{Spec}\, R
\end{tikzcd}
$$

$\square$

CHAPTER 6

Iterated residues

6.1. Comment on Translations

This is more of an orienting remark. Suppose M and N are $\mathscr{O}_X$-modules on a ringed space $(X, \mathscr{O}_X)$, and d, e are integers. According to [**L4**, pp.28–29, (1.5.4)] the functor $F_{N[d]} = (-) \otimes N[d]$ on the homotopy category of complexes of $\mathscr{O}_X$-modules is triangle preserving with the isomorphism $(A^\bullet[1]) \otimes N[d] \xrightarrow{\sim} (A^\bullet \otimes N[d])[1]$ being the identity map ("without the intervention of signs" in the language of [**C1**]). Signs do intervene if the first argument in the tensor product is fixed and the second varies. However, if the fixed first argument is an $\mathscr{O}_X$-module, i.e., a complex concentrated in the 0^{th}-spot, then signs do not intervene. More precisely, $G_M = M \otimes (-)$ is triangle preserving, for the identity isomorphism $M \otimes (B^\bullet[1]) \xrightarrow{\sim} (M \otimes B^\bullet)[1]$. The same sign conventions apply for the derived tensor product on the derived category, see [**L4**, pp.62–63, (2.5.7)].

For complexes of $\mathscr{O}_X$-modules $A^\bullet$ and $B^\bullet$, let

$$\theta^{ij} \colon (A^\bullet[i]) \otimes (B^\bullet[j]) \xrightarrow{\sim} (A^\bullet \otimes B^\bullet)[i+j]$$

be as in [**L4**, pp.28–29, (1.5.4)]. Then the following composite is a composite of *identity maps* and hence is the identity map.

(6.1.1)
$$\begin{aligned} M[e] \otimes N[d] \xrightarrow[\theta^{e0}]{\sim} (M \otimes N[d])[e] &\xrightarrow[\theta^{0d}]{\sim} (M \otimes N)[d][e] \\ &\xrightarrow{=} (M \otimes N)[d+e]. \end{aligned}$$

(Strictly speaking, (6.1.1) is the identity map when the tensor product is in the ordinary category of complexes; over the derived category, the induced map on the homology in degree $-(d+e)$ canonically identifies with the identity map. In particular, if either of M, N is flat as $\mathscr{O}_X$-modules, then (6.1.1), viewed as a derived-category map, also canonically identifies with identity.)

Thus, given a map $\bar{\psi} \colon M \otimes N \to T$ of $\mathscr{O}_X$-modules, we get a map in $\mathbf{D}(X)$

(6.1.2)
$$\psi \colon M[e] \otimes N[d] \to T[d+e]$$

given by $(\bar{\psi}[d+e]) \circ (6.1.1)$. The maps $\bar{\psi}$ and ψ determine each other. Indeed, $\bar{\psi} = \mathrm{H}^{-(d+e)}(\psi)$.

6.2. Iterated generalized fractions

Let R be a (noetherian) ring, $I \subset R$ an ideal generated by $\mathbf{u} = (u_1, \dots, u_d)$. For any R-module M we have a map of complexes

(6.2.1)
$$M[d] \otimes_R K^\bullet_\infty(\mathbf{u}) \longrightarrow \mathrm{H}^d_I(M)[0]$$

defined on 0-cochains by

$$m \otimes \frac{1}{u_1^{\alpha_1} \dots u_d^{\alpha_d}} \mapsto (-1)^d \begin{bmatrix} m \\ u_1^{\alpha_1}, \dots, u_d^{\alpha_d} \end{bmatrix}.$$

This is a map of complexes since every 0-cochain of $M[d] \otimes_R K_\infty^\bullet(\mathbf{u})$ (and of $\mathrm{H}_I^d(M)[0]$) is a 0-cocycle and because $\mathrm{H}_I^d(M)[0]$ is a complex concentrated only in degree 0. In the event M is a free R-module and $\mathbf{u}$ is a quasi-regular sequence (or if $\mathbf{u}$ is locally an M-sequence), (6.2.1) is a quasi-isomorphism. The map (6.2.1) is functorial in M.

Let Mod_R be the category of R-modules. While (6.2.1) is a morphism in the category $\mathbf{C}(\mathrm{Mod}_R)$ there is an analogous map in $\mathbf{D}(\mathrm{Mod}_R)$ described as follows. Since $\mathrm{H}_I^j(M) = 0$ for $j > d$, there is a canonical map in $\mathbf{D}(\mathrm{Mod}_R)$,

$$\phi_{R,I}(M)\colon \mathbf{R}\Gamma_I(M[d]) \longrightarrow \mathrm{H}_I^d(M)[0] \tag{6.2.2}$$

such that $\mathrm{H}^0(\phi_{R,I}(M))$ is the identity map on $\mathrm{H}_I^d(M)$. One checks, using the definition of the generalized fraction $\left[\begin{smallmatrix} m \\ u_1^{\alpha_1}, \dots, u_d^{\alpha_d} \end{smallmatrix}\right]$, that the following diagram commutes in $\mathbf{D}(\mathrm{Mod}_R)$

$$\begin{tikzcd} M[d] \otimes_R K_\infty^\bullet(\mathbf{u}) \arrow[d, "(C.5.2)"', "\wr"] \arrow[dr, "(6.2.1)"] & \\ \mathbf{R}\Gamma_I(M[d]) \arrow[r, "\phi_{R,I}"'] & \mathrm{H}_I^d(M)[0] \end{tikzcd} \tag{6.2.3}$$

Next, suppose S is an R-algebra and $J \subset S$ is an S-ideal generated by $\mathbf{v} = (v_1, \dots, v_e)$. Suppose N is an R-module. We have an isomorphism

$$\mathrm{H}_{IS+J}^{d+e}(M \otimes_R N) \xrightarrow{\sim} \mathrm{H}_I^d(M \otimes_R \mathrm{H}_J^e(N)) \tag{6.2.4}$$

given by

$$\begin{bmatrix} m \otimes n \\ v_1^{\beta_1}, \dots, v_e^{\beta_e}, u_1^{\alpha_1}, \dots, u_d^{\alpha_d} \end{bmatrix} \mapsto \begin{bmatrix} m \otimes \begin{bmatrix} n \\ v_1^{\beta_1}, \dots, v_e^{\beta_e} \end{bmatrix} \\ u_1^{\alpha_1}, \dots, u_d^{\alpha_d} \end{bmatrix} \tag{6.2.5}$$

We claim that the following diagram commutes where we identify $M[d] \otimes_R N[e]$ with $(M \otimes_R N)[d+e]$ as in (6.1.1):

$$\begin{tikzcd} (M[d] \otimes_R N[e]) \otimes_S K_\infty^\bullet(\mathbf{v}, \mathbf{u}, S) \arrow[r, "(6.2.1)"] \arrow[d, equal] & \mathrm{H}_{IS+J}^{d+e}(M \otimes N)[0] \arrow[dd, "(6.2.4)"] \\ M[d] \otimes_R (N[e] \otimes_S K_\infty^\bullet(\mathbf{v}, S)) \otimes_R K_\infty^\bullet(\mathbf{u}, R) \arrow[d, "(6.2.1)"'] & \\ M[d] \otimes_R \mathrm{H}_J^e(N)[0] \otimes_R K_\infty^\bullet(\mathbf{u}, R) \arrow[r, "(6.2.1)"'] & \mathrm{H}_I^d(M \otimes_R \mathrm{H}_J^e(N))[0] \end{tikzcd} \tag{6.2.6}$$

Indeed, consider a 0-cocycle $m \otimes n \otimes \frac{1}{v_1^{\beta_1} \dots v_e^{\beta_e}, u_1^{\alpha_1} \dots u_d^{\alpha_d}}$ of $M[d] \otimes N[e] \otimes K_\infty^\bullet(\mathbf{v}, \mathbf{u}, S)$. Its image along either possible route (east-followed-by-south or south-followed-by-east) is $(-1)^{d+e}\left[\begin{smallmatrix} m \otimes \left[\begin{smallmatrix} n \\ \mathbf{v}^\beta \end{smallmatrix}\right] \\ \mathbf{u}^\alpha \end{smallmatrix}\right]$. This proves that (6.2.6) commutes.

The above is diagram in the category of complexes $\mathbf{C}(\mathrm{Mod}_R)$. This can be upgraded to the following:

PROPOSITION 6.2.7. *The following diagram commutes in* $\mathbf{D}(\mathrm{Mod}_R)$*:*

$$\begin{tikzcd}
\mathbf{R}\Gamma_{IS+J}(M[d]\otimes_R N[e]) \arrow[r, "\phi_{S,IS+J}"] & \mathrm{H}^{d+e}_{IS+K}(M\otimes N)[0] \arrow[ddd, "(6.2.4)"] \\
\mathbf{R}\Gamma_{IS}\mathbf{R}\Gamma_J(M[d]\otimes_R N[e]) \arrow[u, "\wr"] & \\
\mathbf{R}\Gamma_I(M[d]\overset{\mathbf{L}}{\otimes}_R \mathbf{R}\Gamma_J N[e]) \arrow[u, "\wr"] \arrow[d, "\phi_{S,J}"'] & \\
\mathbf{R}\Gamma_I(M[d]\otimes_R \mathrm{H}^e_J(N)[0]) \arrow[r, "\phi_{R,I}"'] & \mathrm{H}^d_I(M\otimes_R \mathrm{H}^e_J(N))[0]
\end{tikzcd}$$

PROOF. This is a straightforward re-interpretation of the commutativity of (6.2.6) using the commutativity of (6.2.3). □

REMARK 6.2.8. Proposition 6.2.7 can be interpreted as saying (6.2.4) is the isomorphism given by the Leray spectral sequence for the composite functor $\Gamma_{IS+J} = \Gamma_I\circ\Gamma_J$. See [**LS**, Proposition (3.3.1)] as well as the correction by the second author.

6.3. Cohen-Macaulay maps and iterated residues

Suppose $X = \mathrm{Spec}\, S$, $Y = \mathrm{Spec}\, R$, $Z = \mathrm{Spec}\, A$ are affine schemes, and $f\colon X\to Y$ is Cohen-Macaulay of relative dimension e, $g\colon Y\to Z$ is Cohen-Macaulay of relative dimension d. Note that we have finite type maps of rings $A\to R$ and $R\to S$. Suppose $I\subset R$ and $J\subset S$ are as in §6.2 with the added condition that the given generators of I and J, namely $\mathbf{u} = (u_1,\dots,u_d)$ and $\mathbf{v} = (v_1,\dots,v_e)$ repectively are quasi-regular, and that $A\to R/I$ and $R\to S/J$ are finite.

Since $A\to R$ and $R\to S$ are flat with Cohen-Macaulay fibres, under our hypotheses, $A\to R/I$ and $R\to S/J$ are finite and flat, i.e., Cohen-Macaulay of relative dimension 0. Let $L = IS+J$, $W_1 = \mathrm{Spec}\, S/J\hookrightarrow X$, $W_2 = \mathrm{Spec}\, R/I\hookrightarrow Y$, and $W = W_1\cap f^{-1}(W_2) = \mathrm{Spec}\, S/L\hookrightarrow X$.

In what follows, for $M\in\mathrm{Mod}_R$ and $N\in\mathrm{Mod}_S$ we make the standard identifications, $\mathrm{H}^d_I(M) = \mathrm{H}^d_{W_2}(Y,\widetilde{M})$, $\mathrm{H}^e_J(N) = \mathrm{H}^e_{W_1}(X,\widetilde{N})$, and $\mathrm{H}^{d+e}_L(N) = \mathrm{H}^{d+e}_W(\widetilde{N})$. We remind the reader that $\omega^{\bullet}_{R/A} = \omega^{\#}_{R/A}[d]$, $\omega^{\bullet}_{S/R} = \omega^{\#}_{S/R}[e]$, and $\omega^{\bullet}_{S/A} = \omega^{\#}_{S/A}[d+e]$.

Finally, let $\widehat{R}$ be the I-adic completion of R, $\widehat{S}$ the L-adic completion of S, and S^* the J-adic completion of S. Let $\widehat{J} = J\widehat{S}$, $\widehat{L} = L\widehat{S}$, and $\widehat{I} = I\widehat{R}$. Note that $\widehat{R}\to\widehat{S}/\widehat{J}$ is finite. Let $\omega^{\#}_{\widehat{S}/A} = \omega^{\#}_{S/A}\otimes_S\widehat{S}$, $\omega^{\#}_{\widehat{S}/\widehat{R}} = \omega^{\#}_{S/R}\otimes_S\widehat{S}$ and $\omega^{\#}_{\widehat{R}/A} = \omega^{\#}_{R/A}\otimes_R\widehat{R}$ The maps $\mathrm{Tr}_{(\widehat{S},\widehat{L})/A}$, $\mathrm{Tr}_{(\widehat{S},\widehat{J})}/\widehat{R}$, and $\mathrm{Tr}_{\widehat{R}/A}$ give rise, on applying the cohomology functor $\mathrm{H}^0(-)$ to maps $\mathrm{tr}^{\#}_{\widehat{S}/A}\colon \mathrm{H}^{d+e}_{\widehat{L}}(\omega^{\#}_{\widehat{S}/A})\to A$, $\mathrm{tr}^{\#}_{\widehat{S}/\widehat{R}}\colon \mathrm{H}^e_{\widehat{J}}(\omega^{\#}_{\widehat{S}/\widehat{R}})\to\widehat{R}$, and $\mathrm{tr}^{\#}_{\widehat{R}/A}\colon \mathrm{H}^d_{\widehat{I}}(\omega^{\#}_{\widehat{R}/A})\to A$.

PROPOSITION 6.3.1. *Let notations be as above.*

(a) *The following diagram commutes, with* $\chi = \chi_{[g,f]}$*:*

$$\begin{tikzcd}
\mathrm{H}^{d+e}_{L}(\omega^{\#}_{R/A} \otimes_R \omega^{\#}_{S/R}) \arrow[rr, "\mathrm{H}^{d+e}_{L}(\chi)"] \arrow[d, "(6.2.4)"', "\wr"] & & \mathrm{H}^{d+e}_{L}(\omega^{\#}_{S/A}) \arrow[dd, "\mathbf{res}^{\#}_{W}"] \\
\mathrm{H}^{d}_{I}(\omega^{\#}_{R/A} \otimes_R \mathrm{H}^{e}_{J}(\omega^{\#}_{S/R})) \arrow[d, "\text{via } \mathbf{res}^{\#}_{W_1}"'] & & \\
\mathrm{H}^{d}_{I}(\omega^{\#}_{R/A}) \arrow[rr, "\mathbf{res}^{\#}_{W_2}"'] & & A
\end{tikzcd}$$

(b) *Let* $\widehat{\chi} = \chi_{[\widehat{S}/\widehat{R}/A]}$*. Then the following diagram commutes:*

$$\begin{tikzcd}
\mathrm{H}^{d+e}_{\widehat{L}}(\omega^{\#}_{\widehat{R}/A} \otimes_{\widehat{R}} \omega^{\#}_{\widehat{S}/\widehat{R}}) \arrow[rr, "\text{via } \widehat{\chi}"] \arrow[d, "(6.2.4)"', "\wr"] & & \mathrm{H}^{d+e}_{\widehat{L}}(\omega^{\#}_{\widehat{S}/A}) \arrow[dd, "\mathrm{tr}^{\#}_{\widehat{S}/A}"] \\
\mathrm{H}^{d}_{\widehat{I}}(\omega^{\#}_{\widehat{R}/A} \otimes_{\widehat{R}} \mathrm{H}^{e}_{\widehat{J}}(\omega^{\#}_{\widehat{S}/\widehat{R}})) \arrow[d, "\text{via } \mathrm{tr}^{\#}_{\widehat{R}/A}"'] & & \\
\mathrm{H}^{d}_{\widehat{I}}(\omega^{\#}_{\widehat{R}/A}) \arrow[rr, "\mathrm{tr}^{\#}_{\widehat{R}/A}"'] & & A
\end{tikzcd}$$

PROOF. Part (a) is mainly a re-statement of Proposition 5.2.17 (b), with Proposition 6.2.7 explaining how (6.2.4) enters into the picture. Before giving more details, we make some observations. First, let J_n denote the S-ideal generated by $(v_1^n, \dots, v_e^n)$. Then S/J_n is finite and flat over R, and hence is Cohen-Macaulay of relative dimension 0 over R. This means that the relative dualizing module for the algebra $R \to S/J_n$, i.e., $\omega^{\#}_{S/R} \otimes_S \wedge^e_S J_n/J_n^2$, is flat over R, whence so its direct limit over n, namely $\mathrm{H}^e_J(\omega^{\#}_{S/R})$. Since f, g and gf are Cohen-Macaulay of relative dimensions e, d, and $d+e$ respectively, the maps $\phi_{S,J}(\omega^{\#}_{S/R})$, $\phi_{R,I}(\omega^{\#}_{R/A})$, and $\phi_{S,L}(\omega^{\#}_{S/A})$ are all isomorphisms. Moreover, since $\mathrm{H}^e_J(\omega^{\#}_{S/R})$ is R-flat, the map $\phi_{R,I}(\omega^{\#}_{R/A} \otimes_R \mathrm{H}^e_J(\omega^{\#}_{S/R}))$ is also an isomorphisms.

Since $\omega^{\#}_{S/R}$ and $\mathrm{H}^e_J(\omega^{\#}_{S/R})$ are both flat over R, we can apply Proposition 6.2.7 with $M = \omega^{\#}_{R/A}$, $N = \omega^{\#}_{S/R}$. Using the isomorphisms $\phi_{S,J}(\omega^{\#}_{S/R})$, $\phi_{R,I}(\omega^{\#}_{R/A})$, $\phi_{S,L}(\omega^{\#}_{S/A})$, $\phi_{R,I}(\omega^{\#}_{R/A} \otimes_R \mathrm{H}^e_J(\omega^{\#}_{S/R}))$, and applying Proposition 6.2.7, our assertion is equivalent to the commutativity of the diagram in part (b) of Proposition 5.2.17. This proves (a)

The proof of (b) is identical, with part(a) of Proposition 5.2.17 replacing part (b) of loc.cit. $\square$

Proposition 6.3.1 gives rise to two related residue formulas. The following is a consequence of part (a) of the proposition and the formula for the map (6.2.4) given in (6.2.5). For $\mu \in \omega^{\#}_{R/A}$ and $\nu \in \omega^{\#}_{S/R}$ and for integers $\alpha_l > 0$, $\beta_k > 0$,

$l \in \{1, \dots, d\}$, $k \in \{1, \dots, e\}$, we have

$$\text{(6.3.2)} \qquad \mathbf{res}^{\#}_{W_2}\begin{bmatrix} \mathbf{res}^{\#}_{W_1}\begin{bmatrix} \nu \\ v_1^{\beta_1}, \dots, v_e^{\beta_e} \end{bmatrix} \mu \\ u_1^{\alpha_1}, \dots, u_d^{\alpha_d} \end{bmatrix} = \mathbf{res}^{\#}_{W}\begin{bmatrix} \chi_{[S/R/A]}(\mu \otimes \nu) \\ v_1^{\beta_1}, \dots, v_e^{\beta_e}, u_1^{\alpha_1}, \dots, u_d^{\alpha_d} \end{bmatrix}$$

Similarly, for $\mu \in \omega^{\#}_{\widehat{R}/A}$ and $\nu \in \omega^{\#}_{\widehat{S}/\widehat{R}}$ and α_l, β_k as above, we have by part (b) of the proposition, and the formula for the map (6.2.4) given in (6.2.5),

$$\text{(6.3.3)} \qquad \mathrm{tr}^{\#}_{\widehat{R}/A}\begin{bmatrix} \mathrm{tr}^{\#}_{\widehat{S}/\widehat{R}}\begin{bmatrix} \nu \\ v_1^{\beta_1}, \dots, v_e^{\beta_e} \end{bmatrix} \mu \\ u_1^{\alpha_1}, \dots, u_d^{\alpha_d} \end{bmatrix} = \mathrm{tr}^{\#}_{\widehat{S}/A}\begin{bmatrix} \chi_{[\widehat{S}/\widehat{R}/A]}(\mu \otimes \nu) \\ v_1^{\beta_1}, \dots, v_e^{\beta_e}, u_1^{\alpha_1}, \dots, u_d^{\alpha_d} \end{bmatrix}$$

REMARK 6.3.4. We will apply part (b) of the Proposition 6.3.1 in Part 2 in the following situation. Let $R = A[u_1, \dots, u_d]$, $S = R[v_1, \dots, v_e]$ where $\mathbf{u} = (u_1, \dots, u_d)$ and $\mathbf{v} = (v_1, \dots, v_e)$ are algebraically independent variables over A and R respectively. Let $I = \mathbf{u}R$, and $J = \mathbf{v}S$. Then $\widehat{R} = A[[\mathbf{u}]]$ and $\widehat{S} = R[[\mathbf{v}]] = A[[\mathbf{u}, \mathbf{v}]]$. See Theorem 12.2.4

Part 2

The concrete theory via Verdier's isomorphism

CHAPTER 7

Overview for Part 2

For this overview, unless otherwise stated, schemes are *ordinary* noetherian schemes. In the main body of Part 2, we use formal schemes as a way around compactifications of separated finite-type maps, so that complications involving compatibilities between different compactifications do not need to be addressed.

The principal aim of Part 2 is to describe explicitly the residues—and the trace, when the map in question is proper—associated with Verdier's isomorphism

$$\Omega^n_{X/Y}[n] \xrightarrow{\sim} f^!\mathscr{O}_Y$$

(see [**V**, p. 397, Thm. 3]) for a smooth separated map $f\colon X \to Y$ of relative dimension n. Recall that the foundations of Grothendieck duality (GD) that we use are the ones initiated by Deligne in [**D1**].

If the map f above is proper, there is an associated trace $\mathrm{tr}_f\colon \mathrm{R}^n f_*\Omega^n_{X/Y} \to \mathscr{O}_Y$ obtained by transferring the source of the map $\mathrm{tr}^{\#}{}_f$ (defined in (3.2.2)) to $\mathrm{R}^n f_*\Omega^n_{X/Y}$ via Verdier's isomorphism. As discussed in the introduction to the book, the task of finding a concrete expression for tr_f is not simple. In this part we take up this task, and believe we give a satisfactory answer to the problem. Briefly, any theory of traces comes with an associated theory of residues, and we show that residues associated with tr_f satisfy the formulas [**RD**, III, § 9], which are stated without proof in *loc.cit.*[1]

The prime object of study in Part 2 of this book is Verdier's isomorphism above. Strictly speaking, the isomorphism given by Verdier in [**V**, p. 397, Thm. 3] is from $f^!\mathscr{O}_Y$ to $\Omega^n_{X/Y}[n]$, and thus, we are talking about the inverse of the map in *loc.cit.* In view of recent results of Lipman and Neeman, this is the *fundamental class map* c_f *associated with* f [**LN2**, p. 152, (4.4.1)], but we use the description given in [**V**] and hence call it the Verdier isomorphism. In [**L3**], Lipman outlines a programme for a global residue theorem via the fundamental class map (see [*ibid.*, § 5.5 and § 5.6]). Part 2 is intimately related to that programme via the just mentioned results of Lipman and Neeman. However, we do not use the results on the fundamental class map of *Ibid.* Since the isomorphism we use (between $\Omega^n_{X/Y}[n]$ and $f^!\mathscr{O}_Y$) is that described by Verdier, we call it the Verdier isomorphism rather than the fundamental class.

We now give a more more detailed description of the contents of Part 2. We are mainly concerned with three (intertwined) aspects:

1. Understanding the abstract traces

$$\mathrm{Tr}_f(\mathscr{O}_Y)\colon \mathbf{R}f_*f^!\mathscr{O}_Y \to \mathscr{O}_Y$$

[1] though all the formulas stated in [**RD**] (labelled (R1)–(R10) there) have been proved with great care by Conrad in [**C1**, Appendix A].

and

$$\mathrm{Tr}_{f,Z}(\mathscr{O}_Y)\colon \mathbf{R}_Z f_* f^! \mathscr{O}_Y \to \mathscr{O}_Y$$

in concrete terms (using differential forms via Verdier's isomorphism) when $f\colon X \to Y$ is smooth and separated. The first map is meaningful when f is proper, as the co-adjoint unit for the adjoint pair $(\mathbf{R}f_*, f^!)$ (cf. (1.1.2)). The second is meaningful when Z is a closed subscheme of X proper over Y (cf. (2.3.1)). If $Z = X$, $\mathrm{Tr}_{f,Z} = \mathrm{Tr}_f$. In fact, we will concentrate on the case when Z is finite over Y, in which case we are talking about abstract residues. The aim to is realise these abstractions concretely when we substitute $\Omega^n_{X/Y}[n]$ for $f^!\mathscr{O}_Y$ via Verdier's isomorphism (n being the relative dimension of f). Understanding $\mathrm{Tr}_{f,Z}$ for such Z is tantamount to understanding Tr_f for f proper via the so-called Residue Theorem.

2. Making the abstract transitivity map

$$\chi_{[g,f]}\colon \mathbf{L}f^*g^!\mathscr{O}_Z \overset{\mathbf{L}}{\otimes}_{\mathscr{O}_X} f^!\mathscr{O}_Y \longrightarrow (gf)^!\mathscr{O}_Z$$

of [**L4**, § 4.9] and (2.3.1) concrete in terms of differential forms (again using Verdier's isomorphism) when $f\colon X \to Y$ and $g\colon Y \to Z$ are separated finite-type maps in certain situations. Our main interest is in the following two situations:

(i) The maps f and g are smooth, say of relative dimensions m and n respectively, and we use Verdier's isomorphisms to identify $g^!\mathscr{O}_Z$, $f^!\mathscr{O}_Y$, and $(gf)^!\mathscr{O}_Z$ with $\Omega^n_{Y/Z}[n]$, $\Omega^m_{X/Y}[m]$, and $\Omega^{m+n}_{X/Z}[m+n]$ respectively. This is closely related to the results in [**LS**].

(ii) The map f is a closed immersion say of codimension d, and the maps g and gf are smooth, say of relative dimensions $n+d$ and n respectively.

In fact these two cases are essentially enough to develop a theory of residues which give the formulas (R1) to (R10) in [**RD**, Chap. III, § 9].

3. Finding a concrete expression for the abstract trace map

$$h_*f^!\mathscr{O}_Z \cong h_*h^!g^!\mathscr{O}_Z \xrightarrow{\mathrm{Tr}_h} g^!\mathscr{O}_Z$$

where $f\colon X \to Z$ and $g\colon Y \to Z$ are smooth separated maps, $h\colon X \to Y$ is a finite flat map and f, g, h satisfy $f = g \circ h$. This concrete expression is in terms of differential forms (via our now familiar way of identifying $f^!\mathscr{O}_Z$ and $g^!\mathscr{O}_Z$ with differential forms). In fact we show that it is the trace of Lipman and Kunz, defined in [**Ku**, § 16]. One consequence is that if $f\colon X \to Y$ is an equidimensional map of relative dimension n such that X and Y are excellent with no embedded points and the smooth locus of f is dense in X, then $\mathrm{H}^{-n}(f^!\mathscr{O}_Y)$ can be identified, via the Verdier isomorphism on the smooth locus of f, with a coherent subsheaf of the sheaf of meromorphic differentials $\wedge^n_{k(X)}\Omega^1_{k(X)/k(Y)}$, namely the sheaf of regular differentials of Kunz and Waldi [**KW**, § 3, § 4]. One therefore recovers the main results in [**HK1**], [**HK2**], [**HS**], and [**LS**] via our approach.

4. Giving explicit formulas for residues in important cases, and complete proofs of the residue formulas R1–R10 in [**RD**, III, §9] using our chosen foundations of GD. The proofs of R1-R10 are given in Chapter 15.

We elaborate on these points in the rest of this introduction.

7.1. The twisted image pseudofunctor $-^!$

GD is concerned with constructing a variance theory, i.e., a pseudofunctor, "upper shriek", which we denote $-^!$, on a suitable subcategory of schemes and finite type maps.[2] For a fixed scheme Z, $Z^!$ is a suitable full subcategory of $\mathbf{D}(Z)$ containing $\mathbf{D}_c(Z)$. We will say more about these subcategories later. For now we wish to paint with broad strokes. Whichever way one approaches the foundations of GD, the resulting pseudofunctor $-^!$ should be *local* (more on that in a moment), stable under, at least, flat base change, and such that when f is *proper*, $f^!$ is right adjoint to $\mathbf{R}f_*$. By *local*, this is what we mean: If $U \to Y$ is an open Y-subscheme of $g\colon V \to Y$ as well as of $h\colon W \to Y$ (g, h of finite type), then $g^!|_U$ and $h^!|_U$ are canonically isomorphic — canonical enough that if we have a third finite type Y-scheme $f\colon X \to Y$ which contains U as an open Y-subscheme, then the isomorphisms between $f^!|_U$, $g^!|_U$, and $h^!|_U$ are compatible. All of this (and much more) can be found in [**L4**] for the theory of $-^!$ initiated in [**D1**]. For schemes with finite Krull dimension, the local nature of upper-shriek was proved by Deligne in [**D1**], and using his flat base change result, by Verdier in [**V**].

Additonally, one wants a theory which specializes to the familiar Serre duality for smooth complete varieties, with the top differential forms playing a critical dualizing role. For a slightly more general situation, this means that from the theory of upper shriek one should recover the duality isomorphisms

$$\mathscr{E}xt^i_f(\mathscr{V},\, \Omega^d_{X/Y}) \cong \mathscr{H}om_Y(\mathrm{R}^{d-i}f_*\mathscr{V},\, \mathscr{O}_Y) \qquad (0 \le i \le d)$$

when $f\colon X \to Y$ is *smooth and proper* of relative dimension d, $\mathscr{V}$ is a finite rank vector bundle on X. This amounts to showing that $f^!\mathscr{O}_Y \cong \Omega^d_{X/Y}[d]$ for such a smooth map f.

7.2. Traces and residues

Recall that if $f\colon X \to Y$ is separated of finite type and Z is closed subscheme of X which is *proper* over Y, then one has a map (the *trace of f along Z*)

$$\mathrm{Tr}_{f,Z}\colon \mathbf{R}_Z f_* f^! \to \mathbf{1}$$

defined in (2.3.1). Note that when $Z = X$ (so that f is proper), then $\mathrm{Tr}_{f,Z} = \mathrm{Tr}_f$.

If f is smooth of relative dimension n, Z as above is cut out by a sequence $\mathbf{t} = (t_1, \dots, t_n) \in \Gamma(X,\, \mathscr{O}_X)$ with $Z \to Y$ finite, and Z_m is the thickening of Z defined by $\mathbf{t}^m = (t_1^m, \dots, t_n^m)$, and say $Y = \operatorname{Spec} A$, then Verdier's isomorphism $\Omega^n_{X/Y} \xrightarrow{\sim} f^!\mathscr{O}_Y$ gives us maps (one for each m)

$$\mathrm{Ext}^n_X(\mathscr{O}_{Z_m},\, \Omega^n_{X/Y}) \longrightarrow A.$$

As Verdier argued in [**V**, bottom of p.399], by passing to the completion of a localisation of A (via flat base change), and making étale base changes, to know the above map (for any m) is to know Tr_f when f is proper. If one passes to the direct limit as $m \to \infty$, then we get a map

$$\mathrm{H}^n_Z(X,\, \Omega^n_{X/Y}) \longrightarrow A.$$

[2] One can have essentially finite-type maps.

The above map is easily seen to be $\mathrm{H}^0(-)$ applied to the composite

$$\mathbf{R}\Gamma_Z(X, \Omega^n_{X/Y}[n]) \xrightarrow{\sim} \mathbf{R}\Gamma_Z(X,\, f^!\mathscr{O}_Y) \xrightarrow{\mathrm{Tr}_{f,Z}} A. \tag{7.2.1}$$

This map, which we denote $\mathbf{res}_Z$, also determines Tr_f if f is proper. We prefer to work with cohomology with supports (rather than with $\mathrm{Ext}^n_X(\mathscr{O}_Z,\, \Omega^n_{X/Y})$), following the general philosophy underlying Lipman's body of work, especially [**L2**]. Berthelot in [**Ber**] also makes the connection between the map on $\mathrm{Ext}^n_X(\mathscr{O}_Z, \Omega^n_{X/Y})$ and the map on cohomology with supports. However Berthelot uses the foundations of GD based on residual complexes.

The relationship between upper shriek and the associated traces is intimate. *To assert that one has a concrete understanding of upper shriek in a particular situation is tantamount to asserting that one understands* $\mathrm{Tr}_{f,Z}$ *for a certain class of closed subschemes* Z *which are proper over the base scheme* Y. For example if f is a Cohen-Macaulay map (i.e., a flat finite type map with Cohen-Macaulay fibres), then to "know" $\mathrm{Tr}_{f,Z}(\mathscr{O}_Y)$ for Z which are finite and flat over Y is to "know" duality for f.

Returning to the case we are discussing (f smooth of relative dimension n), suppose $Z \hookrightarrow X$ is a closed immersion cut out by a sequence of global sections $\mathbf{t} = (t_1, \dots, t_n)$ of $\mathscr{O}_X$, and $Y = \operatorname{Spec} A$. Assume $Z \to Y$ is an isomorphism and further assume that Z is contained in an affine open subscheme $U = \operatorname{Spec} B$ of X (something that can be achieved by shrinking Y, since $Z \to Y$ is an isomorphism). Let

$$\mathbf{res}_Z : \mathrm{H}^n_Z(X,\, \Omega^n_{X/Y}) = \mathrm{H}^n_Z(U,\, \Omega^n_{U/Y}) \longrightarrow A$$

be $\mathrm{H}^0((7.2.1))$. It is well known that elements of $\mathrm{H}^n_{\mathbf{t}B}(\Omega^n_{B/A})$ are finite B-linear combinations of elements of the form $\begin{bmatrix} dt_1\wedge\cdots\wedge t_n \\ t_1^{\beta_1},\dots,t_n^{\beta_n}\end{bmatrix}$ with β_i positive integers. Ideally one would like

$$\mathbf{res}_Z \begin{bmatrix} dt_1 \wedge \cdots \wedge t_n \\ t_1^{\beta_1}, \dots, t_n^{\beta_n} \end{bmatrix} = \begin{cases} 1 & \text{when } \beta_i = 1 \text{ for all } i = 1, \dots, n \\ 0 & \text{otherwise.} \end{cases} \tag{7.2.2}$$

The exact answer depends on the isomorphism $f^!\mathscr{O}_Y \xrightarrow{\sim} \Omega^n_{X/Y}[n]$ chosen. This is at the heart of this part of the book, since our choice is the isomorphism Verdier gives in [**V**, p. 397, Thm. 3]. In fact we show that Verdier's isomorphism does give the above formula in the case being considered, i.e., when Z is a section of f. This is the critical case, and we deduce other residue formulas from this one by either making étale base changes, or base changing f by itself and using the diagonal section $X \hookrightarrow X \times_Y X$ of the first projection (which is to be thought of as the base change of f).

We could obtain the above explicit description of $\mathbf{res}_Z$ when Z is a section of f because of the results in [**S2**]. The main results there state that if $f\colon X \to Y$ is Cohen-Macaulay of relative dimension d, then for any base change $u\colon Y' \to Y$, there is a natural isomorphism $\theta^f_u\colon v^*\omega^{\#}_f \xrightarrow{\sim} \omega^{\#}_g$, where $v\colon X \times_Y Y' \to X$ and $g\colon X \times_Y Y' \to Y'$ are the respective projections. When f is proper, this isomorphism is compatible with traces. If f is smooth (proper or not), then this isomorphism when transferred to $v^*\Omega^n_{X/Y}[n]$ and $\Omega^n_{X\times_Y Y'/Y'}[n]$ is the identity map under the standard identification of differential forms. These are very similar to the main results in [**C1**]. The difference is that in [**S2**] the foundations of GD are based on the one initiated by Deligne in [**D1**], whereas in [**C1**] it is the based on residual complexes. In [**C1**] the identification of differential forms is built into the definition of the base change isomorphism between $v^*\omega_f^{\#}$ and $\omega_g^{\#}$, since the strategy is to embed X into schemes smooth over Y. The challenge in [**C1**] is to show that the result is compatible with traces when f is proper.

In our approach to finding explicit formulas for $\mathbf{res}_Z$, the role played by θ_u^f, *when u is non-flat*, is crucial. Roughly speaking, Verdier's isomorphism can be regarded as the residue formula $\mathbf{res}_\Delta\left[\begin{smallmatrix}\mathrm{d}s_1\wedge\cdots\wedge\mathrm{d}s_n\\ s_1,\dots,s_n\end{smallmatrix}\right]=1$ for the diagonal section Δ in $X\times_Y X$ where the diagonal is cut out by $\mathbf{s}=(s_1,\dots,s_n)$ in $X\times_Y X$. If $Z\hookrightarrow X$ is a section of f, cut out by $t_1,\dots,t_n\in\Gamma(X,\mathscr{O}_X)$ then pulling back the diagonal via the base change $Z\to X$, we get

$$\mathbf{res}_Z\begin{bmatrix}\mathrm{d}t_1\wedge\cdots\wedge\mathrm{d}t_n\\ t_1,\dots,t_n\end{bmatrix}=1. \tag{7.2.3}$$

We can do this because Verdier's isomorphism is compatible with arbitrary base change – the result in [**S2**, p.740, Thm. 2.3.5 (b)] that we alluded to above. The formula (7.2.3) says that (7.2.2) is true when all the β_i are 1. If $X=\mathbb{P}^n_Y$, f the standard projection $\mathbb{P}^n_Y\to Y$, and $Z=\cap_{i=1}^n\{T_i\neq 0\}$, where T_i, $i=0,\dots,n$ are homogeneous co-ordinates on $\mathbb{P}^n_Y$ (and $t_i=T_i/T_0$, for $i=1,\dots,n$), then one can show easily that (7.2.3) implies (7.2.2). The crucial ingredient needed is the simple and elegant computation of Lipman in [**L2**, pp.79–80, Lemma (8.6)]. The proof is essentially carried out in the proof of Proposition 10.2.3 (ii). Since $\mathbf{res}_Z$ depends only on the formal completion of X along Z, therefore if Z and $\mathbf{t}$ satisfy the hypotheses given when stating (7.2.2), then formula (7.2.2) holds. This is the first, and a very important step in our proofs in Chapter 15 of the residue formulas (R1)–(R10) of [**RD**, Chap. III, § 9].

If the closed subscheme Z of X cut out by $\mathbf{t}=(t_1,\dots,t_n)$ is finite over Y (and hence necessarily flat over Y) of constant rank (not necessarily an isomorphism), then it turns out that the right side of (7.2.3) needs to be replaced by $\mathrm{rank}(Z/Y)$.

REMARK 7.2.4. One way to think about formula (7.2.3) is to regard

$$\varphi\mapsto\mathbf{res}_Z\begin{bmatrix}\varphi\mathrm{d}t_1\wedge\cdots\wedge\mathrm{d}t_n\\ t_1,\dots,t_n\end{bmatrix},$$

(for φ a section of $\mathscr{O}_X$ in an open, or even formal, neighbourhood of Z) as the Dirac distribution along Z. Indeed, (with $Y=\mathrm{Spec}(A)$), since Z is a section of f, the completion of X along Z is the power series ring $A[|t_1,\dots,t_n|]$, and according to (7.2.3), the right side is $\varphi(0,\dots,0)$, after developing φ as a power-series in $\mathbf{t}$. If $A=\mathbb{C}$, and field of complex numbers (so that $Z=\{p\}$, a point), this can be interpreted as the fact that the Dolbeault representative of the Cauchy kernel $\mathrm{d}t_1\wedge\cdots\wedge\mathrm{d}t_n/t_1\cdots t_n$ at the point p is the Dirac distribution at p (see [**ST**]).

7.3. Transitivity

Finding concrete expressions (when we have two finite-type separable maps $f\colon X\to Y$ and $g\colon Y\to Z$) for the abstract transitivity map

$$\chi_{[g,f]}\colon \mathbf{L}f^*g^!\mathscr{O}_Z\overset{\mathbf{L}}{\otimes}_{\mathscr{O}_X}f^!\mathscr{O}_Y\longrightarrow(gf)^!\mathscr{O}_Z$$

of [**L4**, § 4.9] and Definition 5.2.16 is perhaps the most important technical task undertaken in Part 2. To establish this, we rely heavily on the abstract transitivity results on formal schemes that we established in the first part of this monograph. As we pointed out earlier (see item (**2.**) on p.58) there are two key situations where concrete manifestations of $\chi_{[-,-]}$ are important. The first situation of importance is when we have two smooth separated maps, $f\colon X \to Y$ and $g\colon Y \to Z$, say of relative dimensions m and n respectively. If we use Verdier's isomorphisms to identify $g^!\mathscr{O}_Z$, $f^!\mathscr{O}_Y$, and $(gf)^!\mathscr{O}_Z$ with $\Omega^n_{Y/Z}[n]$, $\Omega^m_{X/Y}[m]$, and $\Omega^{m+n}_{X/Z}[m+n]$ respectively, then $\chi_{[f,g]}$ transforms to the map $f^*\mu\otimes\nu \mapsto \nu\wedge f^*\mu$ (see Theorem 12.2.4). In fact we show this at the level of formal schemes, and formal schemes enter in an essential way in our proof (via the results on transitivity for residues in (6.3.2) and Theorem 13.1.1) even when X and Y are ordinary schemes. The proof is carried out in Chapter 12.

The second situation of importance occurs when the smooth map $f\colon X \to Y$ factors as $f = \pi\circ i$, where $i\colon X \hookrightarrow P$ is a closed immersion say of codimension d, and $\pi\colon P \to Y$ is smooth of relative dimension $n+d$. The concrete expression for $\chi_{[i,\pi]}$ then is governed by

$$i^*(\eta\wedge \mathrm{d}t_1\wedge\cdots\wedge \mathrm{d}t_d)\otimes \mathbf{1}/\mathbf{t} \mapsto i^*\eta$$

where η is a section of $\Omega^{n+d}_{P/Y}$, $t_i \in \Gamma(P,\, \mathscr{O}_P)$, $i = 1,\dots,d$, are sections which cut out X and $\mathbf{1}/\mathbf{t}$ is a well-defined generating section, depending upon $\mathbf{t} = (t_1,\dots,t_d)$, of the top exterior product $\wedge^d\mathscr{N}$ of the normal bundle $\mathscr{N}$ of X in P, which exterior product, by the *fundamental local isomorphism* is identified with $f^!\mathscr{O}_P[d]$. The proof of this concrete representation of $\chi_{[i,\pi]}$is carried out in §13.2.

In both situations, we need the residue formula (7.2.2) for residues along sections of smooth maps. We turn this around later, and use the concrete expressions for $\chi_{[-,-]}$ to arrive at formulas for $\mathbf{res}_Z$ for smooth maps $f\colon X\to Y$ when $Z \to Y$ is not an isomorphism (but is finite).

There is one interesting way in which (7.2.3) brings in concrete answers. Let A be a ring, and $C = A[T_1,\dots,T_n]/(f_1,\dots,f_n)$ be a finite flat algebra over A. Let $Z = \operatorname{Spec} C$, $X = \mathbb{A}^n_A$ and $Y = \operatorname{Spec} A$. Let $I = \mathbf{f}A[\mathbf{T}]$ where $\mathbf{f} = (f_1,\dots,f_n)$, so that I is the ideal of Z in the polynomial ring $A[\mathbf{T}]$. By the general calculus of generalized fractions, if $p(\mathbf{T}) \in A[\mathbf{T}]$ then the element $\begin{bmatrix} p(\mathbf{T})\mathrm{d}T_1\wedge\cdots\wedge \mathrm{d}T_n \\ f_1,\dots,f_n\end{bmatrix} \in \mathrm{H}^n_I(\Omega^n_{A[\mathbf{T}]/A})$ depends only on the image of $p(\mathbf{T})$ in C. We show that the map

$$c \mapsto \mathbf{res}_Z \begin{bmatrix} p(\mathbf{T})\mathrm{d}T_1\wedge\cdots\wedge \mathrm{d}T_n \\ f_1,\dots f_n\end{bmatrix}$$

with $p(\mathbf{T})$ a pre-image of c, is the Tate trace described in [**MR**, Appendix]. We prove this in Theorem 14.1.7, and (7.2.3) plays an important role. The point is, knowing the residue in a very special situation allows us to deduce formulas for residues in many other situations.

Perhaps the most important way that that (7.2.2) comes into play is that it characterises the Verdier isomorphism (or more accurately the fundamental class). Continuing with the situation where $f\colon X \to Y$ is smooth of relative dimension n, suppose we have some isomorphism $\psi\colon \Omega^n_{X/Y}[n] \xrightarrow{\sim} f^!\mathscr{O}_Y$. When is ψ Verdier's isomorphism? The answer is, if and only if, for every étale base change $u^*Y' \to Y$ and every section Z of the base change map $f'\colon X\times_Y Y' \to Y'$, the composite

$$\mathrm{R}^n_Z f'_*\Omega^n_{X\times_Y Y'/Y'} \xrightarrow[\text{via }\psi]{\sim} \mathrm{H}^0(\mathbf{R}_Z f'_* f'^!\mathscr{O}_{Y'}) \xrightarrow{\mathrm{Tr}_{f',Z}} \mathscr{O}_{Y'}$$

is given by (7.2.2). Note that the first isomorphism involves flat base change for $f^!$. The precise statement is given in Theorem 10.4.8. This characterisation of Verdier's isomorphism allows us to relate the fundamental class with the regular differentials of Kunz. We work this out in Chapter 11. We give a different proof later of the relationship between the fundamental class and regular differentials.

7.4. Trace for finite flat maps

Suppose the smooth map $f\colon X \to Z$ of relative dimension n can be factored as $f = g \circ h$, where $h\colon X \to Y$ is finite and $g\colon Y \to Z$ is smooth of relative dimension n (so that h is in fact flat). Then the composite $h_*f^! \cong h_*h^!g^! \xrightarrow{\mathrm{Tr}_h} g^!$, gives, via Verdier's isomorphism for f and g a map

$$\mathrm{tr}_h\colon h_*\Omega^n_{X/Z} \longrightarrow \Omega^n_{Y/Z}.$$

In [**Ku**], Kunz, based on a suggestion by Lipman (who in turn was influenced by residue formulas stated without proof in [**RD**, Chap. III, § 9]) defined an explicit trace $\sigma_h\colon h_*\Omega^n_{X/Z} \longrightarrow \Omega^n_{Y/Z}$. We show that $\mathrm{tr}_h = \sigma_h$. In fact, we use the two concrete versions of transitivity that we mention above. Assuming h factors as a closed immersion $i\colon X \to P$ followed by a smooth map $\pi\colon P \to Y$, where P is an open subscheme of $\mathbb{A}^{n+d}_Y$ and π the structure map, (a situation we can achieve, retaining finiteness of h, if we pass to completions of local rings of points on Y) then the assertion $\mathrm{tr}_h = \sigma_h$ amounts to the compatibilities between the abstract transitivity maps $\chi_{[h,g]}$, $\chi_{[i,\pi]}$, $\chi_{[\pi,g]}$, and $\chi_{[i,g\pi]}$ given in Proposition-Definition 5.2.4 (ii) or in [**L4**, p. 238]. The map tr_h occurs in formula (R10) for residues, and it is satisfying that there is a more explicit description of it in terms of the Kunz-Lipman trace σ_h.

7.5. Regular Differential Forms

The regular differentials of Kunz and Waldi developed in [**KW**] is a vast generalization of Rosenlicht's differentials for singular curves [**R**]. We have already alluded to the connection between the regular differential forms and Verdier's isomorphism. Regular differntial forms are defined when $f\colon X \to Y$ is a generically smooth equidimensional map between excellent schemes having no embedded points. In such a case, if X^{sm} is the smooth locus of f, and $f^{\mathrm{sm}}\colon X^{\mathrm{sm}} \to Y$ the restriction of f, there is an isomorphism $\Omega^n_{X^{\mathrm{sm}}/Y}[n] \to (f^{\mathrm{sm}})^!$. The isomorphism is based on the construction of regular differentials in [**KW**] and the principal results of [**HS**]. What we show in this part of the book is that this isomorphism is Verdier's isomorphism. We give two proofs. The first uses the characterisation of Verdier's isomorphism via (7.2.2) that we alluded to before. The other, more satisfying, proof relies on the equality of traces $\mathrm{tr}_h = \sigma_h$ for finite flat maps h between schemes smooth over a base that we discussed above. (See §14.3.)

CHAPTER 8

Verdier's isomorphism

8.1. The Definition

Let $f\colon \mathscr{X} \to \mathscr{Y}$ be a smooth map of relative dimension r between (formal) schemes. Assume f is a composite of compactifiable maps. Set $\mathscr{X}'' := \mathscr{X} \times_{\mathscr{Y}} \mathscr{X}$ and let $\Delta\colon \mathscr{X} \to \mathscr{X}''$ be the diagonal immersion, which is closed by our hypotheses. Denote by p_1 and p_2 the two projections from $\mathscr{X}''$ on to $\mathscr{X}$, and by $\mathscr{N}_\Delta$ the locally free $\mathscr{O}_{\mathscr{X}}$-module corresponding to the "normal bundle" of the regular immersion Δ. In other words, if $\mathscr{I}_\Delta$ is the ideal sheaf of $\mathscr{X}$ in $\mathscr{X}''$, then $\mathscr{N}_\Delta = (\mathscr{I}_\Delta/\mathscr{I}_\Delta^2)^*$, the dual of $\mathscr{I}_\Delta/\mathscr{I}_\Delta^2$. As in (C.2.8) and (C.2.10), set

$$\mathscr{N}_\Delta^r = \wedge^r_{\mathscr{O}_{\mathscr{X}}} \mathscr{N}_\Delta$$

and

$$\Delta^{\blacktriangle} = \mathbf{L}\Delta^*(-) \overset{\mathbf{L}}{\otimes}_{\mathscr{O}_{\mathscr{X}}} \mathscr{N}_\Delta^r[-r].$$

We then have an isomorphism

$$f^{\#}\mathscr{O}_{\mathscr{Y}} \overset{\mathbf{L}}{\otimes}_{\mathscr{O}_{\mathscr{X}}} \mathscr{N}_\Delta^r[-r] \xrightarrow{\ \sim\ } \mathscr{O}_{\mathscr{X}} \tag{8.1.1}$$

defined by the commutativity of the following diagram:

$$\begin{tikzcd}
f^{\#}\mathscr{O}_{\mathscr{Y}} \overset{\mathbf{L}}{\otimes}_{\mathscr{O}_{\mathscr{X}}} \mathscr{N}_\Delta^r[-r] \arrow[d, "\wr"', "(8.1.1)"'] & \Delta^{\blacktriangle} p_2^* f^{\#}\mathscr{O}_{\mathscr{Y}} \arrow[l, "\sim"'] \arrow[r, "\sim", "(2.2.2)"'] & \Delta^{\blacktriangle} p_1^{\#}\mathscr{O}_{\mathscr{X}} \arrow[d, "\wr"', "\eta'_\Delta"] \\
\mathscr{O}_{\mathscr{X}} & & \Delta^{\#} p_1^{\#}\mathscr{O}_{\mathscr{X}} \arrow[ll, "\sim"']
\end{tikzcd} \tag{8.1.2}$$

The map η'_Δ is as in (C.2.13). The unlabelled arrow on the top row is the one arising from $\mathbf{L}\Delta^* p_2^* \xrightarrow{\ \sim\ } \mathbf{1}$ and the one on the bottom row from the functorial isomorphism $\Delta^{\#} p_1^{\#} \xrightarrow{\ \sim\ } \mathbf{1}^{\#}_{\mathscr{X}}$.

Writing $\mathscr{L}^*$ for the dual of an invertible $\mathscr{O}_{\mathscr{X}}$-module $\mathscr{L}$ we see that $\mathscr{N}_\Delta^r = \omega_f^*$.

Using this and (8.1.1) one deduces, as Verdier did in [**V**, p. 397, Theorem 3], that $f^{\#}\mathscr{O}_{\mathscr{Y}}$ and $\omega_f[r]$ are isomorphic. However, there is some ambiguity about the exact isomorphism (see the discussion around (7.2) in p. 758 of [**S2**]). But at the very least we note that f is Cohen-Macaulay. We give the isomorphism we will work with in Definition 8.1.5 after some necessary preliminaries.

As usual, let $\omega_f^{\#} = \mathrm{H}^{-r}(f^{\#}\mathscr{O}_{\mathscr{Y}})$ and make the identification

$$f^{\#}\mathscr{O}_{\mathscr{Y}} = \omega_f^{\#}[r].$$

Applying H^0 to (8.1.1) we get an isomorphism

$$\omega_f^{\#} \otimes_{\mathscr{O}_{\mathscr{X}}} \omega_f^* \xrightarrow{\ \sim\ } \mathscr{O}_{\mathscr{X}}. \tag{8.1.3}$$

Let

$$\bar{\mathbf{v}}_f\,(=\bar{\mathbf{v}})\colon \omega_f \xrightarrow{\sim} \omega_f^{\#} \tag{8.1.4}$$

be the canonical isomorphism induced by (8.1.3).

Definition 8.1.5. The *Verdier isomorphism* for the smooth map f is the isomorphism

$$\mathbf{v}_f\,(=\mathbf{v})\colon \omega_f[r] \xrightarrow{\sim} \omega_f^{\#}[r] = f^{\#}\mathscr{O}_{\mathscr{Y}}$$

given by $\mathbf{v}_f = \bar{\mathbf{v}}_f[r]$.

We will often refer to $\bar{\mathbf{v}}_f$ also as the Verdier isomorphism. Indeed $\mathbf{v}_f$ and $\bar{\mathbf{v}}_f$ determine each other.

Remark 8.1.6. The isomorphism $p_2^* f^{\#}\mathscr{O}_{\mathscr{Y}} \xrightarrow{\sim} p_1^{\#}\mathscr{O}_{\mathscr{X}}$ of (2.2.2) induces (on applying H^0) an isomorphism

$$\theta\colon p_2^*\omega_f^{\#} \xrightarrow{\sim} \omega_{p_1}^{\#}.$$

Note that the original isomorphism $p_2^* f^{\#}\mathscr{O}_{\mathscr{Y}} \xrightarrow{\sim} p_1^{\#}\mathscr{O}_{\mathscr{X}}$ is $\theta[r]$ under the identifications we have agreed to make throughout, namely, $f^{\#}\mathscr{O}_{\mathscr{Y}} = \omega_f^{\#}[r]$ and $p_1^{\#}\mathscr{O}_{\mathscr{X}} = \omega_{p_1}^{\#}[r]$. Applying the functor H^0 to the commutative diagram (8.1.2) we get the following commutative diagram, showing the relationship between the pairing (8.1.3) and maps of the form $\tau_h^{\#}$ defined in (3.4.2) (below, h is the identity map).

$$\begin{tikzcd}
\omega_f^{\#}\otimes_{\mathscr{O}_{\mathscr{X}}}\omega_f^{*} \arrow[r, equal] \arrow[d, "(8.1.3)"', "\wr"] & \Delta^*(p_2^*\omega_f^{\#}\otimes_{\mathscr{O}_{\mathscr{X}''}}\Delta_*\omega_f^{*}) \arrow[d, "\text{via }\theta", "\wr"'] \\
\mathscr{O}_{\mathscr{X}} & \Delta^*(\omega_{p_1}^{\#}\otimes_{\mathscr{O}_{\mathscr{X}''}}\Delta_*\omega_f^{*}) \arrow[l, "\sim"', "\tau^{\#}_{\mathbf{1},p_1,\Delta}"]
\end{tikzcd} \tag{8.1.6.1}$$

Here the map on the bottom row is as in (3.4.2), with $h = \mathbf{1}_{\mathscr{X}}$, $i = \Delta$, and $f = p_1$. It is an isomorphism because $h = \mathbf{1}_{\mathscr{X}}$ is an isomorphism.

Definition 8.1.7. Suppose $f\colon \mathscr{X} \to \mathscr{Y}$ is pseudo-proper and smooth of relative dimension r. The *Verdier integral* (or simply the *integral*)

$$\mathrm{tr}_f\colon \mathrm{R}'^{r}_{\mathscr{X}} f_*\omega_f \to \mathscr{O}_{\mathscr{Y}} \tag{8.1.8}$$

is the composite $\mathrm{R}'^{r}_{\mathscr{X}} f_*\omega_f \xrightarrow{\mathbf{v}} \mathrm{R}'^{r}_{\mathscr{X}} f_*\omega_f^{\#} \xrightarrow{\mathrm{tr}_f^{\#}} \mathscr{O}_{\mathscr{Y}}$. If in the above situation, $\mathscr{X} = \mathrm{Spf}(R,\, J)$ and $\mathscr{Y} = \mathrm{Spf}(A, I)$, then we write

$$\mathrm{tr}_{R/A}\colon \mathrm{H}^r_J(\omega_{R/A}) \to A \tag{8.1.9}$$

for the global sections of tr_f. Here $\omega_{R/A}$ is the r-th exterior power of the universally finite module of differentials for the A-algebra R. If we wish to emphasise the adic structure on R and A, we will use the inconvenient notation $\mathrm{tr}_{(R,J)/(A,I)}$ for $\mathrm{tr}_{R/A}$.

Remark 8.1.10. While we have defined tr_f in general, our interest is really in the case where $\mathrm{R}'^{j}_{\mathscr{X}}(\mathscr{F}) = 0$ for every $j > r$ and every $\mathscr{F} \in \mathcal{A}_{\vec{c}}(\mathscr{X})$, for then $(\omega_f, \mathrm{tr}_f)$ represents the functor $\mathrm{Hom}_{\mathscr{Y}}(\mathrm{R}'^{r}_{\mathscr{X}}(-),\, \mathscr{O}_{\mathscr{Y}})$ on coherent $\mathscr{O}_{\mathscr{X}}$-modules (see Corollary 3.2.4). Even here the notion is most useful in this part of the book when $\mathscr{Y}$ is an ordinary scheme and either $\mathscr{X}$ is also ordinary (and hence proper over $\mathscr{Y}$) or else $\mathscr{Y} = \mathrm{Spec}\, A$ and $\mathscr{X} = \mathrm{Spf}\, R$ where R is an adic ring, one of whose defining ideals I is generated by a quasi-regular sequence of length r and such that R/I is finite and flat over A.

8.2. Local description of Verdier's isomorphism

In the above situation suppose $\mathscr{X} = \mathrm{Spf} R$, $\mathscr{Y} = \mathrm{Spf} A$, so that $\mathscr{X}'' = \mathrm{Spf} R''$ where $R'' = R\widehat{\otimes}_A R$ is the complete tensor product of R with itself over A. The diagonal map $\Delta\colon \mathscr{X} \hookrightarrow \mathscr{X}''$ corresponds to the surjective map $R'' \to R$ given by $t_1 \otimes t_2 \mapsto t_1 t_2$. Let us assume that the kernel of this map, i.e., the ideal I of the diagonal immersion, is generated by r elements $\{s_1, \ldots, s_r\}$. Since R is smooth over A of relative dimension r, the sequence $\mathbf{s} = (s_1, \ldots, s_r)$ is necessarily an R''-sequence. This condition on the diagonal is locally (in $\mathscr{X}$ and $\mathscr{X}''$) always achievable.

Let R_1 and R_2 be the two R-algebra structures on R'' corresponding to the projections $p_i\colon \mathscr{X}'' \to X$, $i = 1, 2$. For specificity, if $a \in R$, then the R-algebra structure on R_1' is given by $a(b\otimes c) = (ab)\otimes c$ whilst on R_2 it is given by $a(b\otimes c) = b \otimes (ac)$. Let $\omega^{\#}_{R/A}$, $\omega^{\#}_{R_i/A}$, $\omega_{R/A}$, $\omega_{R_i/A}$ be the global sections of $\omega^{\#}_f$, $\omega^{\#}_{p_i}$, ω_f, ω_{p_i} respectively, where $i \in \{1, 2\}$. Similarly, *Verdier's isomorphism* in this context is the isomorphism

$$\bar{\mathbf{v}}_{R/A} : \omega_{R/A} \xrightarrow{\ \sim\ } \omega^{\#}_{R/A}$$

obtained by taking global sections of $\bar{\mathbf{v}}_f : \omega_f \xrightarrow{\ \sim\ } \omega^{\#}_f$.

The isomorphism (8.1.3) is equivalent to the isomorphism of finitely generated R-modules obtained by taking global sections:

$$\omega^{\#}_{R/A} \otimes_R \omega^{*}_{R/A} \xrightarrow{\ \sim\ } R. \tag{8.2.1}$$

Here is the promised local description of Verdier's isomorphism. The module of differentials $\omega_{R/A} = \wedge^r_R I/I^2$ is a free rank-one R-module with generator

$$\overline{\mathrm{d}\mathbf{s}} := (s_1 + I^2) \wedge \cdots \wedge (s_r + I^2).$$

Let $\mathbf{1}/\mathbf{s}$ be the element of $(\wedge^r_R I/I^2)^* = \omega^*_{R/A} = \mathrm{Hom}_R(\omega_{R/A}, R)$ which sends $\overline{\mathrm{d}\mathbf{s}}$ to 1, i.e., it is the generator of the rank one free module $(\wedge^r_R I/I^2)^*$ which is dual to $\overline{\mathrm{d}\mathbf{s}}$.

PROPOSITION 8.2.2. *In the above situation we have the following:*

(a) *Let $\nu_0(\mathbf{s}) \in \omega^{\#}_{R/A}$ be the unique element such that $\nu_0(\mathbf{s}) \otimes \mathbf{1}/\mathbf{s} \mapsto 1$ under (8.2.1). Verdier's isomorphism $\bar{\mathbf{v}}_{R/A} : \omega_{R/A} \xrightarrow{\ \sim\ } \omega^{\#}_{R/A}$ is given by the formula*

$$\bar{\mathbf{v}}_{R/A}(r\,\overline{\mathrm{d}\mathbf{s}}) = r\,\nu_0(\mathbf{s}) \qquad (r \in R).$$

(b) *Suppose further that the adic rings A and R have discrete topology so that* $\mathrm{Spf}\,A = \mathrm{Spec}\,A$, $\mathrm{Spf}\,R = \mathrm{Spec}\,R$ *and $A \to R$ is of finite type. The following formula holds:*

$$\mathbf{res}^{\#}_{\Delta,\,p_1}\begin{bmatrix} \bar{\mathbf{v}}_{R_1/R}(\mathrm{d}s_1 \wedge \cdots \wedge \mathrm{d}s_r) \\ s_1,\, \ldots,\, s_r \end{bmatrix} = 1.$$

Remarks: Here $\mathrm{d}s_1 \wedge \cdots \wedge \mathrm{d}s_r \in \omega_{R_1/R}$ and the notation $\mathbf{res}^{\#}_{\Delta,\,p_1}$ is to indicate that the residue is to be taken for the map p_1 and not for p_2. The hypotheses in part (b) regarding the adic topologies on A and R is there because we need the result that Verdier's isomorphism is compatible with base change. This is one of the main results of [**S2**] (see [**S2**], p.740, Theorem 2.3.5 (b)]). Unfortunately the results in [**S2**] are for maps between ordinary schemes. Since certain special compactifications are locally used, and these are unavailable for arbitrary formal schemes, we decided it is best not pursue these issues in this book, except in the following special case.

Suppose the base change is flat. Then the proof in [**S2**] works mutatis mutandis, and we see that Verdier's isomorphism is compatible with flat base change whether we are working with ordinary schemes or formal schemes. See Theorem 8.4.1.

PROOF. Part (a) is an immediate consequence of the definition of $\bar{\mathbf{v}}_{R/A}$ in (8.1.4). It remains to prove part (b).

Let us save on notation and write

$$\theta\colon \omega^{\#}_{R/A}\otimes_R R_2 \xrightarrow{\sim} \omega^{\#}_{R_1/R}$$

for the R''-isomorphism corresponding to $\theta\colon p_2^*\omega_f^{\#} \xrightarrow{\sim} \omega^{\#}_{p_1}$ of Remark 8.1.6. Then the affine version of the commutative diagram (8.1.6.1) is the commutative diagram

(8.2.3)
$$\begin{tikzcd}
\omega^{\#}_{R/A}\otimes_R \omega^{*}_{R/A} \arrow[r, equal] \arrow[d, "(8.2.1)"', "\wr"] & (\omega^{\#}_{R/A}\otimes_R R_2)\otimes_{R''}\omega^{*}_{R/A} \arrow[d, "\wr"', "\text{via }\theta"] \\
R & \omega^{\#}_{R_1/R}\otimes_{R''}\omega^{*}_{R/A} \arrow[l, "\sim", "\boldsymbol{\tau}^{\#}_{R/R,R_1}"']
\end{tikzcd}$$

Since $\nu_0(\mathbf{s})\otimes \mathbf{1}/\mathbf{s}\mapsto 1$ under (8.2.1), from the commutative diagram (8.2.3) we get $\boldsymbol{\tau}^{\#}_{R/R,R_1}(\theta(\nu_0\otimes 1)\otimes \mathbf{1}/\mathbf{s}) = 1$. Moreover, by part (a), $\nu_0 = \bar{\mathbf{v}}_{R/A}(\overline{\mathrm{d}\mathbf{s}})$. Thus Proposition 3.5.4 gives us

$$\mathbf{res}^{\#}_{\Delta,\,p_1}\begin{bmatrix}\theta((\bar{\mathbf{v}}_{R/A}(\overline{\mathrm{d}\mathbf{s}}))\otimes 1))\\ s_1,\,\dots,\,s_r\end{bmatrix} = 1.$$

Next, if M and N are finitely generated modules over R and $\varphi\colon M\to N$ a map of R-modules, then we denote the map $\varphi\otimes_R R_i$ by $p_i^*(\varphi)$. One checks easily that in $N\otimes_R R_2$ we have the following equality for $m\in M$ and $\varphi\in \mathrm{Hom}_R(M,\,N)$:

$$(p_2^*(\varphi))(m\otimes 1) = (\varphi(m))\otimes 1.$$

It is immediate that

$$\mathbf{res}^{\#}_{\Delta,\,p_1}\begin{bmatrix}(\theta\circ(p_2^*(\bar{\mathbf{v}}_{R/A})))(\overline{\mathrm{d}\mathbf{s}}\otimes 1))\\ s_1,\,\dots,\,s_r\end{bmatrix} = 1.$$

Now if $s=\sum_i a_i\otimes b_i\in I\subset R''=R\otimes_A R$, then one checks from the definitions that as elements of the R-module $I/I^2=\Omega^1_{R/A}$, we have the equality $s+I^2=\sum_i a_i\mathrm{d}b_i$. This means in particular that in the R''-module $\Omega^1_{R/A}\otimes_R R_2 = \Omega^1_{R_1/R}$ we have $(s+I^2)\otimes 1 = \mathrm{d}s$. It is immediate from here that $\overline{\mathrm{d}\mathbf{s}}\otimes 1 = \mathrm{d}s_1\wedge\cdots\wedge\mathrm{d}s_r$.

We will be done if we can show that $\theta\circ(p_2^*(\bar{\mathbf{v}}_{R/A})) = \bar{\mathbf{v}}_{R_1/R}$. This statement about the compatibility of Verdier's isomorphism with arbitrary base change follows from [**S2**, p.740, Theorem 2.3.5] (see also [*ibid.*, pp.739–740, Remark 2.3.4]). Incidentally, this is where we need our hypothesis that our formal schemes are ordinary schemes and our map is of finite type. ☐

REMARKS 8.2.4. Two observations are worth making.

1) $\bar{\mathbf{v}} = \mathrm{H}^{-r}(\mathbf{v})$.

2) If $\mathscr{U}$ is an open subscheme of $\mathscr{X}$, and $f_{\mathscr{U}}\colon\mathscr{U}\to\mathscr{Y}$ is the structural morphism on $\mathscr{U}$, then we have a natural isomorphism $f^{\#}|_{\mathscr{U}}\xrightarrow{\sim} f^{\#}_{\mathscr{U}}$ from the main results of [**Nay**], whence an isomorphism $\omega^{\#}_f|_{\mathscr{U}}\xrightarrow{\sim}\omega^{\#}_{f_{\mathscr{U}}}$. From the definitions of $\mathbf{v}_f$ and $\mathbf{v}_{f_{\mathscr{U}}}$ it is easy to see that the composition of isomorphisms

$\omega_{f_{\mathscr{U}}} = \omega_f|_{\mathscr{U}} \xrightarrow{\ \sim\ } \omega^{\#}_f|_{\mathscr{U}} \xrightarrow{\ \sim\ } \omega^{\#}_{f_{\mathscr{U}}}$ is $\mathbf{v}_{f_{\mathscr{U}}}$, where the first arrow is $\mathbf{v}_f|_{\mathscr{U}}$ and the second the just mentioned isomorphism.

8.3. Compatibility of Verdier's isomorphism with completions

We now wish to show the compatibility of Verdier's isomorphism with completion. More precisely if $f\colon \mathscr{X} \to \mathscr{Y}$ is a smooth map and $\widehat{f}\colon \mathscr{W} \to \mathscr{Y}$ its "completion" along a closed subscheme of $\mathscr{X}$, then Verdier's isomorphism (i.e., (8.1.5)) for $\widehat{f}$ is the "completion" of the Verdier isomorphism for f. The formal statement is given in Theorem 8.3.2.

Let $f\colon \mathscr{X} \to \mathscr{Y}$ be a smooth map of relative dimension r between formal schemes. Suppose $\mathscr{I}$ is a defining ideal of $\mathscr{X}$ and $\mathscr{J} \subset \mathscr{O}_{\mathscr{X}}$ a coherent ideal containing $\mathscr{I}$ (so that $\mathscr{J}$ is the ideal of an ordinary scheme Z which is a closed subscheme of $\mathscr{X}$). Let $\mathscr{W}$ be the completion of $\mathscr{X}$ along $\mathscr{J}$ (i.e., along Z). Let $\kappa\colon \mathscr{W} \to \mathscr{X}$ be the completion map and $\widehat{f}\colon \mathscr{W} \to \mathscr{Y}$ the composite $\widehat{f} = f \circ \kappa$. We wish to show that the Verdier isomorphism for $\widehat{f}$ "is" κ^* of the Verdier isomorphism for f. As before let $\mathscr{X}'' = \mathscr{X} \times_{\mathscr{Y}} \mathscr{X}$, and $\Delta\colon \mathscr{X} \to \mathscr{X}''$ the diagonal immersion. Let $\mathscr{W}'' = \mathscr{W} \times_{\mathscr{Y}} \mathscr{W}$, and let $\delta\colon \mathscr{W} \to \mathscr{W}''$ be the diagonal immersion.

Let $\widetilde{\kappa}\colon \mathscr{W}'' \to \mathscr{X}''$ be the map $\kappa_2 = \kappa \times \kappa$. As usual, we have projections $p_i\colon \mathscr{X}'' \to \mathscr{X}$ and $\pi_i\colon \mathscr{W}'' \to \mathscr{W}$ for $i = 1, 2$. The following commutative diagrams may help the reader map the relative positions of the schemes and maps involved:

$$\begin{tikzcd}
\mathscr{W}'' \arrow[r, "\kappa_2'"'] \arrow[d, "\kappa_1'"] \arrow[dr, "\widetilde{\kappa}"] \arrow[rr, bend left, "\pi_2"] \arrow[dd, bend right=60, "\pi_1"'] & \bullet \arrow[r, "p_2'"'] \arrow[d, "\kappa_1"] & \mathscr{W} \arrow[d, "\kappa"'] \arrow[dd, bend left=60, "\widehat{f}"] \\
\bullet \arrow[r, "\kappa_2"'] \arrow[d, "p_1'"] & \mathscr{X}'' \arrow[r, "p_2"'] \arrow[d, "p_1"] & \mathscr{X} \arrow[d, "f"'] \\
\mathscr{W} \arrow[r, "\kappa"] \arrow[rr, bend right, "\widehat{f}"'] & \mathscr{X} \arrow[r, "f"] & \mathscr{Y}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathscr{W} \arrow[r, "\kappa"] \arrow[d, "\delta"'] & \mathscr{X} \arrow[d, "\Delta"] \\
\mathscr{W}'' \arrow[r, "\widetilde{\kappa}"'] & \mathscr{X}''
\end{tikzcd}$$

In what follows, let

$$\kappa^* f^{\#} \xrightarrow{\ \sim\ } \widehat{f}^{\#} \tag{8.3.1}$$

be the composite $\mathbf{L}\kappa^* f^{\#} \underset{(2.1.3)}{\xrightarrow{\ \sim\ }} \kappa^{\#} f^{\#} \xrightarrow{\ \sim\ } \widehat{f}^{\#}$, and let $\kappa^*\omega_f[r] \xrightarrow{\ \sim\ } \omega_{\widehat{f}}[r]$ be the one induced by the canonical isomorphsm $\kappa^*\omega_f \xrightarrow{\ \sim\ } \omega_{\widehat{f}}$.

THEOREM 8.3.2. *The following diagram commutes*

$$\begin{tikzcd}
\mathbf{L}\kappa^*\omega_f[r] \arrow[r, "\sim"] \arrow[d, "\mathbf{L}\kappa^*\mathbf{v}_f"', "\wr"] & \kappa^*\omega_f[r] \arrow[r, equal] & \omega_{\widehat{f}}[r] \arrow[d, "\mathbf{v}_{\widehat{f}}", "\wr"'] \\
\mathbf{L}\kappa^*(f^{\#}\mathscr{O}_{\mathscr{Y}}) \arrow[rr, "\sim", "(8.3.1)"'] & & \widehat{f}^{\#}\mathscr{O}_{\mathscr{Y}}
\end{tikzcd}$$

PROOF. Since $\mathbf{v}_f$ and $\mathbf{v}_{\widehat{f}}$ are isomorphisms, we assume $f^{\#}\mathscr{O}_{\mathscr{Y}}$ and $\widehat{f}^{\#}\mathscr{O}_{\mathscr{Y}}$ are complexes which are zero in all degrees except at the $(-r)$-th spot, where each is locally free (in fact invertible). This means we write $h^*(f^{\#}\mathscr{O}_{\mathscr{Y}}) = \mathbf{L}h^*(f^{\#}\mathscr{O}_{\mathscr{Y}})$ (resp. $h^*(\widehat{f}^{\#}\mathscr{O}_{\mathscr{Y}}) = \mathbf{L}h^*(\widehat{f}^{\#}\mathscr{O}_{\mathscr{Y}})$) for any map of schemes to $\mathscr{X}$ (resp. $\mathscr{W}$). Similarly $\mathbf{L}h^*\omega_f[d] = h^*\omega_f[d]$ etc. Let

$$\phi\colon \kappa^*(\mathscr{N}^r_{\Delta}) \xrightarrow{\sim} \mathscr{N}^r_{\delta}$$

be the canonical isomorphism. We have to show that the diagram ♣ below commutes.

$$\begin{tikzcd}
\widehat{f}^{\#}\mathscr{O}_{\mathscr{Y}} \otimes \mathscr{N}^r_{\delta}[-r] \arrow[d, "(8.1.1)"', "\wr"] & \kappa^*f^{\#}\mathscr{O}_{\mathscr{Y}} \otimes \mathscr{N}^r_{\delta}[-r] \arrow[l, "\sim"', "(8.3.1)"] \arrow[d, phantom, "\clubsuit"] & \kappa^*(f^{\#}\mathscr{O}_{\mathscr{Y}} \otimes \mathscr{N}^r_{\Delta}[-r]) \arrow[l, "\sim"', "\text{via } \phi"] \arrow[d, "\text{via } (8.1.1)", "\wr"'] \\
\mathscr{O}_{\mathscr{W}} \arrow[rr, equal] & & \kappa^*\mathscr{O}_{\mathscr{X}}
\end{tikzcd}$$

We expand ♣ as follows:

$$\begin{tikzcd}
\widehat{f}^{\#}\mathscr{O}_{\mathscr{Y}} \otimes \mathscr{N}^r_{\delta}[-r] \arrow[d, equal] & \kappa^*f^{\#}\mathscr{O}_{\mathscr{Y}} \otimes \mathscr{N}^r_{\delta}[-r] \arrow[l, "\sim"', "(8.3.1)"] \arrow[d, equal] & \kappa^*(f^{\#}\mathscr{O}_{\mathscr{Y}} \otimes \mathscr{N}^r_{\Delta}[-r]) \arrow[l, "\sim"', "\text{via } \phi"] \arrow[dd, equal] \\
\delta^{\blacktriangle}(\pi_2^*\widehat{f}^{\#}\mathscr{O}_{\mathscr{Y}}) \arrow[dd, "(2.2.2)"', "\wr"] \arrow[dr, phantom, "\square_1"] & \delta^{\blacktriangle}(\pi_2^*\kappa^*f^{\#}\mathscr{O}_{\mathscr{Y}}) \arrow[l, "\sim"', "(8.3.1)"] \arrow[d, equal] \arrow[r, phantom, "\square_2"] & {} \\
 & \delta^{\blacktriangle}(\widetilde{\kappa}^*p_2^*f^{\#}\mathscr{O}_{\mathscr{Y}}) \arrow[d, "(2.2.2)"', "\wr"] & \kappa^*\Delta^{\blacktriangle}(p_2^*f^{\#}\mathscr{O}_{\mathscr{Y}}) \arrow[l, "\sim"', "(C.4.2)"] \arrow[d, "(2.2.2)", "\wr"'] \\
\delta^{\blacktriangle}(\pi_1^{\#}\mathscr{O}_{\mathscr{W}}) \arrow[d, "\eta'_{\delta}"', "\wr"] & \delta^{\blacktriangle}(\widetilde{\kappa}^*p_1^{\#}\mathscr{O}_{\mathscr{X}}) \arrow[l, "\sim"', "\alpha_1"] \arrow[d, "\eta'_{\delta}"', "\wr"] \arrow[dr, phantom, "\square_3"] & \kappa^*\Delta^{\blacktriangle}(p_1^{\#}\mathscr{O}_{\mathscr{X}}) \arrow[l, "\sim"', "(C.4.2)"] \arrow[d, "\eta'_{\Delta}", "\wr"'] \\
\delta^{\#}(\pi_1^{\#}\mathscr{O}_{\mathscr{W}}) \arrow[d, "\wr"] & \delta^{\#}(\widetilde{\kappa}^*p_1^{\#}\mathscr{O}_{\mathscr{X}}) \arrow[l, "\sim"', "\alpha_2"] \arrow[d, phantom, "\square_4"] & \kappa^*\Delta^{\#}(p_1^{\#}\mathscr{O}_{\mathscr{X}}) \arrow[l, "\sim"', "(2.2.2)"] \arrow[d, "\wr"'] \\
\mathscr{O}_{\mathscr{W}} \arrow[rr, equal] & & \kappa^*\mathscr{O}_{\mathscr{X}}
\end{tikzcd}$$

The maps η'_{Δ} and η'_{δ} are the maps defined in (C.2.13). The maps α_i^{-1} are induced by the isomorphism $\pi_1^{\#}\kappa^*\mathscr{O}_{\mathscr{X}} \xrightarrow{\sim} \widetilde{\kappa}^*p_1^{\#}\mathscr{O}_{\mathscr{X}}$ resulting from the following composite of natural maps

$$\pi_1^\# \kappa^* \xrightarrow{\sim} \kappa_1'^\# p_1'^\# \kappa^* \xrightarrow[(2.1.3)]{\sim} \kappa_1'^\# p_1'^\# \kappa^\# \xrightarrow{\sim} \widetilde{\kappa}^\# p_1^\# \xrightarrow[(2.1.3)]{\sim} \widetilde{\kappa}^* p_1^\#.$$

In the above expansion of ♣, the unlabelled sub-rectangles clearly commute. Subrectangle $\square_2$ commutes by definition of the isomorphism (C.4.2). Proposition C.4.3 gives the commutativity of $\square_3$. For $\square_4$ we apply the outer border of the following diagram on $\mathscr{O}_{\mathscr{X}}$.

$$\begin{tikzcd}
\delta^\# \pi_1^\# \kappa^* \arrow[r] \arrow[dd] & \delta^\# \kappa_1'^\# p_1'^\# \kappa^* \arrow[r, "\kappa^* \cong \kappa^\#"] \arrow[dd] & \delta^\# \kappa_1'^\# p_1'^\# \kappa^\# \arrow[r] \arrow[dd] & \delta^\# \widetilde{\kappa}^\# p_1^\# \arrow[r, "\widetilde{\kappa}^\# \cong \widetilde{\kappa}^*"] \arrow[d] \arrow[dr, phantom, "\square\square_4"] & \delta^\# \widetilde{\kappa}^* p_1^\# \arrow[d, "(2.2.2)"] \\
 & & & \kappa^\# \Delta^\# p_1^\# \arrow[r, "\kappa^\# \cong \kappa^*"'] \arrow[d] & \kappa^* \Delta^\# p_1^\# \arrow[d] \\
\kappa^* \arrow[r, equal] & \kappa^* \arrow[r, "\kappa^* \cong \kappa^\#"'] & \kappa^\# \arrow[r, equal] & \kappa^\# \arrow[r, "\kappa^\# \cong \kappa^*"'] & \kappa^*
\end{tikzcd}$$

The unlabelled arrows are the obvious ones. The rectangle $\square\square_4$ commutes by Lemma A.1.4 while the remaining commute for obvious reasons.

It remains to prove that $\square_1$ commutes. To that end, it suffices to prove that the outer border of the following diagram commutes where the unlabelled arrows are the obvious ones coming from pseudofunctoriality of $(-)^*$ or $(-)^\#$, the ones labelled b-ch are induced by suitable base-change isomorphisms as given in (2.2.2), the ones labelled ${}^\# = {}^*$ are induced by (2.1.3) and the ones labelled γ_i are induced by the composite $\kappa_2'^\# \kappa_1^\# \xrightarrow{\sim} \widetilde{\kappa}^\# \xrightarrow[(2.1.3)]{\sim} \widetilde{\kappa}^*$.

$$\begin{tikzcd}
\pi_2^* \widehat{f}^\# \arrow[rr] \arrow[dddd, "\text{b-ch}"'] & & \kappa_2'^* p_2'^* \kappa^\# f^\# \arrow[rr, "{}^\# = {}^*"] \arrow[d, "\text{b-ch}"'] \arrow[drr, phantom, "\blacktriangle"] & & \kappa_2'^* p_2'^* \kappa^* f^\# \arrow[d] \\
 & & \kappa_2'^* \kappa_1^\# p_2^* f^\# \arrow[rr, "{}^\# = {}^*"] \arrow[dr, "{}^\# = {}^*"] \arrow[dd, "\text{b-ch}"'] & & \kappa_2'^* \kappa_1^* p_2^* f^\# \arrow[d] \arrow[dl, phantom, "\triangle"] \\
 & \ddagger & & \kappa_2'^\# \kappa_1^\# p_2^* f^\# \arrow[r, "\gamma_1"'] \arrow[d, "\text{b-ch}"'] & \widetilde{\kappa}^* p_2^* f^\# \arrow[d, "\text{b-ch}"] \\
 & \kappa_1'^\# \kappa_2^* p_1^\# f^* \arrow[r, "\text{b-ch}"] \arrow[d, "\text{b-ch}"'] \arrow[drr, "{}^\# = {}^*"] \arrow[dr, phantom, "\blacktriangle"] & \kappa_2'^* \kappa_1^\# p_1^\# f^* \arrow[r, "{}^\# = {}^*"] & \kappa_2'^\# \kappa_1^\# p_1^\# f^* \arrow[r, "\gamma_2"'] \arrow[d] & \widetilde{\kappa}^* p_1^\# f^* \\
\pi_1^\# \widehat{f}^* \arrow[r] & \kappa_1'^\# p_1'^\# \kappa^* f^* \arrow[r, "{}^\# = {}^*"] & \kappa_1'^\# p_1'^\# \kappa^\# f^* \arrow[r] & \kappa_1'^\# \kappa_2^\# p_1^\# f^* &
\end{tikzcd}$$

Now ‡ commutes by transitivity of the base-change isomorphism, (see Proposition A.1.1). For the diagrams labelled ▲, we refer to Lemma A.1.4, while △ commutes because of the pseudofunctorial nature of the isomorphism $(-)^\# \cong (-)^*$ of (2.1.3) over the category of formally étale maps. The unlabelled diagrams commute for trivial reasons. $\square$

8.3.3. There is a related result. Suppose $f\colon \mathscr{X} \to \mathscr{Y}$ is a smooth map of relative dimension d in $\mathbb{G}$ and suppose f factors as

$$\begin{tikzcd} \mathscr{X} \arrow[r, "\widehat{f}"] \arrow[rr, bend right, "f"'] & \widehat{\mathscr{Y}} \arrow[r, "\kappa"] & \mathscr{Y} \end{tikzcd}$$

where κ is the completion of $\mathscr{Y}$ along a coherent $\mathscr{O}_{\mathscr{Y}}$-ideal $\mathscr{I}$, and $\widehat{f}$ is smooth (necessarily of relative dimension d). Note that

$$\widehat{\mathscr{Y}} \times_{\mathscr{Y}} \widehat{\mathscr{Y}} = \widehat{\mathscr{Y}}.$$

Consider the commutative diagram of cartesian squares:

$$\begin{tikzcd} \mathscr{X} \arrow[r, equal] \arrow[d, "\widehat{f}"'] \arrow[dr, phantom, "\square"] & \mathscr{X} \arrow[d, "\widehat{f}"] \arrow[dd, bend left, "f"] \\ \widehat{\mathscr{Y}} \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\square"] & \widehat{\mathscr{Y}} \arrow[d, "\kappa"] \\ \widehat{\mathscr{Y}} \arrow[r, "\kappa"'] & \mathscr{Y} \end{tikzcd}$$

From (2.2.2) we conclude that we have an isomorphism

$$(*) \qquad f^{\#} \xrightarrow{\ \sim\ } \mathbf{1}^*_{\mathscr{X}} f^{\#} \xrightarrow{\ \sim\ } \widehat{f}^{\#}\kappa^*.$$

Now clearly, $\omega_f = \omega_{\widehat{f}}$. Call the common $\mathscr{O}_{\mathscr{X}}$-module ω. We have two related isomorphisms, namely, $\bar{\mathbf{v}}_{\widehat{f}}\colon \omega[d] \xrightarrow{\ \sim\ } \widehat{f}^{\#}\mathscr{O}_{\widehat{\mathscr{Y}}}$ and $\bar{\mathbf{v}}_f\colon \omega[d] \xrightarrow{\ \sim\ } f^{\#}\mathscr{O}_{\mathscr{Y}}$. With these notations, we have the following Proposition, related to Theorem 8.3.2:

Proposition 8.3.4. *With notations as above, the following diagram commutes:*

$$\begin{tikzcd} \omega[d] \arrow[rr, "\sim", "\bar{\mathbf{v}}_{\widehat{f}}"'] \arrow[d, "\bar{\mathbf{v}}_f"', "\wr"] & & \widehat{f}^{\#}\mathscr{O}_{\widehat{\mathscr{Y}}} \arrow[d, equal] \\ f^{\#}\mathscr{O}_{\mathscr{Y}} \arrow[rr, "\sim", "(*)"'] & & \widehat{f}^{\#}\kappa^*\mathscr{O}_{\mathscr{Y}}. \end{tikzcd}$$

Proof. We have the following commutative diagram with all squares cartesian:

$$\begin{tikzcd} \mathscr{X}'' \arrow[r, "p_2"] \arrow[d, "p_1"'] \arrow[dr, phantom, "\square"] & \mathscr{X} \arrow[r, equal] \arrow[d, "\widehat{f}"'] \arrow[dr, phantom, "\square"] & \mathscr{X} \arrow[d, "\widehat{f}"] \arrow[dd, bend left, "f"] \\ \mathscr{X} \arrow[r, "\widehat{f}"'] \arrow[d, equal] \arrow[dr, phantom, "\square"] & \widehat{\mathscr{Y}} \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\square"] & \widehat{\mathscr{Y}} \arrow[d, "\kappa"] \\ \mathscr{X} \arrow[r, "\widehat{f}"'] & \widehat{\mathscr{Y}} \arrow[r, "\kappa"'] & \mathscr{Y} \end{tikzcd}$$

We claim that diagram $(**)$ below commutes:

$$(**)\qquad \begin{tikzcd} p_2^* f^{\#} \arrow[r, "\sim", "(2.2.2)"'] \arrow[d, "(*)"', "\wr"] & p_1^{\#} f^* \arrow[d, equal] \\ p_2^* \widehat{f}^{\#} \kappa^* \arrow[r, "\sim", "(2.2.2)"'] & p_1^{\#} \widehat{f}^* \kappa^* \end{tikzcd}$$

Indeed, this follows immediately from the horizontal transitivity of the base-change isomorphism (see Proposition A.1.1) corresponding to the "composite" of base-change diagrams:

$$\begin{tikzcd} \mathscr{X}'' \arrow[r, "p_2"] \arrow[d, "p_1"'] \arrow[dr, phantom, "\square"] & \mathscr{X} \arrow[r, equal] \arrow[d, "\widehat{f}"'] \arrow[dr, phantom, "\square"] & \mathscr{X} \arrow[d, "f"] \\ \mathscr{X} \arrow[r, "\widehat{f}"'] & \widehat{\mathscr{Y}} \arrow[r, "\kappa"'] & \mathscr{Y} \end{tikzcd}$$

We therefore have the following commutative diagram, where the square on the left in induced by $(**)$.

$$\begin{tikzcd} \Delta^*(p_2^* f^{\#} \mathscr{O}_{\mathscr{Y}}) \otimes \omega^{-1}[-d] \arrow[r, "\sim", "(2.2.2)"'] \arrow[d, "(*)"', "\wr"] & \Delta^*(p_1^{\#} \mathscr{O}_{\mathscr{X}}) \otimes \omega^{-1}[-d] \arrow[r, "\sim"] \arrow[d, equal] & \Delta^{\#}(p_1^{\#} \mathscr{O}_{\mathscr{X}}) \arrow[r, "\sim"] \arrow[d, equal] & \mathscr{O}_{\mathscr{X}} \arrow[d, equal] \\ \Delta^*(p_2^* \widehat{f}^{\#} \mathscr{O}_{\widehat{\mathscr{Y}}}) \otimes \omega^{-1}[-d] \arrow[r, "\sim", "(2.2.2)"'] & \Delta^*(p_1^{\#} \widehat{f}^* \mathscr{O}_{\widehat{\mathscr{Y}}}) \otimes \omega^{-1}[-d] \arrow[r, "\sim"] & \Delta^{\#}(p_1^{\#} \mathscr{O}_{\mathscr{X}}) \arrow[r, "\sim"] & \mathscr{O}_{\mathscr{X}} \end{tikzcd}$$

In other words

$$\begin{tikzcd} f^{\#} \mathscr{O}_{\mathscr{Y}} \otimes \omega^{-1}[-d] \arrow[r, "\sim"] \arrow[d, "(*)"', "\wr"] & \mathscr{O}_{\mathscr{X}} \arrow[d, equal] \\ \widehat{f}^{\#} \mathscr{O}_{\widehat{\mathscr{Y}}} \otimes \omega^{-1}[-d] \arrow[r, "\sim"] & \mathscr{O}_{\mathscr{X}} \end{tikzcd}$$

commutes. This is equivalent to the statement of the Proposition. $\square$

8.4. Base change and Verdier's isomorphism

As mentioned earlier, by [**S2**, p.740, Theorem 2.3.5 (a)], for any Cohen-Macaulay map between ordinary schemes $f\colon X \to Y$, and any base change $u\colon Y' \to Y$, with $X' = X \times_Y Y'$, $f'\colon X' \to Y'$ and $v\colon X' \to X$ the base change maps, there is a natural isomorphism $\theta_u^f\colon v^*\omega_f^{\#} \xrightarrow{\sim} \omega_{f'}^{\#}$. In the event f is smooth, then using Verdier's isomorphisms for f and f' to identify $\omega_f^{\#}$ with ω_f and $\omega_{f'}^{\#}$ with $\omega_{f'}$, the map θ_u^f corresponds to the obvious canonical map (see [Ibid., p.740, Theorem 2.3.5 (b)]). The difficulty in transferring this statement to formal schemes is that defining θ_u^f required certain special local compactifications of f which may or may not be available for general formal scheme maps. However these difficulties disappear if the map u is flat, and the proof in loc.cit. works mutatis mutandis. The precise statement is:

THEOREM 8.4.1. *Suppose*

$$\begin{tikzcd} \mathscr{X}' \arrow[r, "v"] \arrow[d, "f'"'] \arrow[dr, phantom, "\square"] & \mathscr{X} \arrow[d, "f"] \\ \mathscr{Y}' \arrow[r, "u"'] & \mathscr{Y} \end{tikzcd}$$

is a cartesian square with f smooth, in $\mathbb{G}$, of relative dimension d, and u flat. Let $\theta\colon v^*f^{\#}\mathscr{O}_{\mathscr{Y}} \xrightarrow{\sim} (f')^{\#}u^*\mathscr{O}_{\mathscr{Y}} = (f')^{\#}\mathscr{O}_{\mathscr{Y}'}$ be the resulting base change isomorphism (see (2.2.2)). Then the isomorphism $\mathbf{v}_{f'}^{-1}\circ\theta\circ v^*(\mathbf{v}_f)\colon v^*\omega_f[d] \xrightarrow{\sim} \omega_{f'}[d]$ is the obvious canonical map.

CHAPTER 9

Residues

9.1. Verdier residue

Let $f\colon X \to Y$ be a smooth map of ordinary schemes of relative dimension r and let $Z \hookrightarrow X$ be a closed subscheme proper over Y. Let $\kappa\colon \mathscr{X} \to X$ be the completion of X along Z and let $\hat{f}\colon \mathscr{X} \to Y$ be composite $\hat{f} = f \circ \kappa$. Analogous to the abstract residue $\mathbf{res}^{\#}_{Z}$ in (3.3.2) one has the *Verdier residue along* Z

$$\mathbf{res}_{Z} : \mathrm{R}^{r} f_{*}\omega_{f} \to \mathscr{O}_{Y} \tag{9.1.1}$$

defined as the composite

$$\mathrm{R}^{r}_{Z} f_{*}\omega_{f} \xrightarrow[(3.3.3)]{\ \sim\ } \mathrm{R}'^{r}_{\mathscr{X}} \widehat{f}_{*}\kappa^{*}\omega_{f} \xrightarrow{\ \sim\ } \mathrm{R}'^{r}_{\mathscr{X}} \widehat{f}_{*}\omega_{\widehat{f}} \xrightarrow{\mathrm{tr}_{\widehat{f}}} \mathscr{O}_{Y}.$$

where the middle isomorphism is induced by the canonical one $\kappa^{*}\omega_{f} \xrightarrow{\sim} \omega_{\widehat{f}}$. By compatibility of the Verdier isomorphism with completions (Theorem 8.3.2), the following diagram commutes (where, as before, $\mathbf{res}^{\#}_{Z}$ is the abstract residue map defined in (3.3.2)):

$$\begin{tikzcd}
\mathrm{R}^{r}_{Z} f_{*}\omega_{f} \arrow[r, "\sim", "\bar{\mathbf{v}}"'] \arrow[dr, bend right, "\mathbf{res}_{Z}"'] & \mathrm{R}^{r}_{Z} f_{*}\kappa^{*}\omega^{\#}_{f} \arrow[d, "\mathbf{res}^{\#}_{Z}"] \\
 & \mathscr{O}_{Y}
\end{tikzcd} \tag{9.1.2}$$

Remark 9.1.3. While we have defined residues in general, our interest is really in the case where $\mathrm{R}'^{j}_{\mathscr{X}}(\mathscr{F}) = 0$ for every $j > r$ and every $\mathscr{F} \in \mathcal{A}_{\vec{c}}(\mathscr{X})$, for then $(\omega_{f}, \mathrm{tr}_{f})$ represents the functor $\mathrm{Hom}_{Y}(\mathrm{R}'^{r}_{\mathscr{X}}(-), \mathscr{O}_{Y})$ on coherent $\mathscr{O}_{\mathscr{X}}$-modules (cf. Corollary 3.2.4). Even here the most useful situation is when $Y = \operatorname{Spec} A$ and $\mathscr{X} = \operatorname{Spf} R$ where R is an adic ring, with a defining ideal I generated by r elements, with R/I finite and flat over A.

The various relationships between the abstract residue, Verdier residue, the trace and the Verdier intergal are captured in the following commutative commutative diagram:

$$\begin{tikzcd}
\mathrm{R}^{r}_{Z} f_{*}\omega_{f} \arrow[r, "\sim", "(3.3.3)"'] \arrow[d, "\mathbf{v}"] \arrow[dd, bend right=60, "\mathbf{res}_{Z}"'] & \mathrm{R}'^{r}_{\mathscr{X}} \widehat{f}_{*}\omega_{\widehat{f}} \arrow[d, "\mathbf{v}"'] \arrow[dd, bend left=60, "\mathrm{tr}_{\widehat{f}}"] \\
\mathrm{R}^{r}_{Z} f_{*}\omega^{\#}_{f} \arrow[r, "\sim", "(A.3.1)"'] \arrow[d, "\mathbf{res}^{\#}_{Z}"] & \mathrm{R}'^{r}_{\mathscr{X}} \widehat{f}_{*}\omega^{\#}_{\widehat{f}} \arrow[d, "\mathrm{tr}^{\#}_{\widehat{f}}"'] \\
\mathscr{O}_{Y} \arrow[r, equal] & \mathscr{O}_{Y}
\end{tikzcd} \tag{9.1.4}$$

In the event $f\colon X \to Y$ is *proper* we have the following commutative diagram

$$
\begin{tikzcd}
\mathrm{R}^r_Z f_*\omega_f \arrow[r] \arrow[d, "\mathbf{v}"] \arrow[dd, bend right=60, "\mathbf{res}_Z"'] & \mathrm{R}^r f_*\omega_f \arrow[d, "\mathbf{v}"'] \arrow[dd, bend left=60, "\mathrm{tr}_f"] \\
\mathrm{R}^r_Z f_*\omega_f^\# \arrow[r] \arrow[d, "\mathbf{res}^\#_Z"] & \mathrm{R}^r f_*\omega_f^\# \arrow[d, "\mathrm{tr}^\#_f"'] \\
\mathscr{O}_Y \arrow[r, equal] & \mathscr{O}_Y
\end{tikzcd}
\tag{9.1.5}
$$

9.2. Some residue formulas

Suppose $A \to R$ is a finite type map of rings which is *smooth*. Set $R'' = R \otimes_A R$. As before, the two R-algebra structures on R'' will be denoted R_1 and R_2, with R_k denoting the algebra corresponding to the projection $p_k\colon X'' := X \times_Y X \to X$ for $k \in \{1, 2\}$. The diagonal map $\Delta\colon X' \hookrightarrow X''$ corresponds to the surjective map $R'' \to R$ given by $t_1 \otimes t_2 \mapsto t_1 t_2$. Suppose the kernel of this map, i.e., the ideal of the diagonal immersion, is generated by r-elements $\{s_1, \ldots, s_r\}$. Since R is smooth over A of relative dimension r, the sequence $\mathbf{s} = (s_1, \ldots, s_r)$ is necessarily a R''-sequence. By part (b) of Proposition 8.2.2 we get the following formula, which is at the heart of much of what we do in this part of the book.

$$
\mathbf{res}_{\Delta, p_1} \begin{bmatrix} \mathrm{d}s_1 \wedge \cdots \wedge \mathrm{d}s_r \\ s_1, \ldots, s_r \end{bmatrix} = 1. \tag{9.2.1}
$$

Proposition 9.2.2. *Let $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} R$ be affine schemes, and suppose $f\colon X \to Y$ is a smooth map of relative dimension r. Suppose further that we have a closed subscheme Z of X such that $Z \to Y$ is an isomorphism and the ideal J of R giving the closed subscheme Z of X is generated by r-elements $\{t_1, \ldots, t_r\}$ of R. Then*

$$
\mathbf{res}_Z \begin{bmatrix} \mathrm{d}t_1 \wedge \cdots \wedge \mathrm{d}t_r \\ t_1, \ldots, t_r \end{bmatrix} = 1. \tag{9.2.3}
$$

Proof. First note that since f is smooth, $\mathbf{t} = (t_1, \ldots, t_r)$ is an R-regular sequence. Next note that the question is local on Y and so we may assume, without loss of generality, that the diagonal immersion $\Delta\colon X \to X''$ is cut out by r-elements $\{s_1, \ldots, s_r\}$ in $R'' = R \otimes_A R$. As in §8.2, we write I for the ideal of the diagonal and use the notations of that subsection. Let $Z = \operatorname{Spec} B$. Let $\sigma\colon Y \to X$ be the section defined by Z, and $i\colon Z \hookrightarrow X$ the natural closed immersion. We have a commutative diagram with all sub-rectangles cartesian.

$$
\begin{tikzcd}
Z \arrow[r, "\sigma_Z"] \arrow[d, hook, "i"'] \arrow[dr, phantom, "\square"] & X \arrow[d, hook, "\Delta"] \\
X \arrow[r, "\sigma_X"] \arrow[d, "f"'] \arrow[dr, phantom, "\square"] & X'' \arrow[d, "p_1"] \\
Y \arrow[r, "\sigma"'] & X
\end{tikzcd}
$$

We now need some results from [**S2**] regarding *non-flat* base-change. Since σ is a closed immersion, the usual flat-base-change results do not apply. Nevertheless, we do have the following. First, there is a base change isomorphism $\theta = \theta^f_\sigma \colon {\sigma_X}^*\omega^\#_f \xrightarrow{\sim} \omega^\#_{p_1}$ as in [Ibid., p.740, Theorem 2.3.5 (a)]. Next, by [Ibid., Prop. 6.2.2, pp.755–756], under this isomorphism, residues are compatible. In other words, the diagram

$$\begin{tikzcd}
\sigma^*\mathrm{R}^r_\Delta {p_1}_*(\omega^\#_{p_1}) \arrow[r, "\sim"] \arrow[d, "\sigma^*\mathbf{res}^\#_\Delta"'] & \mathrm{R}^r_Z f_*({\sigma_X}^*\omega^\#_{p_1}) \arrow[d, "\theta", "\wr"'] \\
\mathscr{O}_{\mathscr{Y}} & \mathrm{R}^r_Z f_*\omega^\#_f \arrow[l, "\mathbf{res}^\#_Z"']
\end{tikzcd}$$

commutes. Finally on replacing $\omega^\#_f$ by ω_f and $\omega^\#_{p_1}$ by ω_{p_1} via $\mathbf{v}_f$ and $\mathbf{v}_{p_1}$, according to [Ibid., p.740, Theorem 2.3.5 (b)], the map θ reduces to the standard identity ${\sigma_X}^*\omega_{p_1} = \omega_f$.

Thus it follows that if u_i, $i = 1, \ldots, r$, are the images of s_i in R under the map $R'' \to R$ corresponding to $\sigma_X : X \to X''$, (so that J is generated by the set $\{u_1, \ldots, u_r\}$) we have (via (9.2.1))

$$\mathbf{res}_Z \begin{bmatrix} \mathrm{d}u_1 \wedge \cdots \wedge \mathrm{d}u_r \\ u_1, \ldots, u_r \end{bmatrix} = 1.$$

Since $\left[\begin{smallmatrix} \mathrm{d}u_1 \wedge\cdots\wedge \mathrm{d}u_r \\ u_1,\ldots,u_r \end{smallmatrix}\right] = \left[\begin{smallmatrix} \mathrm{d}t_1 \wedge\cdots\wedge \mathrm{d}t_r \\ t_1,\ldots,t_r \end{smallmatrix}\right]$, hence the result. $\square$

CHAPTER 10

Residues along sections

Let $f\colon X \to Y$ be a smooth separated map of noetherian schemes of relative dimension r. We begin with some notations and conventions. In general, if we are working over affine schemes (ordinary or formal) we will use the same notations for maps between modules as the corresponding sheaves. For example if $A \to R$ is smooth map of rings of relative dimension r, and I an R ideal generated by a regular sequence $\{t_1, \ldots, t_r\}$ such that $A \to B := R/I$ is finite, then with $X = \operatorname{Spec} R$, $Y = \operatorname{Spec} A$ and $Z = \operatorname{Spec} B$, and $f\colon X \to Y$ the map given by $A \to R$, we will write $\mathbf{res}_{Z}\colon \mathrm{H}^r_Z(X, \omega_f) \to A$ instead of $\Gamma(Y, \mathbf{res}_{Z})$. As another illustration of this principle, in the above situation, if $\omega_{R/A}$ is the A-module given by $\omega_{R/A} = \Gamma(X, \omega_f)$, then we will make no distinction between $\mathrm{H}^r_Z(X, \omega_f)$ and $\mathrm{H}^r_I(\omega_{R/A})$.

10.1. The local cohomology class of a section

Suppose $Y = \operatorname{Spec} A$ and $Z \hookrightarrow X$ is a closed subscheme such that $Z \to Y$ is an isomorphism and Z lies in an open affine subscheme $U = \operatorname{Spec} R$ of X such that Z is given in U by an ideal I which is generated by r elements $t_1, \ldots, t_r$ of R. We have a map

$$\mathbf{res}_{\mathbf{t}}\colon \mathrm{H}^r_Z(X, \omega_f) \to A$$

defined by the formula

$$\mathbf{res}_{\mathbf{t}} \begin{bmatrix} \mathrm{d}t_1 \wedge \cdots \wedge \mathrm{d}t_r \\ t_1^{\alpha_1}, \ldots, t_r^{\alpha_r} \end{bmatrix} = \begin{cases} 1 & \text{when } \alpha_1 = \cdots = \alpha_r = 1 \\ 0 & \text{otherwise.} \end{cases} \tag{10.1.1}$$

This map depends *a priori* on the choice of $\mathbf{t} = (t_1, \ldots, t_r)$, but as we will see later, it is independent of it. It should be pointed out that if Z is also defined (in U) by the vanishing of $s_1, \ldots, s_r$, then by Theorem C.7.2

$$\begin{bmatrix} \mathrm{d}t_1 \wedge \cdots \wedge \mathrm{d}t_r \\ t_1, \ldots, t_r \end{bmatrix} = \begin{bmatrix} \mathrm{d}s_1 \wedge \cdots \wedge \mathrm{d}s_r \\ s_1, \ldots, s_r \end{bmatrix}. \tag{10.1.2}$$

Moreover, there is an A-module direct sum decomposition

$$\mathrm{H}^r_Z(X, \omega_f) = \mathrm{H}^r_I(\omega_{R/A}) = \bigoplus_{\underline{\alpha}} A\left[\begin{smallmatrix} \mathrm{d}t_1 \wedge \cdots \wedge \mathrm{d}t_r \\ t_1^{\alpha_1}, \ldots, t_r^{\alpha_r} \end{smallmatrix}\right]$$

with $\underline{\alpha} = (\alpha_1, \ldots, \alpha_r)$ running over r-tuples of positive integers. The summands are a free A-modules. While this decomposition depends on $\mathbf{t}$, the summand generated by $\left[\begin{smallmatrix} \mathrm{d}t_1 \wedge \cdots \wedge \mathrm{d}t_r \\ t_1, \ldots, t_r \end{smallmatrix}\right]$ is independent of $\mathbf{t}$ by (10.1.2). In what follows, let

$$\theta_{Z} = \begin{bmatrix} \mathrm{d}t_1 \wedge \cdots \wedge \mathrm{d}t_r \\ t_1, \ldots, t_r \end{bmatrix}.$$

10.2. Relative projective space

Let $\mathbb{P} = \mathbb{P}^r_Y$, the relative projective space of relative dimension r over an ordinary scheme Y. We regard $\mathbb{P} = \mathbf{Proj}(\mathscr{O}_Y[T_0, \ldots, T_r])$. Let $\pi\colon \mathbb{P} \to Y$ be the structure map and

$$\int_{\mathbb{P}/Y} : \mathrm{R}^r\pi_*\omega_\pi \xrightarrow{\ \sim\ } \mathscr{O}_Y$$

be the standard trace map (known to be an isomorphism) defined, for example in [**EGA**, III$_1$, 2.1.12] or [**RD**, p.152, Theorem 3.4]. The generating section $\mu = \mu_{\mathbb{P}}$ of $\mathrm{R}^r\pi_*\omega_\pi$ corresponding to the standard section 1 of $\mathscr{O}_Y$ is described as follows. Let $\mathscr{U} = \{U_i \mid i = 0, \ldots, r\}$ be the open cover of $\mathbb{P}$ given by $U_i = \{T_i \neq 0\}$. On $U_0 \cap \cdots \cap U_r$ we have inhomogeneous coordinates $t_i = T_i/T_0$, $i = 1, \ldots, r$ whence a section

$$\check{\mu}_T := \frac{\mathrm{d}t_1 \wedge \cdots \wedge \mathrm{d}t_r}{t_1 \ldots t_r} \in \Gamma(U_0 \cap \cdots \cap U_r, \omega_\pi).$$

We have an isomorphism

$$\mathrm{H}^r(\pi_*\check{\mathcal{C}}^\bullet(\mathscr{U},\ \omega_\pi)) \xrightarrow{\ \sim\ } \mathrm{R}^r\pi_*\omega_\pi$$

and $\check{\mu}_T$ has a natural image in the left side as a Čech cohomology class. Let μ be the corresponding element on the right side. The section μ does not depend on the choice of homogeneous coordinates $T_0, \ldots, T_r$ of $\mathbb{P}$ (cf. [**C1**, p.34, Lemma 2.3.1]) and is the sought after section.

Let Z_0 be the closed subscheme of $\mathbb{P}$ defined by $\{T_i = 0 \mid i = 1, \ldots, r\}$, i.e., the intersection of the relative hyperplanes $H_i = \{T_i = 0\}$, $i = 1, \ldots, r$. Then $Z_0 \to Y$ is an isomorphism. The section $\sigma_0\colon Y \to \mathbb{P}$ defined by Z_0 is the Y-valued point of the Y-scheme $\mathbb{P}$ given by the "homogeneous co-ordinates" $(1, 0, 0, \ldots, 0)$.

Now suppose $Y = \operatorname{Spec} A$. It is well known (see [**L2**, p.74, Prop. (8.4)] for example, the proof of which generalizes to our situation) that the following diagram commutes.

(10.2.1)
$$\begin{tikzcd}
\mathrm{H}^r_{Z_0}(\mathbb{P},\ \omega_\pi) \arrow[r] \arrow[dr, bend right, "\mathbf{res_t}"'] & \mathrm{H}^r(\mathbb{P},\ \omega_\pi) \arrow[d, "\int_{\mathbb{P}/Y}"] \\
& A
\end{tikzcd}$$

We now indicate how the commutativity of (10.2.1) is proved in [**L2**]. For an n-tuple of positive integers $\underline{\alpha} = (\alpha_1, \ldots, \alpha_r)$ one can regard fractions of the form $\mathrm{d}t_1 \wedge \cdots \wedge \mathrm{d}t_r/t_1^{\alpha_1} \cdots t_r^{\alpha_r}$ as r-cocycles in the Čech complex $\check{C}^\bullet(\mathscr{U},\ \omega_\pi) = \Gamma(\mathbb{P},\ \check{\mathcal{C}}^\bullet(\mathscr{U},\ \omega_\pi))$. Let us write $\nu(\underline{\alpha})$ for the image of this fraction in $\mathrm{H}^r(\mathbb{P},\ \omega_\pi)$. (Note that $\nu(1, \ldots, 1) = \mu$.) According to [**L2**, pp.79–80, Lemma (8.6)] the natural map

$$\mathrm{H}^r_{Z_0}(\mathbb{P},\ \omega_\pi) \to \mathrm{H}^r(\mathbb{P},\ \omega_\pi)$$

is described by

$$\begin{bmatrix} \mathrm{d}t_1 \wedge \cdots \wedge \mathrm{d}t_r \\ t_1^{\alpha_1} \cdots t_r^{\alpha_r} \end{bmatrix} \mapsto \nu(\underline{\alpha}) \tag{10.2.2}$$

In particular $\theta_{Z_0} \mapsto \mu = \nu(-1, \ldots, -1)$. It is well known that if $\underline{\alpha} \neq (-1, \ldots, -1)$ the Čech r-cocycle $\mathrm{d}t_1 \wedge \cdots \wedge \mathrm{d}t_r/t_1^{\alpha_1} \cdots t_r^{\alpha_r}$ for the complex $\check{C}^\bullet(\mathscr{U},\ \omega_\pi)$ is a coboundary, whence in this case $\nu(\underline{\alpha}) = 0$. This establishes the commutativity of (10.2.1).

If K_{Z_0} is the kernel of $\mathbf{res}_{\mathbf{t}}$, we have a split short exact sequence of A-modules, with $\mu \mapsto \theta_{Z_0}$ giving the splitting:

$$0 \longrightarrow K_{Z_0} \longrightarrow \mathrm{H}^r_{Z_0}(\mathbb{P}, \omega_\pi) \xrightarrow{\text{canonical}} \mathrm{H}^r(\mathbb{P}, \omega_\pi) \longrightarrow 0.$$

Proposition 10.2.3. *With the above notations we have:*

(i) *The Verdier integral for π equals the standard trace for the relative projective space $\mathbb{P}^r_Y$, i.e.,*

$$\mathrm{tr}_\pi = \int_{\mathbb{P}/Y}.$$

(ii) *Let A be a ring, $\mathbf{t} = (t_1, \dots, t_r)$ analytically independent variables over A, and $J \subset A[[\mathbf{t}]]$ the ideal of $A[[\mathbf{t}]]$ generated by $\mathbf{t}$. Then the Verdier integral $\mathrm{tr}_{A[[\mathbf{t}]]/A} \colon \mathrm{H}^r_J(\omega_{A[[\mathbf{s}]]/A}) \to A$ defined in (8.1.9) is given by*

$$\begin{bmatrix} \mathrm{d}t_1 \wedge \cdots \wedge \mathrm{d}t_r \\ t_1^{\alpha_1}, \dots, t_r^{\alpha_r} \end{bmatrix} \mapsto \begin{cases} 1 & \text{when } \alpha_1 = \cdots = \alpha_r = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Proof. For part (i) without loss of generality we may assume $Y = \operatorname{Spec} A$. Let μ be the canonical generator of the free rank one A-module $\mathrm{H}^r(\mathbb{P}, \omega_\pi)$ corresponding to $1 \in A$ under the isomorphism $\int_{\mathbb{P}/Y} \colon \mathrm{H}^r(\mathbb{P}, \omega_\pi) \xrightarrow{\sim} A$. It is enough to show that $\mathrm{tr}_\pi(\mu) = \int_{\mathbb{P}/Y}(\mu)$, i.e., it is enough to show that $\mathrm{tr}_\pi(\mu) = 1$. According to (10.2.2), the image of $\theta_{Z_0} \in \mathrm{H}^r_{Z_0}(\mathbb{P}, \omega_\pi)$ in $\mathrm{H}^r(\mathbb{P}, \omega_\pi)$ is μ. We have

$$\begin{aligned} \mathrm{tr}_\pi(\mu) &= \mathbf{res}_{Z_0}(\theta_{Z_0}) && \text{(by (9.1.5))} \\ &= 1 && \text{(via Proposition 9.2.2)} \end{aligned}$$

and hence we are done for part (i).

For part (ii), let us agree to write $Y = \operatorname{Spec} A$. Let us write $\mathscr{P} = \mathscr{P}^r_A$ for $\operatorname{Spf} A[[\mathbf{t}]]$, and $\widehat{\pi} \colon \mathscr{P} \to Y$ for the structure map. With $\mathbb{P}$, π, Z_0 as above, we can identify $\mathscr{P}$ with the completion of $\mathbb{P}$ along Z_0. We thus have a completion map $\kappa \colon \mathscr{P} \to \mathbb{P}$, which factors through the open subscheme U_0 of $\mathbb{P}$ where $T_0 \neq 0$ as $\mathscr{P} \to U_0 \subset \mathbb{P}$. Moreover, if U_0 is identified in the usual way with $\operatorname{Spec} A[t_1, \dots, t_r]$ (via $t_i = T_i/T_0$), then the first map in the factorization arises from the inclusion of the polynomial ring $A[\mathbf{t}]$ into the power series ring $A[[\mathbf{t}]]$. Now, by part (a), (9.1.5), and (10.2.1), we have $\mathbf{res}_{\mathbf{t}} = \mathbf{res}_{Z_0}$. Since the composite

$$\mathrm{R}^r_{Z_0}\pi_*\omega_\pi \xrightarrow{\sim} \mathrm{R}'^r_{\mathscr{P}}\widehat{\pi}_*\omega_{\widehat{\pi}} \xrightarrow{\mathrm{tr}_{\widehat{\pi}}} \mathscr{O}_Y$$

is $\mathbf{res}_{Z_0}$ by (9.1.4), and $\mathbf{res}_{Z_0} = \mathbf{res}_{\mathbf{t}}$, by taking global sections we are done. $\square$

10.3. The Verdier residue for sections of smooth maps

Let us return to our smooth map $f \colon X \to Y$ of relative dimension r, and suppose $Y = \operatorname{Spec} A$ and $Z \hookrightarrow X$ as before a closed subscheme such that $Z \to Y$ is an isomorphism, Z lies in affine open set $U = \operatorname{Spec} R$ of X, and Z is cut out in U by the vanishing or r elements $t_1, \dots, t_r$ in R.

Proposition 10.3.1. *In the above situation $\mathbf{res}_{\mathbf{t}} = \mathbf{res}_Z$. In particular, if $\mathbf{s} = (s_1, \dots, s_r)$ is another sequence in R generating the ideal defining Z, then $\mathbf{res}_{\mathbf{t}} = \mathbf{res}_{\mathbf{s}}$.*

Proof. Let I be the ideal generated by $\mathbf{t}$. Suppose I is also generated by $\mathbf{s} = (s_1, \ldots, s_r)$. The completion of R in the I-adic topology is $A[[\mathbf{t}]] = A[[\mathbf{s}]]$ and both are the completion $\widehat{R}$ of R in the I-adic topology. It follows that $\mathrm{tr}_{A[[\mathbf{t}]]/A} = \mathrm{tr}_{\widehat{R}/A} = \mathrm{tr}_{A[[\mathbf{s}]]/A}$. Part (ii) of Proposition 10.2.3 and the relationship between $\mathbf{res}_Z$ and $\mathrm{tr}_{\widehat{R}/A}$ then proves our assertion. $\square$

Consider again the A-module decomposition

$$\mathrm{H}^r_Z(X,\, \omega_f) = \mathrm{H}^r_I(\omega_{R/A}) = \bigoplus_{\underline{\alpha}} A\big[\begin{smallmatrix} \mathrm{d}t_1\wedge\cdots\wedge\mathrm{d}t_r \\ t_1^{\alpha_1},\ldots,t_r^{\alpha_r}\end{smallmatrix}\big]$$

with $\underline{\alpha} = (\alpha_1, \ldots, \alpha_r)$ running over r-tuples of positive integers. Each summand is a free A-module. While this decomposition depends on $\mathbf{t} = (t_1, \ldots, t_r)$, we have seen that the summand generated by $\theta_Z = \big[\begin{smallmatrix} \mathrm{d}t_1\wedge\cdots\wedge\mathrm{d}t_r \\ t_1,\ldots,t_r\end{smallmatrix}\big]$ is independent of $\mathbf{t}$ by (10.1.2). Moreover, since the sum of the remaining summands in the direct sum is the kernel K_Z of $\mathbf{res}_Z$, it too is independent of $\mathbf{t}$. Thus, we have a canonical decompostion of A-modules

$$\mathrm{H}^r_Z(X,\, \omega_f) = K_Z \oplus A\cdot\theta_Z \tag{10.3.2}$$

which is independent of $\mathbf{t}$ with $K_Z = \ker(\mathbf{res}_Z)$.

Remark 10.3.3. Let A and $\mathbf{t} = (t_1, \ldots, t_r)$ be as in Proposition 10.2.3 (ii). Then a little thought shows that for $f \in A[[\mathbf{t}]]$, with $\mu(i_1, \ldots i_r)$ the coefficient of $t_1^{i_1} \cdots t_r^{i_r}$ in the power series expansion of f, one has the formula:

$$\mathrm{tr}_{A[[\mathbf{t}]]/A}\begin{bmatrix} f\cdot \mathrm{d}t_1 \wedge \cdots \wedge \mathrm{d}t_r \\ t_1^{\alpha_1}, \ldots, t_r^{\alpha_r}\end{bmatrix} = \mu(\alpha_1 - 1, \ldots, \alpha_r - 1).$$

In particular, we have

$$\mathrm{tr}_{A[[\mathbf{t}]]/A}\begin{bmatrix} f\cdot \mathrm{d}t_1 \wedge \cdots \wedge \mathrm{d}t_r \\ t_1, \ldots, t_r\end{bmatrix} = f(0, \ldots, 0).$$

Similarly, if A, $\mathbf{t}$, Z, R are as in Proposition 10.3.1, then for any $f \in R$,

$$\mathbf{res}_Z\begin{bmatrix} f\cdot \mathrm{d}t_1 \wedge \cdots \wedge \mathrm{d}t_r \\ t_1, \ldots, t_r\end{bmatrix} = \bar{f}$$

where $\bar{f} \in A$ is the image of f in A under the natural surjection $R \to R/I \cong A$. More generally, given positive integers $\alpha_1, \ldots, \alpha_r$ one can write

$$f = \sum_{i_1,\ldots,i_r} \mu(i_1, \ldots, i_r) t_1^{i_1} \cdots t_r^{i_r} + g,$$

where i_k are non-negative integers such that $\sum_j i_j < \alpha_1 + \cdots + \alpha_r$, $\mu(i_1, \ldots, i_r) \in A$, and $g \in I^{\alpha_1+\cdots+\alpha_r}$. In this case we have

$$\mathbf{res}_Z\begin{bmatrix} f\cdot \mathrm{d}t_1 \wedge \cdots \wedge \mathrm{d}t_r \\ t_1^{\alpha_1}, \ldots, t_r^{\alpha_r}\end{bmatrix} = \mu(\alpha_1 - 1, \ldots, \alpha_r - 1).$$

10.4. A characterisation of the Verdier isomorphism

Now suppose Y is not necessarily affine, and as above we have a closed subscheme $Z \hookrightarrow X$ such that $Z \to Y$ is an isomorphism. Let $z \in Z$ be a point. Pick affine open subschemes U' in X and V' in Y such that $z \in U'$ and $f(U') \subset V'$ and such that $U' \cap Z$ is given in U' by the vanishing of r-elements $t_1, \ldots, t_r \in \Gamma(U', \mathscr{O}_X)$. Let $V = f(U' \cap Z)$. Then V is an affine open subscheme of V' (since it is isomorphic to $U' \cap Z$ which, being a closed subscheme of U' is affine). Moreover $U' \to V'$ is affine, whence $U := f^{-1}(V) \cap U'$ is affine. Note that $U' \cap Z = U \cap Z$, $f(U) = V$, $Z \cap U$ is given by the vanishing of $t_1, \ldots, t_r$ and $Z \cap U \to V = f(Z \cap U)$ is an isomorphism. Thus locally we can reduce to the situation in §10.1. If $Z_U = U \cap Z$, then from (10.1.2), it is clear that θ_{z_U} glue to give a section θ_z of $\mathrm{R}^r_Z f_* \omega_f$:

$$\theta_z \in \Gamma(X,\, \mathrm{R}^r_Z f_* \omega_f).$$

Moreover, the A-module K_Z in (10.3.2) being independent of $\mathbf{t}$ means that its construction globalizes to give a quasi-coherent submodule $\mathscr{K}_Z$ of $\mathrm{R}^r_Z f_* \omega_f$. Finally, since the decomposition (10.3.2) is canonical, it globalizes to give a decompostion:

$$\mathrm{R}^r_Z f_* \omega_f = \mathscr{K}_Z \oplus (\mathscr{O}_Y \cdot \theta_z). \tag{10.4.1}$$

THEOREM 10.4.2. *Let Z be a closed subscheme of X such that $Z \to Y$ is an isomorphism. Then $\mathbf{res}_z$ is the composite*

$$\mathrm{R}^r_Z f_* \omega_f = \mathscr{K}_Z \oplus (\mathscr{O}_Y \cdot \theta_z) \xrightarrow{\textit{projection}} \mathscr{O}_Y \cdot \theta_z \xrightarrow{\sim} \mathscr{O}_Y$$

where the direct sum decomposition is (10.4.1) *and the last isomorphism is $\theta_z \mapsto 1$.*

PROOF. Without loss of generality we may assume $X = \operatorname{Spec} R$, $Z = \operatorname{Spec} R/I$ where I is an ideal of R generated by r elements $\{t_1, \ldots, t_r\}$ and $Y = \operatorname{Spec} A$. The result then follows from Proposition 10.3.1 and the explicit description of $\mathbf{res}_{\mathbf{t}}$. □

Before stating the next theorem we need some notation. If $\psi\colon \omega_f[r] \to f^!\mathscr{O}_Y$ is a map of $\mathscr{O}_X$-modules, then $\bar{\psi}\colon \omega_f \to \omega_f^{\#}$ will denote the map $\bar{\psi} = \mathrm{H}^{-r}(\psi)$. We remind the reader that $\mathbf{v} = \mathbf{v}_f$ denotes the Verdier isomorphism $\omega_f[r] \xrightarrow{\sim} f^!\mathscr{O}_Y$. We alert the reader to one notational issue. In this subsection, for good book-keeping purposes we will write $\bar{\mathbf{v}}\colon \omega_f \xrightarrow{\sim} \omega_f^{\#}$ for $\mathrm{H}^{-r}(\mathbf{v})$. For most of the book we do not put the "bar" over $\mathbf{v}$ for this map, as that abuse of notation is usually harmless. (Cf. also Remark 8.2.4.)

LEMMA 10.4.3. *Let Z be a closed subscheme of X such that $Z \to Y$ is finite and flat. Suppose we have an isomorphism $\psi\colon \omega_f[r] \xrightarrow{\sim} f^!\mathscr{O}_Y$ such that the composite*

$$\mathrm{R}^r_Z f_* \omega_f \xrightarrow[\text{via } \bar{\psi}]{\sim} \mathrm{R}^r_Z f_* \omega_f^{\#} \xrightarrow{\mathbf{res}^{\#}_Z} \mathscr{O}_Y$$

is the residue map $\mathbf{res}_z$. Then there is an open neighbourhood U of Z in X such that $\psi|_U = \mathbf{v}|_U$.

PROOF. It is enough to prove that there is an open neighbourhood U of Z such that $\bar{\psi}|_U = \bar{\mathbf{v}}|_U$. Let $\varphi\colon \omega_f^{\#} \xrightarrow{\sim} \omega_f^{\#}$ be the automorphism given by $\varphi = \bar{\mathbf{v}} \circ \bar{\psi}^{-1}$. Let $\kappa\colon \mathscr{X} = X_{/Z} \to X$ be the completion of X along Z and $\widehat{f} = f \circ \kappa$. By the hypothesis we have $\mathbf{res}^{\#}_z \circ \mathrm{R}^r_Z f_*(\varphi) = \mathbf{res}^{\#}_z$, and hence by definition of $\mathrm{tr}^{\#}_{\widehat{f}}$ we get $\mathrm{tr}^{\#}_{\widehat{f}} \circ \mathrm{R}'^r_{\mathscr{X}} \widehat{f}_*(\kappa^*(\varphi)) = \mathrm{tr}^{\#}_{\widehat{f}}$. Thus by local duality, i.e., Corollary 3.2.4, we see that $\kappa^*(\varphi)$ is the identity map, whence there is an open neighbourhood U of Z such that $\varphi|_U$ is the identity map. □

We need a little more notation in order to state the next Lemma. Consider a cartesian diagram

$$\begin{tikzcd} X' \arrow[r, "v"] \arrow[d, "f'"'] \arrow[dr, phantom, "\square"] & X \arrow[d, "f"] \\ Y' \arrow[r, "u"'] & Y \end{tikzcd} \tag{10.4.4}$$

where f is smooth (and hence Cohen-Macaulay) of relative dimension r. We will use the notation of [**S2**] and denote by

$$\theta_u^f \colon v^*\omega_f^{\#} \xrightarrow{\ \sim\ } \omega_{f'}^{\#}$$

the corresponding base change isomorphism (see [**S2**, p.740, Theorem 2.3.5 (a)]). Now suppose we have an isomorphism $\psi\colon \omega_f[r] \xrightarrow{\ \sim\ } f^!\mathscr{O}_Y$ and suppose $Z \hookrightarrow X'$ is a closed subcheme such that $Z \to Y'$ is an isomorphism. We write

$$\mathbf{res}_{\psi, Z} \colon \mathrm{R}^r_Z f'_*\omega_{f'} \to \mathscr{O}_{Y'} \tag{10.4.5}$$

for the composite:

$$\mathrm{R}^r_Z f'_*\omega_{f'} = \mathrm{R}^r_Z f'_* v^*\omega_f \xrightarrow{\text{via } \bar{\psi}} \mathrm{R}^r_Z f'_* v^*\omega_f^{\#} \xrightarrow[\text{via } \theta_u^f]{\ \sim\ } \mathrm{R}^r_Z f'_*\omega_{f'}^{\#} \xrightarrow{\mathbf{res}_Z^{\#}} \mathscr{O}_{Y'}.$$

In other words $\mathbf{res}_{\psi, Z} = \mathbf{res}_Z^{\#} \circ \mathrm{R}^r_Z f_*(\theta_u^f \circ v^*(\bar{\psi}))$.

Lemma 10.4.6. *Let $u\colon Y' \to Y$ be an étale map, and let X', f', v, θ_u^f be as above. Suppose we have an isomorphism $\psi\colon \omega_f[r] \xrightarrow{\ \sim\ } f^!\mathscr{O}_Y$, and a closed subscheme Z of X' such that $Z \to Y'$ is finite and flat and such that $\mathbf{res}_{\psi, Z} = \mathbf{res}_Z$. Then there is an open neighbourhood U of the locally closed set $v(Z)$ such that $\psi|_U = \mathbf{v}_f|_U$.*

Proof. By the hypothesis on ψ and by (10.4.3) we can find an open neighbourhood V of Z in X' such that $(\theta_u^f \circ v^*(\bar{\psi}))|_V = \bar{\mathbf{v}}_{f'}|_V$. On the other hand, by [**S2**, p.740, Theorem 2.3.5 (b)], we have $\theta_u^f \circ v^*\bar{\mathbf{v}}_f = \bar{\mathbf{v}}_{f'}$. It follows that $v^*(\bar{\psi}))|_V = v^*\bar{\mathbf{v}}_f|_V$. Set $U = v(V)$. Since v is étale, U is open, and $V \to U$ is faithfully flat, whence $\bar{\psi}|_U = \bar{\mathbf{v}}_f|_U$. $\square$

Remark 10.4.7. Since f is smooth, if x is an associated point of X, then $y = f(x)$ is an associated point of Y, and x is a generic point of the fibre $f^{-1}(y)$. This means that if an open subscheme V of X is such that $V \cap f^{-1}(s)$ is dense in $f^{-1}(s)$ for every associated point s of Y, then V is scheme theoretically dense in X, since it contains every associated point of X. We use this fact in what follows.

Theorem 10.4.8. *Let $\psi\colon \omega_f[r] \xrightarrow{\ \sim\ } f^!\mathscr{O}_Y$. A necessary and sufficient condition that ψ is the Verdier isomorphism $\mathbf{v}_f$ is the following:*
For every étale map $u\colon Y' \to Y$ and every closed subscheme Z of X' such that $Z \to Y'$ is an isomorphism, we have $\mathbf{res}_{\psi, Z} = \mathbf{res}_Z$. Here X', f', v are as in diagram (10.4.4).

Proof. For $u\colon Y' \to Y$, $f'\colon X' \to Y'$, $v\colon X' \to X$ as above, according to [**S2**, p.740, Theorem 2.3.5 (b)] we have $\theta_u^f \circ v^*\bar{\mathbf{v}}_f = \bar{\mathbf{v}}_{f'}$. The necessity part of the theorem then follows from Theorem 10.4.2.

Conversely, suppose we have an isomorphism $\psi\colon \omega_f[r] \xrightarrow{\ \sim\ } f^!\mathscr{O}_Y$ satisfying the condition stated in the theorem. We have to show that $\bar{\psi} = \bar{\mathbf{v}}_f$. Fix $y \in Y$. Since

f is smooth, the set W_y of points $x \in f^{-1}(y)$ such that $k(x)$ is finite and separable over $k(y)$, is dense in $f^{-1}(y)$ by [**BLR**, p. 42, §2.2, Cor. 13]. Let x be such a point. We can find an étale map $u\colon Y' \to Y$ such that (with the usual notations) there is a section of f' passing through a point x' satisfying $v(x') = x$ [**BLR**, p. 43, §2.2, Prop. 14]. Let Z be the image of this section. Then Z is closed, and $Z \to Y'$ is an isomorphism, whence by our hypotheses on f and by Lemma 10.4.6 there is an open neighbourhood U of $v(Z)$ on which $\bar{\psi} = \bar{\mathbf{v}}_f$. Since $x \in v(Z)$, this equality holds in an open neighbourhood of x. Varying x over W_y, and varying y over Y, by Remark 10.4.7 the equality holds in a scheme theoretically dense open subset of X and hence everywhere, for ω_f and $\omega_f^{\#}$ are invertible $\mathscr{O}_X$-modules. □

Recall that given a point $x \in X$, closed in its fibre, with $k(x)$ separable over $k(f(x))$, since f is smooth we can find an étale neighbourhood $Y' \to Y$ of $f(x)$ and a section of f' (with the usual notations for base change that we have been following) passing through one of the points of $v^{-1}(x)$. It is immediate that one can find an open cover $\{U_\alpha\}$ of Y, étale surjective maps $u_\alpha\colon Y_\alpha \to U_\alpha$, such that (with $X_\alpha := X \times_Y Y_\alpha$, and $f_\alpha\colon X_\alpha \to Y_\alpha$, $v_\alpha\colon X_\alpha \to X$ the projections) there is a closed subscheme Z_α of X_α which maps isomorphically on to Y_α. Let $Y' = \coprod_\alpha Y_\alpha$, $X' = \coprod_\alpha X_\alpha$, $f' = \coprod_\alpha f_\alpha$, $u = \coprod_\alpha u_\alpha$. Then we have a closed subscheme Z of X' such that $Z \to Y'$ is an isomorphism (take $Z = \coprod_\alpha Z_\alpha$). Note that $u\colon Y' \to Y$ is étale and *surjective*, whence it is *faithfully flat*.

Proposition 10.4.9. *Let $\psi\colon \omega_f[r] \xrightarrow{\sim} f^!\mathscr{O}_Y$ be an isomorphism.*

(a) *If the fibres of f are connected, and Z is a closed subscheme of X such that $Z \to Y$ is an isomorphism and $\mathbf{res}^{\#}_Z \circ \mathrm{R}^r_Z f_*(\bar{\psi}) = \mathbf{res}_Z$, then $\psi = \mathbf{v}_f$.*

(b) *Suppose the fibres of f are geometrically connected. Then $\psi = \mathbf{v}_f$ if and only if there is an étale surjective map $u\colon Y' \to Y$ and (with the usual notation) a closed subscheme Z of $X' = X \times_Y Y'$ with $Z \to Y'$ an isomorphism such that $\mathbf{res}_{\psi,Z} = \mathbf{res}_Z$.*

Proof. For part (a), we note that if $\kappa\colon \mathscr{X} \to X$ is the completion of Z along X, then $\kappa^*\bar{\psi} = \kappa^*\bar{\mathbf{v}}_f$. We therefore have an open subscheme V containing Z such that $\bar{\psi}|_V = \bar{\mathbf{v}}_f|_V$. Since f is smooth, it (locally) has a factorization $f = \pi \circ h$, where h is étale and π is the structural map $\mathbb{A}^r_Y \to Y$. Since the fibres of f are connected, and $f^{-1}(y) \cap V \supset f^{-1}(y) \cap Z \neq \emptyset$, it follows that $V \cap f^{-1}(y)$ is dense in $f^{-1}(y)$. Thus V is scheme-theoretically dense in X by Remark 10.4.7. Now $\bar{\psi}^{-1} \circ \bar{\mathbf{v}}_f$ is the identity automorphism on ω_f on V, which is scheme theoretically dense on X, and ω_f is invertible on X. It follows that it $\bar{\psi}^{-1} \circ \bar{\mathbf{v}}_f$ is the identity automorphism on all of X.

For part (b), first suppose $\psi = \mathbf{v}_f$. By the remarks made above the statement of the theorem, there is an étale surjective map $u\colon Y' \to Y$, and (with the usual meaning attached to X', f' and v) a closed subscheme Z of X' such that $Z \to Y'$ is an isomorphism. Now $\mathbf{res}_{\psi,Z} = \mathbf{res}^{\#}_Z \circ \mathrm{R}^r_Z f_*(\theta^f_u \circ v^*(\bar{\psi})) = \mathbf{res}^{\#}_Z \circ \mathrm{R}^r_Z f_*(\theta^f_u \circ v^*(\bar{\mathbf{v}}_f))$. On the other hand, by [**S2**, p.740, Theorem 2.3.5 (b)], $\mathbf{v}$ behaves well with respect to base change, i.e., $\theta^f_u \circ v^*(\bar{\mathbf{v}}_f) = \bar{\mathbf{v}}_{f'}$. Thus $\mathbf{res}_{\psi,Z} = \mathbf{res}^{\#}_Z \circ \mathrm{R}^r_Z f_*(\bar{\mathbf{v}}_{f'}) = \mathbf{res}_Z$.

Conversely, suppose we have an étale surjective map $u\colon Y' \to Y$ and a closed subscheme Z of $X = X \times_Y Y'$, with $Z \to Y'$ an isomorphism satisfying $\mathbf{res}_{\psi,Z} = \mathbf{res}_Z$. Let $f'\colon X' \to Y'$ and $v\colon X' \to X$ be the projections. Since the fibres of f' are connected, by part (a) we have $\theta^f_u \circ v^*(\bar{\psi}) = \bar{\mathbf{v}}_{f'}$. Now, $\bar{\mathbf{v}}_{f'} = \theta^f_u \circ v^*(\bar{\mathbf{v}}_f)$ (by [**S2**, p.740, Theorem 2.3.5 (b)] again) from which it is immediate that $v^*(\bar{\psi}) =$

$v^*(\bar{\mathbf{v}}_f)$. The map $v\colon X' \to X$ is étale surjective, and hence faithfully flat, giving the result. $\square$

CHAPTER 11

Regular Differential Forms

The results in this section do not affect the results in the rest of the book, and so may be skipped on first reading. These results are here to give a non-trivial application of the characterisation of Verdier's map in the previous section. The main results of this section, connecting the Kunz-Waldi regular differentials with Verdier's isomorphism, are proved again in §14.3 without making use of the results in [**KW**] or the results in this section. There is a fleeting reference to definition of the map (11.4.1) of this section in §14.3 (see in §§14.3.3).

All schemes in this section, unless otherwise stated, are ordinary schemes. The aim is to relate the concrete form of Grothendieck duality via Kunz's regular differential forms to Verdier's isomorphism. In somewhat greater detail, regular differential forms defined for certain types of maps $f\colon X \to Y$ are concrete representations of many aspects of Grothendieck duality. A well-known special case is that of Rosenlicht's differentials on singular curves [**R**]. Kunz defined generalization of these to more general situations (higher dimensions) in a series of papers, and in [**KW**], Kunz and Waldi defined the sheaf of (relative) regular differentials for dominant finite type equidimensional maps $f\colon X \to Y$ between excellent schemes which do not have embedded components. When such an f is generically smooth, this was related to duality theory by Kunz, Lipman, Hübl, Sastry (see [**L2**], [**HK1**], [**HK2**], [**HS**]). All the papers just mentioned work within the framework of a simpler version of Grothendieck duality (one eschewing derived categories) due to Kleiman [**Kl**]. We now review this, taking a slightly revisionist view, in that that we interpret the principal objects (r-dualizing pairs) in terms of the full blown duality theory of Grothendieck.

11.1. Overview of Kleiman's functor

Regular differentials are a vast generalization of the differentials Rosenlicht used for describing describing duality for singular curves [**R**]. To put the theory in context we give a quick account of Kleiman's theory of r-dualizing pairs given in [**Kl**]. Let $f\colon X \to Y$ be a proper map such that $\dim(X \otimes k(y)) \le r$ for every $y \in Y$. For any scheme Z, let Z_{qc} denote the category of quasi-coherent $\mathscr{O}_Z$-modules. According to [*loc. cit.,* pp. 41–42, Definition (1)], an *r-dualizing pair* $(f^K,\, t_f)$ consists of a covariant functor $f^K\colon Y_{qc} \to X_{qc}$ and a natural transformation $t_f\colon \mathrm{R}^r f_* f^K \to \mathbf{1}_{Y_{qc}}$ inducing a bifunctorial isomorphism of quasi-coherent sheaves,

$$f_*\mathscr{H}om_X(\mathscr{F},\, f^K\mathscr{G}) \xrightarrow{\ \sim\ } \mathscr{H}om_Y(\mathrm{R}^r f_*\mathscr{F},\, \mathscr{G})$$

for each $\mathscr{F} \in X_{qc}$ and each $\mathscr{G} \in Y_{qc}$. Kleiman explicitly eschewed derived categories in his paper, and shows the existence of an r-dualizing pair (for f of the kind we are considering) using the special adjoint functor theorem. From our point of view f^K can identified with $\mathrm{H}^{-r}(f^!(-))$. Our hypotheses on f ensure that $\mathrm{H}^j(f^!(-)) = 0$

for $j < -r$, whence we get a map (of functors from Y_{qc} to $\mathbf{D}_{\mathrm{qc}}(X)$) $f^K(-)[r] \to f^!$. The map t_f is then the composite

$$\mathrm{R}^r f_* f^K = \mathrm{H}^0(\mathbf{R}f_* f^K(-)[r]) \longrightarrow \mathrm{H}^0(\mathbf{R}f_* f^! \mathscr{O}_Y) \xrightarrow{\mathrm{Tr}_f} \mathrm{H}^0(\mathscr{O}_Y) = \mathscr{O}_Y.$$

When f is not proper, f^K still makes sense (even if f is not compactifiable, i.e., even if f is not separated), since $\mathrm{H}^n(f^!)$ makes sense for every integer n, even if $f^!$ is not defined (see comment above Definition 3.1.2), and hence one can set $f^K = \mathrm{H}^{-r}(f^!)$. If f is Cohen-Macaulay of relative dimension r, $f^K\mathscr{O}_Y = \omega_f^{\#}$, and if further f is smooth we have, via Verdier's isomorphism $\omega_f \xrightarrow{\sim} f^K\mathscr{O}_Y$. In the proper, Cohen-Macaulay case we have $(\omega_f^{\#}, \mathrm{tr}^{\#}_f) = (f^K\mathscr{O}_Y, t_f(\mathscr{O}_Y))$. If f is in addition smooth, we have a unique isomorphism of pairs $(\omega_f, \mathrm{tr}_f) \xrightarrow{\sim} (f^K\mathscr{O}_Y, t_f(\mathscr{O}_Y))$.

11.2. Regular Differentials

Let $f\colon X \to Y$ be a finite-type map. Following Kunz in [**Ku**, B.17] we say it is *equidimensional of dimension* r if

- the generic points of X are mapped to the generic points of Y, and
- the non-empty fibres of f are such that the irreducible components of these fibres are all of dimension r.

Now suppose the map $f\colon X \to Y$ satisfies the following conditions

- X, Y are excellent schemes, and neither has embedded points amongst their associated points;
- f is equidimensional of dimension r, and
- the smooth locus of f is scheme-theoretically dense in X (which, given our hypotheses, means that the smooth locus of f contains all the generic points of X).

Next let X_0 be the artinian scheme

$$X_0 = \coprod_s \operatorname{Spec} \mathscr{O}_{X,s}$$

where s runs through the set of associated (= generic in this case) points and $i_X\colon X_0 \to X$ the natural affine map. Similarly, we have the artinian scheme Y_0 constructed out of the generic points of Y, and an affine map $i_Y\colon Y_0 \to Y$. We write

$$k(X) = i_{X*}\mathscr{O}_{X_0}$$

where as before s runs over generic points of X. The sheaf of relative meromorphic r-forms $\Omega^r_{k(X)/k(Y)}$ on X is then the quasi-coherent $\mathscr{O}_X$-module given by the formula

$$\Omega^r_{k(X)/k(Y)} = i_{X*}\Omega^r_{X_0/Y_0} = \omega_f \otimes_{\mathscr{O}_X} k(X).$$

Under our hypotheses on f the $\mathscr{O}_X$-module of *regular differentials* ω_f^{reg} (denoted $\omega^r_{X/Y}$ in [**HK1**], [**HK2**], and [**HS**]) is defined in [**KW**, § 3, § 4]. It is coherent and is an $\mathscr{O}_X$ submodule of the module of meromorphic r-differentials $\Omega^r_{k(X)/k(Y)}$, and hence is torsion-free. On the smooth locus X^s of f, writing $f^s\colon X^s \to Y$ for the smooth map obtained by restricting f, we have $\omega_f^{\mathrm{reg}}|_{X^s} = \omega_{f^s}^{\mathrm{reg}} = \omega_{f^s}$.

When f is *proper* we have a trace map (denoted $\int_{X/Y}$ in [**HK1**], [**HK2**], [**HS**])

$$\int_{\boldsymbol{f}}^{\mathbf{reg}}\colon \mathrm{R}^r f_*\omega_f^{\mathrm{reg}} \to \mathscr{O}_Y.$$

This map is defined when f is *projective* in [**HK1**], and is generalized to proper f in [**HS**]. One of the main results of [**HS**] is that the resulting map $\omega_f^{\mathrm{reg}} \to f^K\mathscr{O}_Y$ is an isomorphism (a fact proved in [**HK2**] for projective maps f). There is also a notion of a residue map $\mathrm{R}^r_Z f_*\omega_f^{\mathrm{reg}} \to \mathscr{O}_Y$ (denoted $\int_{X/Y,Z}$ in [**HK1**], [**HK2**], and [**HS**]) for certain special closed subschemes Z of X which are finite over Y (see [**HK1**, pp.77–78, Assumption 4.3 and Theorem 4.4]).

To avoid notational confusion we denote this

$$\mathbf{res}_Z^{\mathrm{reg}}\colon \mathrm{R}^r_Z f_*\omega_f^{\mathrm{reg}} \to \mathscr{O}_Y.$$

11.3. Summary of the main result of [HS]

The complete statement concerning $\omega_f^{\mathrm{reg}}(=\omega^r_{X/Y})$, $\int_{\boldsymbol{f}}^{\mathbf{reg}}(=\int_{X/Y})$ and $\mathbf{res}_Z^{\mathrm{reg}}(=\int_{X/Y,Z})$ can be found in [**HS**, pp.750–752, Theorem]. In brief, here are the main points of this result:

(i) One has a canonical isomorphism $\varphi = \varphi_f\colon \omega_f^{\mathrm{reg}} \xrightarrow{\sim} f^K\mathscr{O}_Y$ such that when f is proper φ is the unique isomorphism for which the diagram

$$\begin{tikzcd} \mathrm{R}^r f_*\omega_f^{\mathrm{reg}} \arrow[r, "\text{via } \varphi"] \arrow[dr, bend right, "\int_{\boldsymbol{f}}^{\mathbf{reg}}"'] & \mathrm{R}^r f_* f^K\mathscr{O}_Y \arrow[d, "t_f(\mathscr{O}_Y)"] \\ & \mathscr{O}_Y \end{tikzcd}$$

commutes [**HS**, pp.750–751, (i) (The Duality Theorem)].

(ii) The isomorphism φ_f is compatible with open immersions into X. In greater detail, if $j\colon U \to X$ is an open immersion, as submodules of the $\mathscr{O}_X$ module $\Omega_{k(U)/k(Y)}$, $i^*\omega_f^{\mathrm{reg}} = \omega_{fi}^{\mathrm{reg}}$ and the diagram

$$\begin{tikzcd} i^*\omega_f^{\mathrm{reg}} \arrow[r, "\sim", "i^*\varphi_f"'] \arrow[d, equal] & i^*f^K\mathscr{O}_Y \arrow[d, "\wr"] \\ \omega_{fi}^{\mathrm{reg}} \arrow[r, "\sim", "\varphi_{fi}"'] & (fi)^K\mathscr{O}_Y \end{tikzcd}$$

commutes [Ibid, pp.750–751, (i) and (ii)].

(iii) If Z is a closed subscheme of X satisfying Assumption 4.3 of [**HK1**, p.77] then the diagram

$$\begin{tikzcd} \mathrm{R}^r_Z f_*\omega_f^{\mathrm{reg}} \arrow[r, "\text{canonical}"] \arrow[dr, bend right, "\mathbf{res}_Z^{\mathrm{reg}}"'] & \mathrm{R}^r f_*\omega_f^{\mathrm{reg}} \arrow[d, "\int_{\boldsymbol{f}}^{\mathbf{reg}}"] \\ & \mathscr{O}_Y \end{tikzcd}$$

commutes [**HS**, p.752, (iii) (The Residue Theorem)].

(iv) If Z is a closed subscheme of X such that Z lies in the smooth locus of f and $Z \to Y$ is an isomorphism, then $\mathbf{res}_Z^{\mathrm{reg}} = \mathbf{res}_Z$ (see [**HK1**, p.62, Cor. 1.13] and [**HK1**, p.78, 4.4] as well as the formulae in Remark 10.3.3).

(v) The map φ is compatible with flat base change to excellent schemes without embedded associated points [**KW**, 3.13] and [**HS**, pp.751–752, (ii) and (iv)]. In particular φ is compatible with étale base change.

11.4. Regular Differentials and Verdier

Now suppose f is smooth. Then $f^K\mathscr{O}_Y = \omega_f^\#$ and $\omega_f^{\mathrm{reg}} = \omega_f$. Let $\psi = \varphi[r]$. Identifying $f^!\mathscr{O}_Y$ with $\omega_f^\#[r]$ we have an isomorphism

$$\psi\colon \omega_f[r] \xrightarrow{\sim} f^!\mathscr{O}_Y.$$

Then using the notations of §10.4, we have $\varphi = \bar{\psi}$. In light of the properties listed above for φ_f and ω_f^{reg} we see that if $u\colon Y' \to Y$ is an étale map and Z is a closed subscheme of $X' = X \times_Y Y'$ such that $Z \to Y'$ is an isomorphism, then $\mathbf{res}_Z^{\mathrm{reg}} = \mathbf{res}_Z$. However, the left side is the map $\mathbf{res}_{\psi,Z}$ of (10.4.5), whence we conclude from Theorem 10.4.8 that $\psi = \mathbf{v}_f$, where the right side is the Verdier isomorphism of (8.1.5).

Our next observation is one that was made by J. Lipman in pp. 33–34 of [**L2**] for varieties over fields in his discussion leading to Lemma (2.2) of *ibid.* Suppose f is as in the previous subsection, and U is the smooth locus of f. Let $j\colon U \to X$ be the open immersion and $g = f\circ j\colon U \to Y$ the resulting smooth map. By our hypotheses, U contains all the associated points of X, whence it is scheme-theoretically dense. Without getting into the notions of canonical structures and dualizing structures, we have a composition

$$f^K\mathscr{O}_Y \hookrightarrow j_*g^K\mathscr{O}_Y \xrightarrow{j_*\bar{\mathbf{v}}_{\mathbf{g}}^{-1}} j_*\omega_g \hookrightarrow \Omega^r_{k(X)/k(Y)} \tag{11.4.1}$$

with every arrow an inclusion since $f^K\mathscr{O}_Y$, $g^K\mathscr{O}_Y$ and ω_g are torsion free and $j^*f^K\mathscr{O}_Y \xrightarrow{\sim} g^K\mathscr{O}_Y$. The image of $f^K\mathscr{O}_Y$ in $\Omega^r_{k(X)/k(Y)}$ must be ω_f^{reg} since $\bar{\mathbf{v}}_{\mathbf{g}}$ is φ_g of item (1) of §11.3. In greater detail, if $\bar{\omega}$ is the image of $f^K\mathscr{O}_Y$ in $\Omega^r_{k(X)/k(Y)}$ under (11.4.1), and $\alpha\colon f^K\mathscr{O}_Y \xrightarrow{\sim} \bar{\omega}$ the resulting isomorphism, then we have an isomorphism $\beta\colon \omega_f^{\mathrm{reg}} \xrightarrow{\sim} \bar{\omega}$ such that $\alpha = \beta\circ\varphi_f$. Now $j^*\beta = \mathbf{1}_{\omega_g}$ since $\varphi_g = \bar{\mathbf{v}}_g$. Since U is scheme theoretically dense in X and the sheaves involved are torsion free, the assertion follows. In other words Verdier's isomorphism gives us the regular differential forms of Kunz and Waldi, as well as the dualizing structure on them.

Here is the formal statement of the result(s) we just proved.

THEOREM 11.4.2. *Let $f\colon X \to Y$ be a finite type map between excellent schemes such that X and Y have no embedded points, f is equidimensional of dimension r, and the smooth locus of f contains all the associated points of X (i.e., the smooth locus of X is scheme-theoretically dense in X).*

(a) *If f is smooth then the map φ_f of item* (i) *in §11.3 is the Verdier isomorphism $\bar{\mathbf{v}}_f$ defined in* (8.1.1).

(b) *If $j\colon U \to X$ is the open immersion from the smooth locus of f to X, and $g\colon U \to Y$ is the composite $g = f\circ i$, then the module of regular differential r-forms ω_f^{reg} of Kunz and Waldi* [**KW**, § 3, § 4] *is the image of $f^K\mathscr{O}_Y$ under injective composite* (11.4.1). *Moreover the resulting isomorphism $f^K\mathscr{O}_Y \xrightarrow{\sim} \omega_f^{\mathrm{reg}}$ is inverse of the map φ_f.*

CHAPTER 12

Transitivity for smooth maps

12.1. The map $\zeta_{g,f}$ between differential forms

Suppose $f\colon \mathscr{X} \to \mathscr{Y}$ and $g\colon \mathscr{Y} \to \mathscr{Z}$ are maps in $\mathbb{G}$, with f a smooth map of relative dimension e , and g a smooth map of relative dimension d. We have a map of differential forms

$$\zeta_{g,f}\colon f^*\omega_g[d] \otimes_{\mathscr{O}_{\mathscr{X}}} \omega_f[e] \longrightarrow \omega_{gf}[d+e] \tag{12.1.1}$$

defined by the commutativity of the following diagram

$$\begin{tikzcd} f^*\omega_g[d] \otimes_{\mathscr{O}_{\mathscr{X}}} \omega_f[e] \arrow[rr, "\zeta_{g,f}"] \arrow[d, "f^*\mathbf{v}_g \otimes \mathbf{v}_f"', "\wr"] & & \omega_{gf}[d+e] \arrow[d, "\mathbf{v}_{gf}", "\wr"'] \\ f^*g^{\#}\mathscr{O}_{\mathscr{Z}} \overset{\mathbf{L}}{\otimes}_{\mathscr{O}_{\mathscr{Y}}} f^{\#}\mathscr{O}_{\mathscr{Y}} \arrow[rr, "\chi_{[g,f]}"'] & & (gf)^{\#}\mathscr{O}_{\mathscr{Z}} \end{tikzcd}$$

where $\chi_{[g,f]}\colon f^*g^{\#}\mathscr{O}_{\mathscr{Z}} \overset{\mathbf{L}}{\otimes}_{\mathscr{O}_{\mathscr{Y}}} f^{\#}\mathscr{O}_{\mathscr{Y}} \to (gf)^{\#}\mathscr{O}_{\mathscr{Z}}$ is the map defined in Definition 5.2.16.

Proposition 12.1.2. *The following hold:*

(a) (Flat Base Change) *Suppose*

$$\begin{tikzcd} \mathscr{U} \arrow[r, "u"] \arrow[d, "p"'] \arrow[dr, phantom, "\square"] & \mathscr{X} \arrow[d, "f"] \\ \mathscr{V} \arrow[r, "v"'] \arrow[d, "q"'] \arrow[dr, phantom, "\square"] & \mathscr{Y} \arrow[d, "g"] \\ \mathscr{W} \arrow[r, "w"'] & \mathscr{Z} \end{tikzcd}$$

is a cartesian square with u flat, f and g smooth and in $\mathbb{G}$. Then

$$u^*\zeta_{g,f} = \zeta_{g,f}.$$

(b) *Suppose $f\colon \mathscr{X} \to \mathscr{Y}$ and $g\colon \mathscr{Y} \to \mathscr{Z}$ are smooth maps and in $\mathbb{G}$. Let $\kappa\colon \mathscr{X}^* \to \mathscr{X}$ be the completion of $\mathscr{X}$ with respect to an open coherent ideal. Then*

$$\zeta_{g,f\kappa} = \kappa^*\zeta_{g,f}.$$

(c) *Suppose $\mathscr{X} \xrightarrow{f} \mathscr{Y}_1 \xrightarrow{\kappa} \mathscr{Y}_2 \xrightarrow{g} \mathscr{Z}$ is a sequence of maps in $\mathbb{G}$ with f and g smooth, and κ a completion map with respect to an open coherent ideal. Then*

$$\zeta_{g,\kappa f} = \zeta_{g\kappa,f}.$$

(d) *Suppose*

$$\begin{tikzcd} \widehat{\mathscr{X}} \arrow[r, "\kappa_2"] \arrow[d, "\widehat{f}"'] & \mathscr{X} \arrow[d, "f"] \\ \widehat{\mathscr{Y}} \arrow[r, "\kappa_1"] \arrow[dr, "\widehat{g}"'] & \mathscr{Y} \arrow[d, "g"] \\ & \mathscr{Z} \end{tikzcd}$$

is a commutative diagram in $\mathbb{G}$ *with* f *and* g *smooth, and* κ_1 *and* κ_2 *completions with respect to open coherent ideals of* $\mathscr{O}_{\mathscr{Y}}$ *and* $\mathscr{O}_{\mathscr{X}}$ *respectively. Then*

$$\kappa_2^*\zeta_{g,f} = \zeta_{\widehat{g},\widehat{f}}.$$

PROOF. Follows from the properties for $\chi_{g,f}$ listed in §5.2, Theorem 8.3.2, Theorem 8.3.4, and the fact that the Verdier isomorphism is compatible with flat base change. ☐

12.2. The map $\varphi_{g,f}$ between differential forms

For a smooth map between ordinary schemes $f\colon X \to Y$ of relative dimension d, $\omega_f := \wedge^d_{\mathscr{O}_X}\Omega^1_{X/Y}$. Let X, Y, and Z be ordinary schemes. Suppose $f\colon X \to Y$ is a smooth map of schemes of relative dimension d and $g\colon Y \to Z$ is smooth of relative dimension e. Let

$$\bar{\varphi}_{g,f}\colon f^*\omega_g \otimes \omega_f \xrightarrow{\ \sim\ } \omega_{gf} \tag{12.2.1}$$

be the map which is locally given by

$$f^*(\mathrm{d}t_1 \wedge \cdots \wedge \mathrm{d}t_e) \otimes \mathrm{d}s_1 \wedge \cdots \wedge \mathrm{d}s_d \mapsto \mathrm{d}s_1 \wedge \cdots \wedge \mathrm{d}s_d \wedge \mathrm{d}t_1 \wedge \cdots \wedge \mathrm{d}t_e.$$

Here $\mathbf{t} = (t_1, \ldots, t_e)$ and $\mathbf{s} = (s_1, \ldots, s_d)$ are local relative "co-ordinates", i.e., $\mathbf{t}$ gives an étale map $U \to \mathbb{A}^e_Z$ on an open subscheme U of Y, and on an open sunscheme V of $f^{-1}(U)$, $\mathbf{s}$ gives an étale map $V \to \mathbb{A}^d_Y$. The local map given above (i.e., $f^*(\mathrm{d}\mathbf{t}) \otimes \mathrm{d}\mathbf{s} \mapsto \mathrm{d}\mathbf{s} \wedge \mathrm{d}\mathbf{t}$) is independent of these local relative co-ordinates and hence globalises to give $\bar{\varphi}_{g,f}$.

Using the recipe that gives us ψ in (6.1.2) from $\bar{\psi}$ we get a well defined isomorphism in $\mathbf{D}_{\mathrm{c}}(X)$

$$\varphi_{g,f}\colon f^*\omega_g[e] \otimes_{\mathscr{O}_X} \omega_f[d] \xrightarrow{\ \sim\ } \omega_{gf}[d+e]. \tag{12.2.2}$$

Note that

$$\mathrm{H}^{-(d+e)}(\varphi_{g,f}) = \bar{\varphi}_{g,f} \tag{12.2.3}$$

and hence one can go back and forth between $\bar{\varphi}_{g,f}$ and $\varphi_{g,f}$.

Here is the main theorem:

THEOREM 12.2.4. *Let* $f\colon \mathscr{X} \to \mathscr{Y}$ *and* $g\colon \mathscr{Y} \to \mathscr{Z}$ *be maps in* $\mathbb{G}$ *which are smooth. Then*

$$\zeta_{g,f} = \varphi_{g,f}.$$

PROOF. We divide the proof into cases.

Case 1. Let A be a noetherian ring, $u_1, \ldots, u_d$, $v_1, \ldots, v_e$ analytically independent variables over A, and consider the A-algebras R, S, and T given by $R = A[[u_1, \ldots, u_d]]$, $S = R[[v_1, \ldots, v_e]] = A[[u_1, \ldots, u_d, v_1, \ldots, v_e]]$. Let $I = \mathbf{u}R$

be the R-ideal generated by $\mathbf{u} = (u_1, \dots, u_d)$, $J = \mathbf{v}S$ the S-ideal generated by $\mathbf{v} = (v_1, \dots, v_e)$, and $L = IS + J$. In other words, J is the S-ideal generated by $(\mathbf{u}, \mathbf{v})$.

Suppose $\mathscr{X} = \mathrm{Spf}(S, L)$, $\mathscr{Y} = \mathrm{Spf}(R, I)$, $\mathscr{Z} = Z = \mathrm{Spec}\, A$, and that our smooth maps $f\colon \mathscr{X} \to \mathscr{Y}$, $g\colon \mathscr{Y} \to \mathscr{Z} = Z$ are the natural maps corresponding to the maps of adic rings $(R, I) \to (S, L)$ and $(A, 0) \to (R, I)$.

We have additional schemes, namely $Y = \mathrm{Spec}\, R$, and $\mathscr{V} = \mathrm{Spf}(S, J)$. The natural maps between the adic rings involved give us a commutative diagram with the square on top being a cartesian square:

$$\begin{tikzcd}
\mathscr{X} \arrow[r, "\kappa'"] \arrow[d, "f"'] \arrow[dr, phantom, "\square"] & \mathscr{V} \arrow[d, "p"] \\
\mathscr{Y} \arrow[r, "\kappa"] \arrow[dr, "g"'] & Y^* \arrow[d, "q"] \\
& Z
\end{tikzcd}$$

The maps κ and $\kappa'$are completion maps and f, g, p, q are the obvious maps. Note that p, f, and g are smooth and pseudo-proper (however this is not true for the map q, which is not of pseudo-finite-type unless $d = 0$).

The rank one free $\mathscr{O}_{\mathscr{X}}$-modules ω_f and ω_{gf} correspond to the universal finite S-module of differentials $\omega_{S/R} := \widehat{\Omega}^e_{S/R}$ and $\omega_{S/A} := \widehat{\Omega}^{d+e}_{S/A}$. The rank one free $\mathscr{O}_{\mathscr{Y}}$-module ω_g corresponds to the universal finite R-module of degree d differentials $\omega_{R/A} := \widehat{\Omega}^d_{R/A}$. Thus

$$\begin{aligned}
\omega_{S/R} &= S\,\mathrm{d}v_1 \wedge \cdots \wedge \mathrm{d}v_e,\\
\omega_{S/A} &= S\,\mathrm{d}u_1 \wedge \cdots \wedge \mathrm{d}u_d \wedge \mathrm{d}v_1 \wedge \cdots \wedge \mathrm{d}v_e,\\
\omega_{R/A} &= R\,\mathrm{d}u_1 \wedge \cdots \wedge \mathrm{d}u_d.
\end{aligned}$$

The S-module $\omega_{S/R}$ gives us a rank one free $\mathscr{O}_{\mathscr{V}}$-module. A little thought shows us that this module is in fact ω_p. Define ω_q as the rank one free $\mathscr{O}_Y$-module corresponding to $\omega_{R/A}$. The equations $\Gamma(\mathscr{X}, \omega_f) = \omega_{S/R} = \Gamma(\mathscr{V}, \omega_p)$ and $\Gamma(\mathscr{Y}, \omega_g) = \omega_{R/A} = \Gamma(Y, \omega_q)$ can be re-written as

$$\omega_f = (\kappa')^*\omega_p \text{ and } \omega_g = \kappa^*\omega_q.$$

Write $\bar{\varphi}$ and φ for (the global sections of) the maps $\bar{\varphi}_{g,f}$ and $\varphi_{g,f}$. Then the S-module isomorphism

$$\bar{\varphi}\colon \omega_{R/A} \otimes_R \omega_{S/R} \xrightarrow{\sim} \omega_{S/A}$$

is given by $\bar{\varphi}(\mathrm{d}\mathbf{u} \otimes \mathrm{d}\mathbf{v}) = \mathrm{d}\mathbf{v} \wedge \mathrm{d}\mathbf{u}$. We have the following formula, where $\mathrm{tr}_{A[[\mathbf{u},\mathbf{v}]]/A}$ and $\mathrm{tr}_{R[[\mathbf{v}]]/R}$ are as in (8.1.9).

(∗)

$$\mathrm{tr}_{A[[\mathbf{u}]]/A}\begin{bmatrix}\mathrm{tr}_{R[[\mathbf{v}]]/R}\begin{bmatrix}\mathrm{d}\mathbf{v}\\ v_1^{\beta_1}, \dots, v_e^{\beta_e}\end{bmatrix}\mathrm{d}\mathbf{u}\\ u_1^{\alpha_1}, \dots, u_e^{\alpha_e}\end{bmatrix} = \mathrm{tr}_{A[[\mathbf{u},\mathbf{v}]]/A}\begin{bmatrix}\bar{\varphi}(\mathrm{d}\mathbf{u} \otimes \mathrm{d}\mathbf{v})\\ v_1^{\beta_1}, \dots, v_e^{\beta_e}, u_1^{\alpha_1}, \dots, u_d^{\alpha_d}\end{bmatrix}.$$

Indeed, if any of the α_l's or β_k's is not equal to 1, then both sides equal zero. If $\alpha_l = \beta_k = 1$ for $l = 1, \dots, d$, $k = 1, \dots, e$, both sides equal 1. This means that the following diagram commutes.

$$\begin{tikzcd}
\mathrm{H}_L^{d+e}(\omega_{R/A} \otimes_R \omega_{S/R}) \arrow[r, "\sim", "\bar{\varphi}"'] \arrow[d, "(6.2.4)"', "\wr"] & \mathrm{H}_L^{d+e}(\omega_{S/A}) \arrow[dd, "\mathrm{tr}_{A[[\mathbf{u},\,\mathbf{v}]]/A}"] \\
\mathrm{H}_I^{d}(\omega_{R/A} \otimes_R \mathrm{H}_J^e(\omega_{S/R})) \arrow[d, "\text{via } \mathrm{tr}_{R[[\mathbf{v}]]/R}"'] & \\
\mathrm{H}_I^{d}(\omega_{R/A}) \arrow[r, "\mathrm{tr}_{A[[\mathbf{u}]]/A}"'] & A
\end{tikzcd}$$

If, in the above diagram, we replace $\bar{\varphi}$ by $\bar{\zeta}_{g,f}$, then by Proposition 6.3.1 (b), the resulting diagram commutes. (See also Remark 6.3.4.) By the universal property of the pair $(\omega^{\#}_{S/A}, \mathrm{tr}_{S/A})$ we see that $\bar{\varphi} = \bar{\zeta}_{g,f}$, i.e., $\varphi_{g,f} = \zeta_{g,f}$.

Case 2. Suppose we have a section $\sigma\colon \mathscr{Z} \to \mathscr{X}$ and $\tau := f \circ \sigma$, and $\mathscr{X}$ and $\mathscr{Y}$ are the completions of $\mathscr{X}$ and $\mathscr{Y}$ along the closed subschemes given by the closed immersions $\sigma\colon \mathscr{Z} \hookrightarrow \mathscr{X}$ and $\tau\colon \mathscr{Z} \hookrightarrow \mathscr{Y}$ respectively. More precisely, if $\mathscr{I}_1 \subset \mathscr{O}_{\mathscr{X}}$ and $\mathscr{I}_2 \subset \mathscr{O}_{\mathscr{Y}}$ are the coherent ideals giving the embeddings of $\mathscr{Z}$ into $\mathscr{X}$ and $\mathscr{Y}$ (via σ and τ), and $\mathscr{I} \subset \mathscr{O}_{\mathscr{Z}}$ is an ideal of definition of $\mathscr{X}$, then $\mathscr{I}\mathscr{O}_{\mathscr{X}} + \mathscr{I}_1$ and $\mathscr{I}\mathscr{O}_{\mathscr{Y}} + \mathscr{I}_2$ are ideals of definition of $\mathscr{X}$ and $\mathscr{Y}$ respectively. Since the source and target of $\zeta_{g,f}$ and $\varphi_{g,f}$ are concentrated in one degree, the question of their equality is a local question on $\mathscr{X}$ and hence, without loss of generality, we may assume that the schemes involved are affine, say $\mathscr{X} = \mathrm{Spf}(S, L)$, $\mathscr{Y} = \mathrm{Spf}(R, I)$ and $\mathscr{Z} = \mathrm{Spf}(A, I_0)$ respectively. In fact we may assume that τ and σ are given by regular sequences $(u_1, \dots, u_d)$ and $(u_1, \dots, y_d, v_1, \dots, v_e)$ respectively, and $\mathbf{u}$ is analytically independent over A, and $\mathbf{v}$ is analytically independent over R. We then have a cartesian diagram (where the power series rings $A[[u_1, \dots, u_d]] = A[[\mathbf{u}]]$ and $R[[v_1, \dots, v_e]] = R[[\mathbf{v}]]$ are given the adic topologies from the ideals $(u_1, \dots, u_d)$ and $(v_1, \dots, v_e)$ repsectively)

$$\begin{tikzcd}
\mathscr{X} \arrow[r, "v"] \arrow[d, "f"'] \arrow[dr, phantom, "\square"] & \mathrm{Spf}\, R[[\mathbf{v}]] \arrow[d, "p"] \\
\mathscr{Y} \arrow[r, "u"] \arrow[d, "g"'] \arrow[dr, phantom, "\square"] & \mathrm{Spf}\, A[[\mathbf{u}]] \arrow[d, "q"] \\
\mathscr{Z} \arrow[r, "w"'] & \mathrm{Spec}\, A
\end{tikzcd}$$

with the horizontal arrows being the natural ones. Note that w is flat being a completion map. Therefore flat base change applies (see Proposition 12.1.2 (a)) and we have $\zeta_{g,f} = v^*\zeta_{q,p}$. Clearly $\varphi_{g,f} = v^*\varphi_{q,p}$ from the explicit description of $\varphi_{q,p}$ and $\varphi_{g,f}$. By Case 1, we have $\zeta_{q,p} = \varphi_{q,p}$. Applying v^* to both sides, we get the result for this case.

Case 3 (The General Case). In the general case, let $\mathscr{Y} \times_{\mathscr{Z}} \mathscr{X} = \mathscr{Q}$, $\mathscr{X} \times_{\mathscr{Z}} \mathscr{X} = \mathscr{P}$, and let $p\colon \mathscr{P} \to \mathscr{Q}$, $q\colon \mathscr{Q} \to \mathscr{X}$ be the base changes of f and g, and let $\pi_i\colon \mathscr{P} \to \mathscr{X}$ and π_2 be the projections $\mathscr{X} \times_{\mathscr{Z}} \mathscr{X} \to \mathscr{X}$, with $\pi_1 = q \circ p$. It $\Delta\colon \mathscr{X} \to \mathscr{P}$ is the diagonal immersion, then let $\kappa\colon \mathscr{U} \to \mathscr{X}$ be the completion of $\mathscr{X}$ with respect to $\Delta(\mathscr{X})$, and let $\kappa'\colon \mathscr{V} \to \mathscr{Y}$ be the completion of $\mathscr{Y}$ along $(p \circ \Delta)(\mathscr{X})$. We have a natural map $\widehat{p}\colon \mathscr{U} \to \mathscr{V}$ such that $\kappa' \circ \widehat{p} = p \circ \kappa$. Let

$\widehat{q} = q \circ \kappa'$ and let $\delta\colon \mathscr{X} \to \mathscr{U}$ be the natural closed immersion. We then have a commutative diagram with the two rectangles on the right being cartesian:

$$\begin{tikzcd}
\mathscr{X} \arrow[dr, "\delta"] \arrow[drrr, bend left, "\mathbf{1}_{\mathscr{X}}"] \arrow[dddrr, bend right, "\mathbf{1}_{\mathscr{X}}"'] & & & \\
& \mathscr{U} \arrow[r, "\kappa"] \arrow[d, "\widehat{p}"'] & \mathscr{P} \arrow[r, "\pi_2"] \arrow[d, "p"] \arrow[dr, phantom, "\square"] & \mathscr{X} \arrow[d, "f"] \\
& \mathscr{V} \arrow[r, "\kappa'"] \arrow[dr, "\widehat{q}"'] & \mathscr{Q} \arrow[r] \arrow[d, "q"] \arrow[dr, phantom, "\square"] & \mathscr{Y} \arrow[d, "g"] \\
& & \mathscr{X} \arrow[r, "f"'] & \mathscr{Z}
\end{tikzcd}$$

Now, using the explicit formula for $\varphi_{g,f}$, $\varphi_{q,p}$, and $\varphi_{\widehat{q},\widehat{p}}$, we see that $\pi_2^*\varphi_{g,f} = \varphi_{q,p}$ and $\kappa^*\varphi_{q,p} = \varphi_{\widehat{q},\widehat{p}}$. Thus $\kappa^*\pi_2^*\varphi_{g,f} = \varphi_{\widehat{q},\widehat{p}}$.

A similar relationship holds for the $\zeta_{\bullet,\bullet}$ maps. Indeed, by Proposition 12.1.2 (a) we have $\pi_2^*\zeta_{g,f} = \zeta_{q,p}$ and by Proposition 12.1.2 (d) we have $\kappa^*\zeta_{q,p} = \zeta_{\widehat{q},\widehat{p}}$, giving $\kappa^*\pi_2^*\zeta_{g,f} = \zeta_{\widehat{q},\widehat{p}}$. On the other hand, by Case 2 considered above, we have $\zeta_{\widehat{q},\widehat{p}} = \varphi_{\widehat{q},\widehat{p}}$. Thus

$$\kappa^*\pi_2^*\zeta_{g,f} = \kappa^*\pi_2^*\varphi_{g,f}.$$

Applying δ^* to both sides of this equation, and noting that $\pi_2 \circ \kappa \circ \delta = \mathbf{1}_{\mathscr{X}}$, we get the result. $\square$

CHAPTER 13

Applications of Transitivity

13.1. Iterated residues

Suppose $f\colon X \to Y$ is smooth of relative dimension e, $g\colon Y \to Z$ smooth of relative dimension d, $W_1 \hookrightarrow X$ a closed subscheme, finite and flat over Y, $W_2 \hookrightarrow Y$ a closed subscheme which is finite and flat over Z. Let $W = W_1 \cap f^{-1}(W_2) \hookrightarrow X$. Suppose further that W_1 is cut out by a quasi-regular sequence $\mathbf{v} = (v_1, \dots, v_e)$ in S and W_2 is cut out by a quasi-regular sequence $\mathbf{u} = (u_1, \dots, u_d)$ in R.

Theorem 13.1.1. *In the above situation, for $\nu \in \Gamma(\mathscr{O}_Y, \omega_g)$, $\mu \in \Gamma(\mathscr{O}_X, \omega_f)$, we have*

$$\mathbf{res}_{W_2}\begin{bmatrix}\mathbf{res}_{W_1}\begin{bmatrix}\mu \\ v_1^{\beta_1}, \dots, v_e^{\beta_e}\end{bmatrix}\nu \\ u_1^{\alpha_1}, \dots, u_d^{\alpha_d}\end{bmatrix} = \mathbf{res}_{W}\begin{bmatrix}\mu \wedge f^*\nu \\ v_1^{\beta_1}, \dots, v_e^{\beta_e}, u_1^{\alpha_1}, \dots, u_d^{\alpha_d}\end{bmatrix}$$

where, for notational simplicity, we denote the image of u_i in S also by u_i.

Proof. Recall from the definition of $\zeta_{g,f}$ in (12.1.1) that $\zeta_{g,f}$ is the transform of $\chi_{[g,\,f]}$ after applying Verdier's isomorphism to $f^!\mathscr{O}_Y$, $g^!\mathscr{O}_Z$ and $(gf)^!\mathscr{O}_Z$. From Theorem 12.2.4 and (6.3.2) we get

$$\mathbf{res}_{W_2}\begin{bmatrix}\mathbf{res}_{W_1}\begin{bmatrix}\mu \\ v_1^{\beta_1}, \dots, v_e^{\beta_e}\end{bmatrix}\nu \\ u_1^{\alpha_1}, \dots, u_d^{\alpha_d}\end{bmatrix} = \mathbf{res}_{W}\begin{bmatrix}\bar{\varphi}_{g,f}(\nu \otimes \mu) \\ v_1^{\beta_1}, \dots, v_e^{\beta_e}, u_1^{\alpha_1}, \dots, u_d^{\alpha_d}\end{bmatrix}.$$

The result then follows from the definition of $\bar{\varphi}_{g,f}$ in (12.2.1). $\square$

13.2. The Restriction Formula

An important application of our transitivity result is the so-called *Restriction Formula*, namely the formula in Corollary 13.2.7 below. The formula is related to the following problem. Suppose

(13.2.1)
$$\begin{tikzcd} X \arrow[r, hook, "i"] \arrow[dr, "f"'] & P \arrow[d, "\pi"] \\ & Y \end{tikzcd}$$

is a commutative diagram of ordinary schemes, with π and f smooth and separated and i a closed immersion. Let the relative dimension of π be $n = d + e$ and the relative dimension of f be e. As usual, let $\mathscr{N}_i^d$ be the d-th exterior power of the normal bundle $\mathscr{N}_i$ of X in P. We have, via Verdier's isomorphism and the isomorphism η_i' of (C.2.13), an isomorphism

(13.2.2)
$$a_{X/P}\colon i^*\omega_\pi[n] \otimes_{\mathscr{O}_X} \mathscr{N}_i^d[-d] \xrightarrow{\ \sim\ } \omega_f[e],$$

defined as the composite

$$
\begin{aligned}
i^*\omega_\pi[n] \otimes_{\mathscr{O}_X} \mathscr{N}_i^d[-d] &\xrightarrow{\eta_i'} i^!\omega_\pi[n] \\
&\xrightarrow{\mathbf{v}_\pi} i^!\pi^!\mathscr{O}_Y = i^!\omega_\pi^{\#}[n] \\
&\xrightarrow{\sim} f^!\mathscr{O}_Y = \omega_f^{\#}[e] \\
&\xrightarrow{\mathbf{v}_f^{-1}} \omega_f[e].
\end{aligned} \tag{13.2.2}
$$

The question then is, what is the concrete form of $a_{X/P}$ in terms of local relative coordinates? We answer the question in Theorem 13.2.6 below.

We leave it to the reader to check that $a_{X/P}$ is compatible with open immersions into Y and P. Indeed every map in the composition defining $a_{X/P}$ is well behaved with respect to open immersions into Y and P. Thus we may assume that P and Y are affine and that X is defined by an ideal generated by a quasi-regular sequence, which is part of a relative system of co-ordinates for $\pi\colon P \to Y$.

We now assume that $Y = \operatorname{Spec} A$, $P = \operatorname{Spec} S$, and $X = \operatorname{Spec} R$ where $R = S/I$, and I is generated by a quasi-regular sequence $\mathbf{t} = (t_1, \dots, t_d)$ in S, and there is an étale map $A[T_1, \dots, T_d, V, \dots, V_e] \to S$, (where T_l, $l = 1, \dots, d$, and V_k, $k = 1, \dots, e$ are algebraically independent variables), and t_i is the image of T_i for $i = 1, \dots, d$. This can always be achieved by shrinking Y and P (see [**BLR**, pp. 39–40, Prop. 7(c)]). Now every $\mu \in \omega_{S/A}$ can be written uniquely as

$$\mu = \mathrm{d}t_1 \wedge \cdots \wedge \mathrm{d}t_d \wedge \nu \tag{13.2.3}$$

with $\nu \in \wedge_S^e \Omega^1_{S/A}$. Define

$$b_{X/P}\colon i^*\omega_\pi \otimes_{\mathscr{O}_X} \mathscr{N}_i^d \xrightarrow{\sim} \omega_f \tag{13.2.4}$$

by the formula

$$\mu \otimes 1/\mathbf{t} \mapsto i^*\nu$$

where μ and ν are related by (13.2.3). We should clarify that $i^*\nu \in \omega_{R/A}$ is the pull-back of ν as a differential form. In other words, $i^*\nu$ is the image of ν under the composite of maps $\wedge_S^e \Omega^1_{S/A} \to R \otimes_S \wedge_S^e \Omega^1_{S/A} \to \wedge_R^e \Omega^1_{R/A} = \omega_{R/A}$.

In what follows, let

$$\bar{a}_{X/P}\colon i^*\omega_\pi \otimes_{\mathscr{O}_X} \mathscr{N}_i^d \to \omega_f \tag{13.2.5}$$

be the map

$$\bar{a}_{X/P} = \mathrm{H}^0(a_{X/P}).$$

The notation follows the conventions we have been using throughout, and as observed earlier, $a_{X/P}$ can be recovered from $\bar{a}_{X/P}$ (see §6.1).

THEOREM 13.2.6. *Under the above assumptions on i, π, f, A, S, and R, we have*

$$\bar{a}_{X/P} = b_{X/P}.$$

PROOF. Since the results we have established have been stated in terms of the abstract dualizing sheaves of the form $\omega_f^{\#}$, rather than in terms of ω_f, it is convenient for us to have an analogue of $a_{X/P}$ taking values in $\omega_f^{\#}[e]$. To that end, suppose $k\colon W \hookrightarrow P$ is a regular immersion of codimension $m \le n$, and that $g = \pi \circ k$ is flat over Y, so that $g\colon W \to Y$ is Cohen-Macaulay of relative dimension $n - m$. Define

$$a^{\#}_{W/P}\colon k^*\omega_\pi[n] \otimes_{\mathscr{O}_W} \mathscr{N}_k^m[-m] \xrightarrow{\sim} \omega_g^{\#}[n-m]$$

as the composite:

$$\begin{aligned} k^*\omega_\pi[n]\otimes_{\mathscr{O}_W}\mathscr{N}_k^m[-m] &\xrightarrow{\eta'_k} k^!\omega_\pi[n] \\ &\xrightarrow{\mathbf{v}_\pi} k^!\pi^!\mathscr{O}_Y = k^!\omega_\pi^\#[n] \\ &\xrightarrow{\ \sim\ } g^!\mathscr{O}_Y = \omega_g^\#[n-m] \end{aligned}$$

(Similarly we have an isomorphism $a^\#_{W/X}$ for a regular immersion $W \hookrightarrow X$ of Y-schemes such that $W \to Y$ is flat.) If $W \to Y$ is finite (in addition to being flat), so that $m = n$, and W is given by the vanishing of $\mathbf{v} = (v_1, \dots, v_n)$, then by the definition of the map $\boldsymbol{\tau}^\#_{g,\pi,k}$ in (3.4.2), we have:

$$(\dagger) \qquad \mathrm{tr}^\#_g \circ h_*(a^\#_{W/X}) = \boldsymbol{\tau}^\#_{g,\pi,k}.$$

By definition of $a_{X/P}$, it is clear that $\mathbf{v}_f \circ a_{X/P} = a^\#_{X/P}$.

We first prove that the map $\bar{a}_{X/P}$ is compatible with base change. In greater detail, suppose $u\colon Y' \to Y$ is a map, $X' := X\times_Y Y'$, $P' := P\times_Y Y'$, and let $f'\colon X' \to Y'$, $\pi'\colon P' \to Y'$, $i'\colon X' \to P'$, $w\colon P' \to P$, $v\colon X' \to X$ be the resulting maps obtained from base change. We then clearly have $v^*(i^*\omega_\pi \otimes_{\mathscr{O}_X} \mathscr{N}_i^d) = i'^*\omega_{\pi'} \otimes_{\mathscr{O}_{X'}} \mathscr{N}_{i'}^d$ and $v^*\omega_f = \omega_{f'}$. We claim that $v^*\bar{a}_{X/P} = \bar{a}_{X'/P'}$. In what follows, $Y' = \operatorname{Spec} A'$, $R' = R\otimes_A A'$, $S' = S\otimes_A A'$.

Consider the base change isomorphism $\theta = \theta^f_u\colon v^*\omega^\#_f \xrightarrow{\ \sim\ } \omega^\#_{f'}$ of part (a) of [**S2**, p.740, Theorem 2.3.5 (a)]. According to loc.cit. (b) we have a commutative diagram

$$\begin{tikzcd} v^*\omega_f \arrow[rr, "\sim", "v^*(\bar{\mathbf{v}}_f)"'] \arrow[d, equal] & & v^*\omega^\#_f \arrow[d, "\theta", "\wr"'] \\ \omega_{f'} \arrow[rr, "\sim", "\bar{\mathbf{v}}_{f'}"'] & & \omega^\#_{f'} \end{tikzcd}$$

Since, $\mathbf{v}_f \circ a_{X/P} = a^\#_{X/P}$, from the above diagram we see that it is enough to show that $\theta \circ v^* a^\#_{X/P} = a^\#_{X'/P'}$ in order to show that $v^*a_{X/P} = a_{X'/P'}$.

Let η_i and η'_i be the maps defined in (C.2.11) and (C.2.13) of Appendix C. By definition, $\eta'_i = $ (B.1.2) $\circ\, \eta_i$. It follows that $\bar{a}^\#_{X/P}$ is the composite of isomorphisms:

$$i^*\omega_\pi \otimes_{\mathscr{O}_X} \mathscr{N}_i^d \xrightarrow[\text{(C.2.7)}]{\ \sim\ } \mathscr{E}xt^d_{\mathscr{O}_P}(\mathscr{O}_X,\, \omega_\pi) \xrightarrow[\text{(B.1.2)}]{\ \sim\ } \mathrm{H}^0(i^!\omega^\#_\pi[n]) \xrightarrow{\ \sim\ } \omega^\#_f.$$

Let the composite of the last two maps in the above composition be denoted $c_{X/P}\colon \mathscr{E}xt^d_{\mathscr{O}_P}(\mathscr{O}_X,\, \omega_\pi) \xrightarrow{\ \sim\ } \omega^\#_f$. Consider the diagram

$$(\ddagger) \qquad \begin{tikzcd} v^*(i^*\omega_\pi \otimes_{\mathscr{O}_X} \mathscr{N}_i^d) \arrow[r, "\sim", "\text{(C.2.7)}"'] \arrow[d, equal] & v^*\,\mathscr{E}xt^d_{\mathscr{O}_P}(\mathscr{O}_X,\, \omega_\pi) \arrow[r, "\sim", "c_{X/P}"'] \arrow[d, "\wr"] & v^*\omega_f \arrow[d, "\theta", "\wr"'] \\ i'^*\omega_{\pi'} \otimes_{\mathscr{O}_{X'}} \mathscr{N}_{i'}^d \arrow[r, "\sim", "\text{(C.2.7)}"'] & \mathscr{E}xt^d_{\mathscr{O}_{P'}}(\mathscr{O}_{X'},\, \omega_{\pi'}) \arrow[r, "\sim", "c_{X'/P'}"'] & \omega_{f'} \end{tikzcd}$$

where the isomorphism in the middle is the natural one, which we now describe. Let $Q^\bullet \to R$ be a projective resolution of the S-module R. Then $Q^\bullet \otimes_R R' =$

$Q^\bullet \otimes_A A' \to R \otimes_A A' = R'$ is an S'-projective resolution of the S'-module R'. Now

$$\begin{aligned}\mathrm{Hom}^\bullet_S(Q^\bullet,\, \omega_{S/A}[d]) \otimes_A A' &= \mathrm{Hom}^\bullet_{S'}(Q^\bullet \otimes_A A',\, \omega_{S/A}[d] \otimes_A A') \\ &= \mathrm{Hom}^\bullet_{S'}(Q^\bullet \otimes_A A',\, \omega_{S'/A'}[d]).\end{aligned}$$

Since $\mathbf{t}$ is a quasi-regular sequence in S, we can (and will) pick $Q^\bullet$ to be the version of the Koszul homology complex on $\mathbf{t}$ such that $\mathrm{Hom}^\bullet_S(Q^\bullet,\, S) = K^\bullet(\mathbf{t})$, and the equality $\mathrm{Hom}^\bullet_S(Q^\bullet,\, \omega_{S/A}[d]) \otimes_A A' = \mathrm{Hom}^\bullet_{S'}(Q^\bullet \otimes_A A',\, \omega_{S'/A'}[d])$ reduces to the well-known equality $\omega_{S/A}[d] \otimes_S K^\bullet(\mathbf{t}) \otimes_S S' = \omega_{S'/A'}[d] \otimes_{S'} K^\bullet(\mathbf{t}')$, where $\mathbf{t}' = \mathbf{t} \otimes 1$. By right-exactness of tensor products, we get:

$$\begin{aligned}\mathrm{H}^0(\omega_{S/A}[d] \otimes_S K^\bullet(\mathbf{t})) \otimes_S S' &= \mathrm{H}^0(\omega_{S/A}[d] \otimes_S K^\bullet(\mathbf{t}) \otimes_S S') \\ &= \mathrm{H}^0(\omega_{S'/A'}[d] \otimes_{S'} K^\bullet(\mathbf{t}')).\end{aligned}$$

The isomorphism $v^* \mathscr{E}xt^d_{\mathscr{O}_P}(\mathscr{O}_X,\, \omega_\pi) \xrightarrow{\sim} \mathscr{E}xt^d_{\mathscr{O}_{P'}}(\mathscr{O}_{X'},\, \omega_{\pi'})$ then follows from the isomorphism in (C.2.3). (See also the proof of Lemma 1 of [**L1**, pp.39–40] as well as [**S2**, p.762, (8.9)] for the case when $X \hookrightarrow P$ is not necessarily a regular immersion, but R is relatively Cohen-Macaulay over A). The description of the isomorphism we have given also shows that the rectangle on the left in diagram (‡) above commutes. The rectangle on the right commutes by [**S2**, p.741, Theorem 2.3.6]. Thus diagram (‡) commutes and $\theta^f_u \circ v^* \bar{a}^\#_{X/P} = \bar{a}^\#_{X'/P'}$, i.e., $v^* \bar{a}_{X/P} = \bar{a}_{X'/P'}$.

Next suppose we have a closed subscheme $j\colon Z \hookrightarrow X$ such that (a) $h = f \circ j\colon Z \to Y$ is an isomorphism, and (b), if $\bar{L} \subset R$ is the ideal of R which defines Z, then $\bar{L}$ is generated by a quasi-regular sequence $\bar{\mathbf{u}} = (\bar{u}_1, \ldots, \bar{u}_e)$. Let $u_i \in S$ be lifts of $\bar{u}_i \in R$ for $i = 1, \ldots, e$. Let L be the ideal generated by $(\mathbf{t}, \mathbf{u})$. Then L is the ideal defining the closed immersion $ij\colon Z \hookrightarrow P$. Let $B = \Gamma(Z,\, \mathscr{O}_Z)$. For a sequence of positive integeres $\mathbf{m} = (m_1, \ldots, m_e)$, let $\mathbf{u}^{\mathbf{m}} = (u_1^{m_1},\, \ldots,\, u_e^{m_e})$, $\bar{\mathbf{u}}^{\mathbf{m}} = (\bar{u}_1^{m_1},\, \ldots,\, \bar{u}_e^{m_e})$, $L_{\mathbf{m}}$ the R-ideal generated by $\bar{\mathbf{u}}^{\mathbf{m}}$, $B_{\mathbf{m}} = R/L_{\mathbf{m}}$, $Z_{\mathbf{m}} = \operatorname{Spec} B_{\mathbf{m}}$ and $j_{\mathbf{m}}\colon Z_{\mathbf{m}} \hookrightarrow X$ the natural closed immersion. Let $h_{\mathbf{m}}\colon Z_{\mathbf{m}} \to Y$ be the finite flat map $h_{\mathbf{m}} = f \circ j_{\mathbf{m}}$. Finally let $\kappa\colon \mathscr{X} \to X$ be the completion of X along Z.

For $\mu \in \omega_{S/A}$, and positive integers m_i, $i = 1, \ldots, e$, it is easy to see that

$$(*) \qquad \mathbf{res}_{Z,\pi} \begin{bmatrix} \mu \\ t_1,\, \ldots,\, t_d, u_1^{m_1},\, \ldots,\, u_e^{m_e} \end{bmatrix} = \mathbf{res}_{Z,f} \begin{bmatrix} b_{X/P}(\mu \otimes 1/\mathbf{t}) \\ \bar{u}_1^{m_1},\, \ldots,\, \bar{u}_e^{m_e} \end{bmatrix}.$$

Indeed, we can write μ in a unique manner as $\mu = f\mathrm{d}t_1 \wedge \cdots \wedge \mathrm{d}t_d \wedge \mathrm{d}u_1 \wedge \cdots \wedge \mathrm{d}u_e$, with $f \in S$. Then $b_{X/P}(\mu \otimes 1/\mathbf{t}) = \bar{f} \wedge \mathrm{d}\bar{u}_1 \wedge \cdots \wedge \mathrm{d}\bar{u}_e$, where $\bar{f}$ is the image of f in R. Both sides of $(*)$ are then realised as the coefficient of $u_1^{m_1-1}u_2^{m_2-1} \ldots u_e^{m_e-1}$ in the power series expansion of f, whence$(*)$ holds. On the other hand, according to Proposition C.6.6,

$$\bar{a}^\#_{Z_{\mathbf{m}}/P}(\mu \otimes 1/(\mathbf{t}, \mathbf{u}^{\mathbf{m}})) = \bar{a}^\#_{Z_{\mathbf{m}}/X}(\bar{a}_{X/P}(\mu \otimes 1/\mathbf{t})) \otimes 1/\bar{\mathbf{u}}^{\mathbf{m}}).$$

Apply $\mathrm{tr}^\#_{h_{\mathbf{m}}} \circ h_{\mathbf{m}*}$ to both sides. By (†) and Proposition 3.5.4, this yields,

$$(**) \qquad \mathbf{res}_{Z,\pi} \begin{bmatrix} \mu \\ t_1,\, \ldots,\, t_d, u_1^{m_1},\, \ldots,\, u_e^{m_e} \end{bmatrix} = \mathbf{res}_{Z,f} \begin{bmatrix} \bar{a}_{X/P}(\mu \otimes 1/\mathbf{t}) \\ \bar{u}_1^{m_1},\, \ldots,\, \bar{u}_e^{m_e} \end{bmatrix}.$$

From $(*)$ and $(**)$ we conclude that $\mathbf{res}_Z\big[{\bar{a}_{X/P}(\mu\otimes 1/\mathbf{t}) \atop \bar{\mathbf{u}}^{\mathbf{m}}}\big] = \mathbf{res}_Z\big[{b_{X/P}(\mu\otimes 1/\mathbf{t}) \atop \bar{\mathbf{u}}^{\mathbf{m}}}\big]$. Now apply local duality, i.e. Corollary 3.2.4, to conclude that $\kappa^*\bar{a}_{X/P} = \kappa^* b_{X/P}$. This means that on a Zariski open neighbourhood of Z, $\bar{a}_{X/P} = b_{X/P}$. We point out that the hypothesis that Z be defined globally by the vanishing of a quasi-regular sequence is not necessary to reach this conclusion, since j is a regular immersion and locally, one can arrange this. In other words, if we have a section of f, then in an open neighbourhood U of the image of the section, $\bar{a}_{X/P}|_U = b_{X/P}|_U$.

In the general case, let $X'' = X \times_Y X$, $P' = P \times_Y X$, and consider the cartesian square

$$\begin{tikzcd} X'' \arrow[r, "p_2"] \arrow[d, "p_1"'] \arrow[dr, phantom, "\square"] & X \arrow[d, "f"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

We know that $p_2^*\bar{a}_{X/P} = \bar{a}_{X''/P'}$. It is clear from the description of $b_{X/P}$ that it is compatible with arbitrary base change and hence $p_2^* b_{X/P} = b_{X''/P'}$ Then by what we have proven, there is a Zariski open subscheme V of X'' containing the diagonal such that

$$p_2^*\bar{a}_{X/P}|_V = p_2^* b_{X/P}|_V.$$

Let $\Delta\colon X \hookrightarrow V$ be the map induced by the diagonal immersion $X \hookrightarrow X''$. Applying Δ^* to both sides of the displayed equation above, we see that $\bar{a}_{X/P} = b_{X/P}$. $\square$

Corollary 13.2.7. *Let* $\mathbf{v} = (v_1, \dots, v_e) \in \Gamma(P, \mathscr{O}_P) = S$*,* J *the ideal in* S *generated by* $(\mathbf{t}, \mathbf{v})$*,* $Z = \operatorname{Spec} S/J$*, and* v_i' *the restriction of* v_i *to* Z *for* $i = 1, \dots, e$*. If* $Z \to Y$ *is finite and flat, then*

$$\mathbf{res}_{Z,\pi}\begin{bmatrix} \mathrm{d}t_1 \wedge \cdots \wedge \mathrm{d}t_d \wedge \nu \\ t_1, \dots, t_d, v_1, \dots, v_e \end{bmatrix} = \mathbf{res}_{Z,f}\begin{bmatrix} i^*\nu \\ v_1', \dots, v_e' \end{bmatrix}.$$

for $\nu \in \wedge^e \Omega^1_{S/A}$*.*

Proof. Theorem 13.2.6 together with Proposition C.6.6 yields

$$\bar{a}^{\#}_{Z/P}(\mathrm{d}t_1 \wedge \cdots \wedge \mathrm{d}t_d \wedge \nu \otimes 1/(\mathbf{t}, \mathbf{v})) = \bar{a}^{\#}_{Z/X}(i^*\nu \otimes 1/\mathbf{v}')$$

where $\bar{a}^{\#}_{Z/P}$ and $\bar{a}^{\#}_{Z/X}$ are as in the proof of Theorem 13.2.6 and $\mathbf{v}'$ is $(v_1', \dots, v_e')$. Let $h\colon Z \to Y$ be the composite $Z \hookrightarrow X \xrightarrow{f} Y$. Applying $\mathrm{tr}^{\#}_h \circ h_*$ to both sides, we get the result. (See Proposition 3.5.4.) $\square$

13.2.8. Quasi-finite maps. Suppose the map $\pi\colon P \to Y$ in (13.2.1) factors as $P \xrightarrow{p} W \xrightarrow{g} Y$, with p smooth of relative dimension d, and g smooth of relative dimension e, and assume $h = p \circ \pi$ is *quasi-finite*. In other words we have a commutative diagram of ordinary schemes

$$\begin{tikzcd} X \arrow[r, hook, "i"] \arrow[d, "h"'] & P \arrow[d, "\pi"] \arrow[dl, "p"] \\ W \arrow[r, "g"'] & Y \end{tikzcd}$$

with h quasi-finite and $f = g \circ h = p \circ i$, and with p, π, g and f smooth of relative dimensions d, $d + e$, e and e, respectively. To lighten notation, we write

$$\mathscr{N} = \mathscr{N}_i^d = (\wedge^d_{\mathscr{O}_X} \mathscr{I}/\mathscr{I}^2)^*$$

where $\mathscr{I}$ is the quasi-coherent ideal sheaf in $\mathscr{O}_P$ defining $i\colon X \hookrightarrow P$.

Since h is quasi-finite and flat over W (the latter because p is smooth, and i is a local complete intersection map), for quasi-coherent $\mathscr{O}_W$-module $\mathscr{F}$, $h^!\mathscr{F}$ can be identified with $\mathrm{H}^0(h^!\mathscr{F})$ in the standard way, and we will do so in what follows. With this convention, we have three isomorphisms which we now describe. First, we clearly have

$$h^!\omega_g \xrightarrow{\ \sim\ } \omega_f \tag{13.2.8.1}$$

via the isomorphism $h^!g^! \xrightarrow{\ \sim\ } f^!$, and Verdier's isomorphisms for f and g.

Next, for a quasi-coherent $\mathscr{O}_W$-module $\mathscr{F}$, we have the transitivity isomorphism

$$\chi^h(\mathscr{F}, \mathscr{O}_W)\colon h^*\mathscr{F} \otimes_{\mathscr{O}_X} h^!\mathscr{O}_W \xrightarrow{\ \sim\ } h^!\mathscr{F} \tag{13.2.8.2}$$

of (5.2.1). Since we are dealing with ordinary schemes, taking account of our choice of order of tensor product, this is the same as the map $\chi^h_{\mathscr{F},\mathscr{O}_W}$ of [**L4**, p. 231, (4.9.1.1)]. The map $\chi^h(\mathscr{F}, \mathscr{O}_W)$ is an isomorphism since h is flat and hence perfect [**L4**, pp. 234–235, Thm. 4.9.4].

Finally, we have an isomorphism

$$i^*\omega_p \otimes \mathscr{N} \xrightarrow{\ \sim\ } h^!\mathscr{O}_W \tag{13.2.8.3}$$

given by $\eta_i'(\omega_p[d])\colon i^\blacktriangle(\omega_p[d]) \xrightarrow{\ \sim\ } i^!\omega_p$ of (C.2.13), the isomorphism $i^!p^! \xrightarrow{\ \sim\ } h^!$, and Verdier's isomorphism $\bar{\mathbf{v}}\colon \omega_p[d] \xrightarrow{\ \sim\ } p^!\mathscr{O}_W$.

These three isomorphisms are related in the following way.

Proposition 13.2.8.4. *The following diagram of isomorphisms commutes*

$$\begin{tikzcd}
i^*(p^*\omega_g \otimes_{\mathscr{O}_P} \omega_p) \otimes_{\mathscr{O}_X} \mathscr{N} \arrow[r, "\bar{\varphi}_{g,p}", "\sim"'] \arrow[d, equal] & i^*\omega_\pi \otimes_{\mathscr{O}_X} \mathscr{N} \arrow[r, "\bar{a}_{X/P}", "\sim"'] & \omega_f \\
h^*\omega_g \otimes_{\mathscr{O}_X} (i^*\omega_p \otimes_{\mathscr{O}_X} \mathscr{N}) \arrow[d, "\mathbf{1}\otimes(13.2.8.3)"', "\wr"] & & \\
h^*\omega_g \otimes_{\mathscr{O}_X} h^!\mathscr{O}_W \arrow[rr, "\sim", "(13.2.8.2)"'] & & h^!\omega_g \arrow[uu, "\wr"', "(13.2.8.1)"]
\end{tikzcd}$$

where $\bar{\varphi}_{g,\,p}$ is the explicit map described in (12.2.1) *and $\bar{a}_{X/P}$ is the map* (13.2.5) *described locally, via Theorem* 13.2.6*, by the explicit map $b_{X/P}$ in* (13.2.4)*.*

Proof. The essential point is that other than (13.2.8.1), all other maps in the diagram are various avatars of transitivity maps. The map η_i' which is used in the definition of (13.2.8.3) is

$$\chi^i(-, \mathscr{O}_P)\colon \mathbf{L}i^*(-) \overset{\mathbf{L}}{\otimes}_{\mathscr{O}_X} i^!\mathscr{O}_P \xrightarrow{\ \sim\ } i^!$$

with $\mathscr{N}[-d]$ substituted for $i^!\mathscr{O}_P$, via the canonical isomorphism

$$\eta'(\mathscr{O}_P)\colon \mathscr{N}[d] \xrightarrow{\ \sim\ } i^!\mathscr{O}_P$$

(see also eqrefdiag:1-tensor-Tr for another way of looking at this).

Next, according to Theorem 12.2.4, and the definition in (12.1.1), the map $\bar{\varphi}_{g,\,p}$ is H^{d+e} of the composite (after substituting $g^!\mathscr{O}_Y$, $\pi^!\mathscr{O}_Y$, $p^!\mathscr{O}_W$ with $\omega_g[e]$, $\omega_\pi[d{+}e]$, and $\omega_p[d]$ respectively, via Verdier's isomorphisms):

$$p^*g^!\mathscr{O}_Y \otimes_{\mathscr{O}_P} p^!\mathscr{O}_W \xrightarrow{\ \sim\ } p^!g^!\mathscr{O}_Y \xrightarrow{\ \sim\ } \pi^!\mathscr{O}_Y,$$

where the first arrow is the transitivity map $\chi^p(g^!\mathscr{O}_Y, \mathscr{O}_W)$, which is an isomorphism since p is flat and hence perfect.

The map $\bar{a}_{X/P}$ is, according to (13.2.2) and (13.2.2) (after the usual Verdier substitutions and the substitution $\mathscr{N}[-d] \xrightarrow{\sim} i^!\mathscr{O}_P$), H^d applied to the composite

$$\mathbf{L}i^*\pi^!\mathscr{O}_Y \overset{\mathbf{L}}{\otimes}_{\mathscr{O}_X} i^!\mathscr{O}_P \xrightarrow{\sim} i^!\pi^!\mathscr{O}_Y \xrightarrow{\sim} f^!\mathscr{O}_Y$$

where the first arrow is the transitivity map $\chi^i(\pi^!\mathscr{O}_Y, \mathscr{O}_P)$. This is an isomorphism since a regular immersion is a perfect map.

Finally, (13.2.8.1) is by definition $\chi^h(\omega_g, \mathscr{O}_W)$.

Consider the diagram below, in which the arrows are either natural ones arising from the pseudofunctorial nature of $-^!$ or from abstract transitivity maps, and in which:

$$i^* = \mathbf{L}i^* \text{ and } \otimes = \overset{\mathbf{L}}{\otimes}.$$

$$\begin{tikzcd}
i^*(p^*g^!\mathscr{O}_Y \otimes p^!\mathscr{O}_W) \otimes i^!\mathscr{O}_P \arrow[r, "\sim"] \arrow[d, equal] & i^*(p^!g^!\mathscr{O}_Y) \otimes i^!\mathscr{O}_P \arrow[r, "\sim"] & i^*\pi^!\mathscr{O}_Y \otimes i^!\mathscr{O}_P \arrow[d, "\wr"] \\
h^*(g^!\mathscr{O}_Y) \otimes (i^*p^!\mathscr{O}_W \otimes i^!\mathscr{O}_P) \arrow[dd, "\wr"'] & & i^!\pi^!\mathscr{O}_Y \arrow[d, "\wr"] \\
& & f^!\mathscr{O}_Y \\
h^*(g^!\mathscr{O}_Y) \otimes i^!p^!\mathscr{O}_W \arrow[r, "\sim"] & h^*(g^!\mathscr{O}_Y) \otimes h^!\mathscr{O}_W \arrow[r, "\sim"] & h^!g^!\mathscr{O}_Y \arrow[u, "\wr"']
\end{tikzcd}$$

The diagram commutes by Prop.-Def 5.2.4 (ii) and [**L4**, p. 238]. The Proposition follows. $\square$

CHAPTER 14

Traces of differential forms for finite maps

14.1. Tate traces

Let A be a ring, and C an A-algebra which is finite and free as an A-module. We have the *canonical trace*

$$\mathrm{Trc}_{C/A}\colon C \to A \tag{14.1.1}$$

given by the composite

$$C \longrightarrow \mathrm{End}_A(C,\, C) \longrightarrow A$$

where the first arrow is the map $c \mapsto (x \mapsto cx)$ and the second the standard trace of an endomorphism of a finite free A-module.

If the C-module $\mathrm{Hom}_A(C,\, A)$ is a free C-module of rank one (this happens if and only if, in addition to C being a finite free A-module, its fibres are Gorenstein) then, following Kunz in [**Ku**], we regard any free generator of $\mathrm{Hom}_A(C,\, A)$ as a "trace" for the A-algebra C (cf. [**Ku**, F8 (b), pp. 362–363]). If there is one, then clearly we have exactly as many as the units of C. We point out that the canonical trace, $\mathrm{Trc}_{C/A}$, need not be a trace in this sense on C. Indeed if A and C are fields and C is a purely inseparable extension of A, then $\mathrm{Trc}_{C/A} = 0$ and hence cannot be a free generator of $\mathrm{Hom}_A(C, A)$.

Tate studies the existence and characterisation of traces in an important situation which includes the case of C being a complete intersection algebra over A.

In the rest of this sub-section we make the following assumptions and use the following notations. The A-algebra C (which is free of finite rank as an A-module) is such that the canonical map $A \to C$ factors as

$$A \longrightarrow B \xrightarrow{\pi} C$$

with π a surjective map, the kernel I of π generated by a regular B-sequence $\mathbf{f} = (f_1, \dots, f_n)$, and the kernel J of the canonical map

$$s\colon B \otimes_A C \longrightarrow C$$

is generated by a $B \otimes_A C$-sequence $\mathbf{g} = (g_1, \dots, g_n)$. In somewhat greater detail, if $m\colon C \otimes_A C \to C$ is the A-algebra map $c \otimes c' \mapsto cc'$, then s is the composition

$$B \otimes_A C \xrightarrow{\pi \otimes \mathbf{1}_C} C \otimes_A C \xrightarrow{\ m\ } C.$$

Note that $f_i \otimes 1 \in J$ and hence we have $h_{ij} \in B \otimes_A C$ such that $f_i \otimes 1 = \sum_{j=1}^{n} h_{ij} g_j$ for $i = 1, \dots, n$. Let

$$\Delta = \det(h_{ij}), \tag{14.1.2}$$

and

$$\overline{\Delta} = (\pi \otimes \mathbf{1}_C)(\Delta). \tag{14.1.3}$$

Set $\overline{J} = \ker m = (\pi \otimes \mathbf{1}_C)(J)$. We have the following commutative diagram with

$$\check{s}\colon C \otimes_A B \to C$$

being the composite $m \circ (1_C \otimes \pi)$.

$$
\begin{tikzcd}
 & & & C \\
C \arrow[r] & C \otimes_A B \arrow[r, "1_C \otimes \pi"] \arrow[urr, bend left, "s^\vee"] & C \otimes C \arrow[ur, "m"] & \\
B \arrow[u, "\pi"] \arrow[r] & B \otimes_A B \arrow[u, "\pi \otimes 1_B"] \arrow[r, "1_B \otimes \pi"'] & B \otimes_A C \arrow[u, "\pi \otimes 1_C"] \arrow[uur, bend right, "s"'] & \\
A \arrow[u] \arrow[r] & B \arrow[u] \arrow[r, "\pi"'] & C \arrow[u] &
\end{tikzcd}
\tag{14.1.4}
$$

In the above situation it is shown in [**MR**, Appendix] that traces exist (i.e., $\mathrm{Hom}_A(C, A)$ is a rank one free C-module) and there is a canonical free generator (i.e., a trace) $\lambda = \lambda(\mathbf{f}, \mathbf{g})$ of $\mathrm{Hom}_A(C, A)$. We summarise the results of Tate as given in [**MR**, Appendix] in the following two theorems in which we make the standard indentifications $B \otimes_A \mathrm{Hom}_A(C, A) = \mathrm{Hom}_B(B \otimes_A C, B)$ and $C \otimes_A \mathrm{Hom}_A(C, A) = \mathrm{Hom}_C(C \otimes_A C, C)$. Under these identifications it is clear that

$$\pi \circ (1_B \otimes \phi) = (1_C \otimes \phi) \circ (\pi \otimes 1_C) \qquad (\phi \in \mathrm{Hom}_A(C, A)). \tag{14.1.5}$$

THEOREM 14.1.6. (Tate) [**MR**, p.231, Lemma (A.10)] *The map*

$$t\colon \mathrm{Hom}_A(C, A) \longrightarrow C$$

given by

$$\phi \mapsto \pi((1_B \otimes \phi)(\Delta)) = (1_C \otimes \phi)(\overline{\Delta})$$

is an isomorphism of C-modules.

In *loc.cit.* the description of t is $\phi \mapsto \pi((1_B \otimes \phi)(\Delta))$. Using (14.1.5) it is clear that t can also be described as $\phi \mapsto (1_C \otimes \phi)(\overline{\Delta})$.

The results in [**MR**, Appendix] are perhaps more useful when stated in the following way.

THEOREM 14.1.7. (Tate) *Let $\lambda = \lambda(\mathbf{f}, \mathbf{g})$ be the free C-module generator of* $\mathrm{Hom}_A(C, A)$ *given by*

$$\lambda = t^{-1}(1)$$

where $t\colon \mathrm{Hom}_A(C, A) \xrightarrow{\sim} C$ *is the isomorphism in Theorem 14.1.6.*

(a) *If $\phi \in \mathrm{Hom}_A(C, A)$, then the constant of proportionality $c \in C$ such that $\phi = c\lambda$, is given by*

$$c = \pi((1_B \otimes \phi)(\Delta) = (1_C \otimes \phi)(\overline{\Delta}).$$

(b) *If $\psi \in \mathrm{Hom}_B(B \otimes_A C, B)$ and $\phi \in \mathrm{Hom}_A(C, A)$ are such that $1_B \otimes \phi - \psi \in$ $J\mathrm{Hom}_B(B \otimes_A C, B)$, then*

$$\pi((1_B \otimes \phi)(\Delta)) = \pi\psi(\Delta).$$

(c) *If* $\psi \in \mathrm{Hom}_C(C\otimes_A C,\, C)$ *and* $\phi \in \mathrm{Hom}_A(C,\, A)$ *are such that* $1_C\otimes\phi-\psi \in \overline{J}\mathrm{Hom}_C(C\otimes_A C,\, C)$, *then*

$$(1_C\otimes\phi)(\overline{\Delta})) = \psi(\overline{\Delta}).$$

(d) *If* $\mathrm{Trc}_{C/A}\colon C\to A$ *is the canonical trace given in* (14.1.1), *then*

$$\mathrm{Trc}_{C/A} = m(\overline{\Delta})\lambda.$$

PROOF. These are all results in [**MR**, Appendix], stated in perhaps a different way. Part (a) is [*ibid*, pp. 229–230, 3. of Theorem (A.3)] (together with (14.1.5)). Part (b) is an immediate consequence of [*ibid*, p. 230, Lemma (A.9)] and (c) is the same, together with (14.1.5). Part (d) is [*ibid*, pp. 229–230, 4. of Theorem (A.3)]. ☐

The first application of Tate's result we give is the following (this is (R6) of [**RD**, p. 198] but for our version of residues).

THEOREM 14.1.8. *In the above situation, suppose* B *is smooth of relative dimension* n *over* A, $f\colon X\to Y$ *the corresponding smooth map from* $X=\mathrm{Spec}\,B$ *to* $Y=\mathrm{Spec}\,A$, *and* $Z=\mathrm{Spec}\,C$. *Then*

$$\mathbf{res}_{Z,f}\begin{bmatrix} b\,\mathrm{d}f_1\wedge\cdots\wedge\mathrm{d}f_n\\ f_1,\dots,f_n\end{bmatrix} = \mathrm{Trc}_{C/A}(b|_Z).$$

PROOF. It is important to keep diagram (14.1.4) in mind when following this proof. There is an annoying issue that Δ is defined in terms of $f_i\otimes 1_C$ and g_i, but in dealing with the base change $1_C\otimes\phi$, for $\phi\in\mathrm{Hom}_A(C,\,A)$, the natural elements that show up are $1_C\otimes f_i\in C\otimes_A B$. One has to do somewhat careful book-keeping to avoid confusion. Since $C\otimes_A B$ and $B\otimes_A C$ play different roles, let us agree to write $x^\vee$ for the element of $C\otimes_A B$ corresponding to $x\in B\otimes_A C$ under the standard isomorphism between $B\otimes_A C$ and $C\otimes_A B$.

In what follows, the C-algebra structures on $C\otimes_A B$ and $C\otimes_A C$ are $c\mapsto c\otimes 1_B$ and $c\mapsto c\otimes 1_C$ respectively. Let $N=(I/I^2)^*$ and $N_C = C\otimes_A N$. Let $h\colon Z\to Y$ be the natural finite flat map corresponding to $A\to C$ and $i\colon Z\hookrightarrow X$ the natural closed immersion, with normal bundle $\mathscr{N}$. If $\boldsymbol{\tau_{C/A}}\colon \Omega^n_{B/A}\otimes_B\wedge^n_C N\to A$ is the map arising from the composite (all isomorphisms being the obvious ones, e.g., the fundamental local isomorphism, Verdier's isomorphism, ...)

$$h_*(i^*(\Omega^n_{X/Y})\otimes_Z\wedge^n\mathscr{N})\xrightarrow{\ \sim\ }\mathrm{H}^0(h_*(i^!f^!\mathscr{O}_Y))\xrightarrow{\ \sim\ }\mathrm{H}^0(h_*h^!\mathscr{O}_Y)\xrightarrow{\mathrm{H}^0(\mathrm{Tr}_h)}\mathscr{O}_Y$$

then $(\Omega^n_{B/A}\otimes_B\wedge^n_C N, \boldsymbol{\tau_{C/A}})$ represents the functor $M\mapsto\mathrm{Hom}_A(M,\,A)$ from finite C-modules to finite A-modules, whence we have an isomorphism of C-modules

$$\Phi\colon\Omega^n_{B/A}\otimes_C\wedge^n_C N\xrightarrow{\ \sim\ }\mathrm{Hom}_A(C,\,A)$$

with $\boldsymbol{\tau_{C/A}}$ corresponding to "evaluation at 1" under this isomorphism. According to Proposition 3.5.4, we have

$$\boldsymbol{\tau_{C/A}}(\mu\otimes\mathbf{1/f}) = \mathbf{res}_Z\begin{bmatrix}\mu\\ f_1,\dots f_n\end{bmatrix}\qquad(\mu\in\Omega^n_{B/A}).$$

Thus

$$\Phi(\mu\otimes\mathbf{1/f})(c) = \mathbf{res}_Z\begin{bmatrix}b\cdot\mu\\ f_1,\dots,f_n\end{bmatrix}\qquad(c\in C)$$

where $b\in B$ is any pre-image of c. If $b\in I$, then $b\,\mathrm{d}f_1\wedge\cdots\wedge\mathrm{d}f_n\otimes 1/\mathbf{f}=0$ in $\Omega^n_{B/A}\otimes_B\wedge^n N$ and hence the right side of the above displayed formula is well-defined as a function of $c\in C$.

Similarly we have an isomorphism of $C\otimes_A C$-modules

$$\Phi'\colon \Omega^n_{(C\otimes_A B)/C}\otimes\wedge^n N_C \xrightarrow{\ \sim\ } \mathrm{Hom}_C(C\otimes_A C, C)$$

given by

$$\Phi'(\nu\otimes\mathbf{1}/(1_C\otimes\mathbf{f}))(x)=\mathbf{res}_{Z\times_Y Z,p}\begin{bmatrix}\widetilde{x}\cdot\mu\\(1_C\otimes f_1),\dots,(1_C\otimes f_n)\end{bmatrix}\qquad (x\in C\otimes_A C)$$

where $\widetilde{x}\in C\otimes_A B$ is any pre-image of x and $p\colon Z\times_Y X\to Z$ is the natural projection.

Let $s^\vee\colon C\otimes_A B\to C$ be as in (14.1.4), i.e., $s^\vee=m\circ(1_C\otimes\pi)$. Then $J^\vee:=\ker s^\vee$ is generated by $g_1^\vee,\dots,g_n^\vee$.

Let $\phi\in\mathrm{Hom}_A(C,\,A)$ and $\psi\in\mathrm{Hom}_C(C\otimes_A C,\,C)$ be the maps defined by

$$\phi=\Phi(\mathrm{d}f_1\wedge\cdots\wedge\mathrm{d}f_n\otimes\mathbf{1}/\mathbf{f}),$$

and

$$\psi=\Phi'((\Delta^\vee\cdot\mathrm{d}g_1^\vee\wedge\cdots\wedge\mathrm{d}g_n^\vee)\otimes\mathbf{1}/(1_C\otimes\mathbf{f})).$$

We have to show that $\phi=\mathrm{Trc}_{C/A}$. By Theorem 14.1.7 (d), this is equivalent to showing that $(1_C\otimes\phi)(\overline{\Delta})=m(\overline{\Delta})$. It easier to show that $\psi(\overline{\Delta})=m(\overline{\Delta})$, and we can reduce to this via Theorem 14.1.7 (c). The details are as follows. First, we claim that $1_C\otimes\phi-\psi\in\overline{J}\mathrm{Hom}_C(C\otimes_A C,\,C)$ so that Theorem 14.1.7 (c) applies. Before we prove the claim, we point out that

$$1_C\otimes\phi=\Phi'((\mathrm{d}(1_C\otimes f_1)\wedge\cdots\wedge\mathrm{d}1_C\otimes f_n)\otimes\mathbf{1}/(1_C\otimes\mathbf{f}).$$

Since $1_C\otimes f_i=\sum_j h_{ij}^\vee g_j^\vee$ we have

$$\mathrm{d}(1_C\otimes f_1)\wedge\cdots\wedge\mathrm{d}(1_C\otimes f_n)=\mu+\Delta^\vee\,\mathrm{d}g_1^\vee\wedge\cdots\wedge\mathrm{d}g_n^\vee$$

where $\mu\in J^\vee\Omega^n_{(C\otimes_A B)/C}$ (for $h_{ij}^\vee\in J^\vee$). It follows that

$$1_C\otimes\phi-\psi=\Phi'(\mu\otimes(\mathbf{1}/(1_C\otimes\mathbf{f})))\in\overline{J}\mathrm{Hom}_C(C\otimes_A C,\,C)$$

as claimed.

We then have, with $\delta\in C\otimes_A B$ a lift of $\overline{\Delta}\in C\otimes_A C$,

$$\begin{aligned}(1_C\otimes\phi)(\overline{\Delta})=\psi(\overline{\Delta})&=\mathbf{res}_{Z\times_Y Z,p}\begin{bmatrix}\delta\,\Delta^\vee\,\mathrm{d}g_1^\vee\wedge\cdots\wedge\mathrm{d}g_n^\vee\\(1_C\otimes f_1),\dots,(1_C\otimes f_n)\end{bmatrix}\\ (*)\qquad\qquad &=\mathbf{res}_{Z,p}\begin{bmatrix}\delta\,\mathrm{d}g_1^\vee\wedge\cdots\wedge\mathrm{d}g_n^\vee\\g_1^\vee,\dots,g_n^\vee\end{bmatrix}\\ &=s^\vee(\delta)\\ &=m(\overline{\Delta}).\end{aligned}$$

In the above sequence, the first equality is from Theorem 14.1.7 (c), the one in the second line from Theorem 3.5.5, the third from the fact that the composite $Z\xrightarrow{\text{via } s^\vee} Z\times_Y X\xrightarrow{p} Z$ is an isomorphism, which means the formulae in Remark 10.3.3 apply. The last equality is from the definition of $s^\vee$ as $m\circ(1_C\otimes\pi)$. From $(*)$ and Theorem 14.1.7 (d) we get that $\phi=\mathrm{Trc}_{C/A}$, and from this the Theorem follows. □

REMARKS 14.1.9. 1) The above proof would be easier if one could show that $\Delta^\vee$ is a pre-image of $\overline{\Delta}$ under $C\otimes_A B\xrightarrow{1_C\otimes\pi} C\otimes_A C$. But there is no guarantee it is so. However, in the special case where B is a polynomial ring over A, something like this be arranged as the proof Proposition 14.1.10 below shows.

2) If B is flat over A, then λ is stable under any base change of A. In somewhat greater detail, if $A \to A'$ is a map of rings, B', C', $\mathbf{f}'$, and $\mathbf{g}'$ the obvious base changes of B, C, $\mathbf{f}$, and $\mathbf{g}$, then, under the identification $\mathrm{Hom}_{A'}(C' A') = A' \otimes_A \mathrm{Hom}_A(C, A)$, we have $\lambda(\mathbf{f}', \mathbf{g}') = 1 \otimes \lambda(\mathbf{f}, \mathbf{g})$. This is because, if B is flat over A, then $\mathbf{f}'$ and $\mathbf{g}'$ are regular sequences.

PROPOSITION 14.1.10. *Let $q \in A[T_1, \dots, T_n] = A[\mathbf{T}]$. Suppose B is the A-algebra $B = A[\mathbf{T}]_q$. For $i = 1, \dots, n$ let $\gamma_i = \pi(T_i)$ and*

$$g_i = T_i \otimes 1_C - 1_B \otimes \gamma_i.$$

Let $Z = \operatorname{Spec} C$. Then $\lambda = \lambda(\mathbf{f}, \mathbf{g}) \in \mathrm{Hom}_A(C, A)$ is given by

$$\lambda(c) = \mathbf{res}_Z \begin{bmatrix} b\,\mathrm{d}T_1 \wedge \cdots \wedge \mathrm{d}T_n \\ f_1, \dots, f_n \end{bmatrix} \qquad (c \in C)$$

where $b \in B = A[T_1, \dots, T_n]$ is any pre-image of c.

PROOF. It is straightforward to see that the g_i, as defined in the Proposition, generate $J = \ker s$, and form a regular $B \otimes_A C$-sequence. As before, let $X = \operatorname{Spec} B$, $Y = \operatorname{Spec} A$, $Z = \operatorname{Spec} C$, and let $p\colon Z \times_Y X \to Z$ be the projection map. As we did earlier, we need to distinguish between $B \otimes_A C$ and $C \otimes_A B$, and so between $X \times_Y Z$ and $Z \times_Y X$, and p corresponds to the map $C \to C \otimes_A B$ given by $c \mapsto c \otimes 1$.

For the proof of the theorem, it is simpler to regard the two copies of B in Diagram (14.1.4), the one in the middle of the bottom row, and the one in the middle of the left column, as two different copies of $A[\mathbf{T}]_q$, say $A[X_1, \dots, X_n]_{q(X_1,\dots,X_n)} = A[\mathbf{X}]_{q(\mathbf{X})}$ and $A[Y_1, \dots, Y_n]_{q(Y_1,\dots,Y_n)} = A[\mathbf{Y}]_{q(\mathbf{Y})}$ respectively. Then $B \otimes_A B$ can be regarded as $A[\mathbf{X}, \mathbf{Y}]_{q(\mathbf{X})q(\mathbf{Y})}$. Moreover, $B \otimes_A C$ is then identified with $C[\mathbf{Y}]_{q(\mathbf{Y})}$ and $C \otimes_A B$ with $C[\mathbf{X}]_{q(\mathbf{X})}$. Diagram (14.1.4) translates to

$$\begin{tikzcd}
 & & & C \\
C \arrow[r] & C[\mathbf{X}]_{q(\mathbf{X})} \arrow[r, "\pi_1''"] \arrow[urr, bend left, "s^\vee"] & C \otimes C \arrow[ur, "m"] & \\
A[\mathbf{Y}]_{q(\mathbf{Y})} \arrow[r] \arrow[u, "\pi_2"] & A[\mathbf{X}, \mathbf{Y}]_{q(\mathbf{X})q(\mathbf{Y})} \arrow[r, "\pi_1'"'] \arrow[u, "\pi_2'"] & C[\mathbf{Y}]_{q(\mathbf{Y})} \arrow[u, "\pi_2''"] \arrow[uur, bend right, "s"'] & \\
A \arrow[r] \arrow[u] & A[\mathbf{X}]_{q(\mathbf{X})} \arrow[r, "\pi_1"'] \arrow[u] & C \arrow[u] &
\end{tikzcd}$$

Here π_1 is the map $X_i \mapsto \gamma_i$, and π_2 is $Y_i \mapsto \gamma_i$. The maps π_1' and π_1'' are the base changes of π_1, and π_2', π_2'' the base changes of π_2. We point out that

$$\pi_1''\left(\sum_{\underline{i}} c_{\underline{i}} \mathbf{X}^{\underline{i}}\right) = \sum_{\underline{i}} c_{\underline{i}} \otimes \boldsymbol{\gamma}^{\underline{i}}$$

and

$$\pi_2''\left(\sum_{\underline{i}} c_{\underline{i}} \mathbf{Y}^{\underline{i}}\right) = \sum_{\underline{i}} \boldsymbol{\gamma}^{\underline{i}} \otimes c_{\underline{i}}.$$

For any $h \in B = A[\mathbf{T}]_q$, the element $h \otimes 1_B$ (resp. $h \otimes 1_C$) is identified with the element $h(\mathbf{Y})$ of $A[\mathbf{X}, \mathbf{Y}]_{q(\mathbf{X})q(\mathbf{Y})} = B \otimes_A B$ (resp. the element $h(\mathbf{Y})$ of $C[\mathbf{Y}]_{q(\mathbf{Y})} =$

$B\otimes_A C$), whereas $1_B\otimes h$ and $1_C\otimes h$ are identified with $h(\mathbf{X})$ (regarded as elements of $A[\mathbf{X},\,\mathbf{Y}]_{q(\mathbf{X})q(\mathbf{Y})}$ and of $C[\mathbf{X}]_{q(\mathbf{X})}$ respectively). Finally $s(\sum_{\underline{i}} c_{\underline{i}}\mathbf{Y}^{\underline{i}}) = \sum_{\underline{i}} c_{\underline{i}}\boldsymbol{\gamma}^{\underline{i}}$ and $s^\vee(\sum_{\underline{i}} c_{\underline{i}}\mathbf{X}^{\underline{i}}) = \sum_{\underline{i}} c_{\underline{i}}\boldsymbol{\gamma}^{\underline{i}}$. It follows that

$$g_i = Y_i - \gamma_i, \quad \text{and} \quad g_i^\vee = X_i - \gamma_i \qquad (i = 1,\dots n).$$

Now there exist $h_{ij}(\mathbf{X},\mathbf{Y}) \in A[\mathbf{X},\,\mathbf{Y}]$ such that

$$f_i(\mathbf{X}) - f_i(\mathbf{Y}) = \sum_j h_{ij}(\mathbf{X},\mathbf{Y})(X_j - Y_j).$$

Then $f_i(\mathbf{Y}) = \sum_j h_{ij}(\boldsymbol{\gamma},\mathbf{Y})(Y_j-\gamma_j)$ and $f_i(\mathbf{X}) = \sum_j h_{ij}(\mathbf{X},\boldsymbol{\gamma})(X_j-\gamma_j)$. Let

$$\delta(\mathbf{X},\,\mathbf{Y}) = \det\left(h_{ij}(\mathbf{X},\mathbf{Y})\right).$$

If Δ is defined as in (14.1.2), then

$$\Delta = \delta(\boldsymbol{\gamma},\,\mathbf{Y}).$$

Note that

$$(*) \qquad \overline{\Delta} = \pi_2''(\Delta) = \pi_1''(\delta(\mathbf{X},\,\boldsymbol{\gamma})).$$

On the other hand, since $f_i(\mathbf{X}) = \sum_{i,j} h_{ij}(\mathbf{X},\boldsymbol{\gamma})(X_i-\gamma_i)$, according to Theorem 3.5.5 we have

$$(**) \qquad \mathbf{res}_{Z\times_Y Z,\,p}\begin{bmatrix}\delta(\mathbf{X},\,\boldsymbol{\gamma})\mu\\ f_1(\mathbf{X}),\dots,f_n(\mathbf{X})\end{bmatrix} = \mathbf{res}_Z\begin{bmatrix}\mu\\ g_1^\vee,\dots,g_n^\vee\end{bmatrix} \qquad \left(\mu\in\Omega^n_{C[\mathbf{X}]/C}\right).$$

Let $\phi\colon C\to A$ be defined by

$$\phi(c) = \mathbf{res}_Z\begin{bmatrix} b\,\mathrm{d}T_1\wedge\dots\wedge\mathrm{d}T_n\\ f_1,\dots,f_n\end{bmatrix} \qquad (c\in C)$$

where $b\in B = A[\mathbf{T}]$ is any element in $\pi^{-1}(c)$. Since $\mathrm{d}X_i = \mathrm{d}(X_i-\gamma_i)$, therefore for $x\in C\otimes_A C$ and $\widetilde{x}\in C[\mathbf{X}]$ such that $\pi_1''(\widetilde{x}) = x$, we have

$$(1_C\otimes\phi)(x) = \mathbf{res}_{Z\times_Y Z,p}\begin{bmatrix}\widetilde{x}\ \mathrm{d}(X_1-\gamma_1)\wedge\dots\wedge\mathrm{d}(X_n-\gamma_n)\\ f_1(\mathbf{X}),\dots,f_n(\mathbf{X})\end{bmatrix}.$$

By $(*)$ and $(**)$ we get

$$(1_C\otimes\phi)(\overline{\Delta}) = \mathbf{res}_Z\begin{bmatrix}\mathrm{d}g_1^\vee\wedge\dots\wedge\mathrm{d}g_n^\vee\\ g_1^\vee,\dots,g_n^\vee\end{bmatrix} = 1.$$

Theorem 14.1.7 (a) then gives $\phi = \lambda$. □

14.2. Traces of differential forms

Suppose we have a commutative diagram of ordinary schemes

$$(14.2.1) \qquad \begin{tikzcd} X \arrow[d, "h"'] \arrow[dr, bend left, "f"] & \\ Y \arrow[r, "g"] & Z \end{tikzcd}$$

with f and g smooth of relative dimension n and h a finite map (necessarily flat). The composite $h_*f^!\mathscr{O}_Z \xrightarrow{\sim} h_*h^!g^!\mathscr{O}_Z \xrightarrow{\mathrm{Tr}_h} g^!\mathscr{O}_Z$ gives a $\mathscr{O}_Y$-map map (after applying Verdier's isomorphism to $f^!\mathscr{O}_Z$ and $g^!\mathscr{O}_Z$ and applying $\mathrm{H}^{-n}(-)$)

$$(14.2.2) \qquad \mathrm{tr}_h\colon h_*\omega_f \longrightarrow \omega_g.$$

A note of caution. We have used the symbol tr_p earlier for the trace map $\mathrm{R}^m p_*\omega_f \to \mathscr{O}_W$ for a smooth proper map $p\colon V \to W$ of relative dimension m. The context will make the meaning of the symbol clear.

PROPOSITION 14.2.3. *Let W be a closed subscheme of Y, proper over Z, and let $W' = h^{-1}(W)$. Assume g (and hence f) is separated. Then the following diagram commutes:*

$$\begin{tikzcd}
\mathrm{R}^n_{W'} f_*\omega_f \arrow[d, "\mathbf{res}_{W'}"'] & \mathrm{R}^n_W g_* h_*\omega_f \arrow[l, "\sim"'] \arrow[d, "\text{via } \mathrm{tr}_h"] \\
\mathscr{O}_Z & \mathrm{R}^n_W g_*\omega_g \arrow[l, "\mathbf{res}_W"]
\end{tikzcd}$$

PROOF. By Nagata's compactification [N] we have an open immersion $u\colon Y \to \overline{Y}$ together with a proper map $\bar{g}\colon \overline{Y} \to Z$ such that $\bar{g}\circ u = g$. By Zariski's Main Theorem the quasi-finite map $u\circ h\colon X \to \overline{Y}$ can be completed to a finite map, i.e., we can find an open immersion $v\colon X \to \overline{X}$ and a finite map $\bar{h}\colon \overline{X} \to \overline{Y}$ such that $u\circ h = \bar{h}\circ v$. Moreover, we may assume X is scheme-theoretically dense in $\overline{X}$ so that $\bar{h}^{-1}(u(Y)) = v(X)$. Let $\bar{f} = \bar{g}\circ\bar{h}$. We have a composite

$$(\dagger) \qquad \bar{h}_*\bar{f}^! \xrightarrow{\ \sim\ } \bar{h}_*\bar{h}^!\bar{g}^! \xrightarrow{\mathrm{Tr}_{\bar{h}}} \bar{g}^!.$$

Consider the commutative diagram

$$\begin{tikzcd}
\mathbf{R}g_*\mathbf{R}\varGamma_W\omega_g[n] \arrow[r, "\sim"] & \mathbf{R}\bar{g}_*\mathbf{R}\varGamma_{u(W)}\bar{g}^!\mathscr{O}_Z \arrow[r] & \mathbf{R}\bar{g}_*\bar{g}^!\mathscr{O}_Z \arrow[r, "\mathrm{Tr}_{\bar{g}}"] & \mathscr{O}_Z \arrow[ddd, equal] \\
\mathbf{R}g_*\mathbf{R}\varGamma_W h_*\omega_f[n] \arrow[u, "\mathrm{tr}_h"] \arrow[r, "\sim"] \arrow[d, "\wr"'] & \mathbf{R}\bar{g}_*\mathbf{R}\varGamma_{u(W)}\bar{h}_*\bar{f}^!\mathscr{O}_Z \arrow[u, "(\dagger)"] \arrow[r] \arrow[d, "\wr"'] & \mathbf{R}\bar{g}_*\bar{h}_*\bar{f}^!\mathscr{O}_Z \arrow[u, "(\dagger)"] \arrow[d, equal] & \\
\mathbf{R}g_*\mathbf{R}h_*\mathbf{R}\varGamma_{W'}\omega_f[n] \arrow[r, "\sim"] \arrow[d, "\wr"'] & \mathbf{R}\bar{g}_*\mathbf{R}\bar{h}_*\mathbf{R}\varGamma_{v(W')}\bar{f}^!\mathscr{O}_Z \arrow[r] \arrow[d, "\wr"'] & \mathbf{R}\bar{g}_*\mathbf{R}\bar{h}_*\bar{f}^!\mathscr{O}_Z \arrow[d, "\wr"'] & \\
\mathbf{R}f_*\mathbf{R}\varGamma_{W'}\omega_f[n] \arrow[r, "\sim"] & \mathbf{R}\bar{f}_*\mathbf{R}\varGamma_{v(W')}\bar{f}^!\mathscr{O}_Z \arrow[r] & \mathbf{R}\bar{f}_*\bar{f}^!\mathscr{O}_Z \arrow[r, "\mathrm{Tr}_{\bar{f}}"'] & \mathscr{O}_Z
\end{tikzcd}$$

The rectangle on the right commutes by definition of $(\dagger)$ (especially of the isomorphism $\bar{h}^!\bar{g}^! \xrightarrow{\ \sim\ } \bar{f}^!$ which drives $(\dagger)$).

Applying $\mathrm{H}^0(-)$ to the above diagram we get the asserted result. $\square$

PROPOSITION 14.2.4. *Let f, g, h be as above, and suppose $u\colon Z' \to Z$ is a map of ordinary schemes. Let*

$$\begin{tikzcd}
X' \arrow[r, "w"] \arrow[d, "h'"'] \arrow[dd, bend right=60, "f'"'] \arrow[dr, phantom, "\square"] & X \arrow[d, "h"] \arrow[dd, bend left=60, "f"] \\
Y' \arrow[r, "v"] \arrow[d, "g'"'] \arrow[dr, phantom, "\square"] & Y \arrow[d, "g"] \\
Z' \arrow[r, "u"'] & Z
\end{tikzcd}$$

be the corresponding base change diagram. Then $v^*\mathrm{tr}_h = \mathrm{tr}_{h'}$.

PROOF. By [**EGA**, IV$_3$, (13.3.2)], Y can be covered by open subschemes U such that $U \to Y$ is the composite of a quasi-finite map $U \to \mathbb{P}^n_Z$ followed by the structural map $\mathbb{P}^n_Z \to Z$. Since the question is local on Y, we replace Y by U if necessary, and assume we have a quasi-finite map $Y \to \mathbb{P}^n_Z$. Using Zariski's Main Theorem we can find a finite map $\overline{Y} \to \mathbb{P}^n_Z$ such that Y is an open $\mathbb{P}^n_Z$-subscheme of $\overline{Y}$.

Since h is finite, the composite $X \to Y \hookrightarrow \overline{Y}$ is quasi-finite, and another application of Zariski's Main Theorem tells us that $X \to \overline{Y}$ factors as an open immersion $X \hookrightarrow \overline{X}$ followed by a finite map $\bar{h}\colon \overline{X} \to \overline{Y}$. Replacing $\overline{X}$ by the scheme theoretic closure of its open subscheme X if necessary, we may assume that X is scheme theoretically dense in $\overline{X}$. This forces $X = \bar{h}^{-1}(Y)$. We thus have a cartesian diagram, with horizontal arrows being open immersions

$$\begin{tikzcd} X \arrow[r, hook] \arrow[d, "h"'] \arrow[dr, phantom, "\square"] & \overline{X} \arrow[d, "\bar{h}"] \\ Y \arrow[r, hook] & \overline{Y} \end{tikzcd}$$

We write $\bar{g}\colon \overline{Y} \to Z$ for the composite $\overline{Y} \to \mathbb{P}^n_Z \to Z$, and set $\bar{f} = \bar{g} \circ \bar{h}$. The important point is that $\bar{f}\colon \overline{X} \to Z$ and $\bar{g}\colon \overline{Y} \to Z$ are proper over Z and *the fibres of* $\bar{f}$ *and* $\bar{g}$ *have dimension* $\leq n$. This means $\mathrm{H}^j(\bar{f}^!\mathscr{O}_Z) = \mathrm{H}^j(\bar{g}^!\mathscr{O}_Z) = 0$ for $j < -n$. It follows that if $\omega^\#_{\bar{f}} := \mathrm{H}^{-n}(\bar{f}^!\mathscr{O}_Z)$, and if $\mathrm{tr}^\#_{\bar{f}}\colon \mathrm{R}^n\bar{f}_*\omega^\#_{\bar{f}} \to \mathscr{O}_Z$ is the map induced by $\mathrm{Tr}_{\bar{f}}(\mathscr{O}_Z)\colon \mathbf{R}\bar{f}_*\bar{f}^!\mathscr{O}_Z \to \mathscr{O}_Z$ then $(\omega^\#_{\bar{f}}, \mathrm{tr}^\#_{\bar{f}})$ represents the functor $\mathscr{F} \mapsto \mathrm{Hom}_Z(\mathrm{R}^n\bar{f}_*\mathscr{F},\, \mathscr{O}_Z)$ of quasi-coherent sheaves $\mathscr{F}$ on X (see Remark 3.2.5 for this argument). Along these lines, if $\omega^\#_{\bar{g}} := \mathrm{H}^{-n}(\bar{g}^!\mathscr{O}_Z)$, and $\mathrm{tr}^\#_{\bar{g}}\colon \mathrm{R}^n\bar{g}_*\omega^\#_{\bar{g}} \to \mathscr{O}_Z$, the map induced by $\mathrm{Tr}_{\bar{g}}(\mathscr{O}_Z)$, then one can make a similar statement about $(\omega^\#_{\bar{g}}, \mathrm{tr}^\#_{\bar{g}})$.

Let $\overline{X}' = X \times_Z Z'$, $\overline{Y}' = \overline{Y} \times_Z Z'$, and let $\bar{f}'$, $\bar{g}'$, $\bar{h}'$, $\bar{u}$, $\bar{v}$ be the obvious base changes of f, g, h, u, and v, respectively. Let $\mathrm{tr}^\#_h\colon h_*\omega^\#_f \to \omega^\#_g$ be the obvious analogue of tr_h, namely

$$\mathrm{tr}^\#_h = \mathrm{H}^{-n}\Big(h_*f^!\mathscr{O}_Z \xleftarrow{\ \sim\ } h_*h^!g^!\mathscr{O}_Z \xrightarrow{\mathrm{Tr}_h} g^!\mathscr{O}_Z\Big). \tag{14.2.4.1}$$

Similarly define $\mathrm{tr}^\#_{h'}$, $\mathrm{tr}^\#_{\bar{h}}$, and $\mathrm{tr}^\#_{\bar{h}'}$. Since $\bar{g}$ and $\bar{f}$ are proper, $\mathrm{tr}^\#_{\bar{h}}\colon \bar{h}_*\omega^\#_{\bar{f}} \to \omega^\#_{\bar{g}}$ has the following alternative description: It is the unique element of $\mathrm{Hom}_Y(\mathrm{R}^n\bar{g}_*\bar{h}_*\omega^\#_{\bar{f}},\, \mathscr{O}_Z)$ corresponding, via adjointness, to the composite

$$\mathrm{R}^n\bar{g}_*\bar{h}_*\omega^\#_{\bar{f}} \xleftarrow{\ \sim\ } \mathrm{R}^n\bar{f}_*\omega^\#_{\bar{f}} \xrightarrow{\mathrm{tr}^\#_{\bar{f}}} \mathscr{O}_Z.$$

Let $\theta^{\bar{f}}_u\colon \bar{w}^*\omega^\#_{\bar{f}} \to \omega^\#_{\bar{f}'}$ and $\theta^{\bar{g}}_u\colon \bar{v}^*\omega^\#_{\bar{g}} \to \omega^\#_{\bar{g}'}$ be the base change isomorphisms defined in [**S2**, pp. 738–739, Rmk. 2.3.2, especially (2.5)]. We claim that the following diagram commutes:

$$\begin{tikzcd} \bar{v}^*\bar{h}_*\omega^\#_{\bar{f}} \arrow[r, equal] \arrow[d, "\bar{v}^*\mathrm{tr}^\#_{\bar{h}}"'] & \bar{h}'_*\bar{w}^*\omega^\#_{\bar{f}} \arrow[r, "\bar{h}'_*\theta^{\bar{f}}_u"] & \bar{h}'_*\omega^\#_{\bar{f}'} \arrow[d, "\mathrm{tr}^\#_{\bar{h}'}"] \\ \bar{v}^*\omega^\#_{\bar{g}} \arrow[rr, "\theta^{\bar{g}}_u"'] & & \omega^\#_{\bar{g}'}. \end{tikzcd} \tag{$*$}$$

Suppose $(*)$ commutes. Restricting $(*)$ to Y, and using the Verdier isomorphisms for f, $\bar f$, g, and $\bar g$ and [**S2**, p. 739, Theorem 2.3.3, especially (c)] (which states that via these isomorphisms θ^f_u and θ^g_u are the identity maps) we get $v^*\mathrm{tr}_h = \mathrm{tr}_{h'}$ as we wish. The commutativity of $(*)$ is equivalent to

$$(\dagger)\qquad \mathrm{tr}^{\#}_{\bar g'}(\mathrm{R}^n\bar g'_*(\mathrm{tr}^{\#}_{\bar h'}\circ \bar h'_*\theta^{\bar f}_u)) = \mathrm{tr}^{\#}_{\bar g'}(\mathrm{R}^n\bar g'_*(\theta^{\bar g}_u\circ \bar v^*\mathrm{tr}^{\#}_{\bar h}))$$

The proof of $(\dagger)$ rests on the fact that the following diagram of functors commutes

$$(\ddagger)\qquad \begin{tikzcd}
u^*\mathrm{R}^n\bar f \arrow[r, "\sim"] \arrow[dd, "\wr"'] & u^*\mathrm{R}^n\bar g_*\bar h_* \arrow[dr] & \\
& & \mathrm{R}^n\bar g'_*\bar v^*\bar h_* \\
\mathrm{R}^n\bar f'_*\bar w^* \arrow[r, "\sim"] & \mathrm{R}^n\bar g'_*\bar h'_*\bar w^* \arrow[ur, equal] &
\end{tikzcd}$$

In greater detail, consider the following diagram:

$$\begin{tikzcd}
u^*\mathrm{R}^n\bar f_*\omega^{\#}_{\bar f} \arrow[r, equal] \arrow[dddd, "u^*\mathrm{tr}^{\#}_{\bar f}"'] & u^*\mathrm{R}^n\bar f_*\omega^{\#}_{\bar f} \arrow[rrr, "\sim"] \arrow[d, "\wr"'] & & & u^*\mathrm{R}^n\bar g_*\bar h_*\omega^{\#}_{\bar f} \arrow[d, equal] \\
& \mathrm{R}^n\bar f'_*\bar w^*\omega^{\#}_{\bar f} \arrow[r, "\sim"] \arrow[d, "\theta^{\bar f}_u"'] & \mathrm{R}^n\bar g'_*\bar h'_*\bar w^*\omega^{\#}_{\bar f} \arrow[r, equal] \arrow[d, "\theta^{\bar f}_u"] & \mathrm{R}^n\bar g'_*\bar v^*\bar h_*\omega^{\#}_{\bar f} \arrow[dd, "\mathrm{tr}^{\#}_{\bar h}"'] & u^*\mathrm{R}^n\bar g_*\bar h_*\omega^{\#}_{\bar f} \arrow[l, "\sim"'] \arrow[dd, "\mathrm{tr}^{\#}_{\bar h}"] \\
& \mathrm{R}^n\bar f'_*\omega^{\#}_{\bar f'} \arrow[r, "\sim"] \arrow[dd, "\mathrm{tr}^{\#}_{\bar f'}"'] & \mathrm{R}^n\bar g'_*\bar h'_*\omega^{\#}_{\bar f'} \arrow[d, "\mathrm{tr}^{\#}_{\bar h'}"] \arrow[r, phantom, "\blacksquare"] & & \\
& & \mathrm{R}^n\bar g'_*\omega^{\#}_{\bar g'} \arrow[d, "\mathrm{tr}^{\#}_{\bar g'}"] & \mathrm{R}^n\bar g'_*\bar v^*\omega^{\#}_{\bar g} \arrow[l, "\theta^{\bar g}_u"] & u^*\mathrm{R}^n\bar g_*\omega^{\#}_{\bar g} \arrow[l, "\sim"'] \arrow[d, "\mathrm{tr}^{\#}_{\bar g}"] \\
u^*\mathscr{O}_Z \arrow[r, equal] & \mathscr{O}_{Z'} \arrow[r, equal] & \mathscr{O}_{Z'} \arrow[rr, equal] & & u^*\mathscr{O}_Z
\end{tikzcd}$$

The outer border commutes because of our alternate description of $\mathrm{tr}^{\#}_{\bar h}$. The rectangle on the left commutes because of the definition of $\theta^{\bar f}_u$. The rectangle on the lower right commutes because of the definition of $\theta^{\bar g}_u$ The remaining rectangle bordering the bottom edge commutes because of the alternate description of $\mathrm{tr}^{\#}_{\bar h'}$. The rectangle on the top right is simply $(\ddagger)$ and so commutes. All other rectangles, save $\blacksquare$, commute for functorial reasons. Consider $\blacksquare$. We have two possible routes from its northeast vertex to $\mathscr{O}_{Z'}$ lying directly below its southwest vertex, namely, south followed by west followed by south, and west followed by south all the way. We have to show that the two routes give the same map. This follows from the fact that all the subrectangles (except possibly $\blacksquare$) and the outer border commute. This establishes $(\dagger)$ and hence the theorem. $\square$

We wish to understand (14.2.2) more explicitly. For that we need to work more locally, with affine schemes, and often in a "punctual way", i.e., by working with completions of local rings at points. With this in mind, let us assume that we are in the situation of diagram (14.2.1), with a small change in hypothesis, namely we assume h is separated and quasi-finite, rather than finite. The maps f and g remain smooth of relative dimension n.

$$\begin{tikzcd} X \arrow[d, "h"'] \arrow[dr, bend left, "f"] & \\ Y \arrow[r, "g"] & Z \end{tikzcd}$$

We are interested in duality for h in terms of ω_g and ω_f "at a point $x \in X$". To that end we make the following further assumptions.

- $Z = \operatorname{Spec} A$
- $Y = \operatorname{Spec} R$ and $X = \operatorname{Spec} S$.

Let $y \in Y$ assume $h^{-1}(y)$ consists of exactly one point x.

Let $R' = \widehat{\mathscr{O}_y}$ be the completion of the local ring $\mathscr{O}_{Y,y}$, $S' = \widehat{\mathscr{O}_x}$ the completion of $\mathscr{O}_{X,x}$, and set $Y' = \operatorname{Spec} R'$, $X' = \operatorname{Spec} S'$. Since $h^{-1}(y) = \{x\}$, we have a cartesian square

$$\begin{tikzcd} X' \arrow[r, "v"] \arrow[d, "h'"'] \arrow[dr, phantom, "\square"] & X \arrow[d, "h"] \\ Y' \arrow[r, "u"'] & Y \end{tikzcd}$$

with h' *finite*, even though h need not be finite.

To lighten notation, write $\omega_R = \omega_{R/A}$, and $\omega_S = \omega_{S/A}$. Set $\omega_{R'} = \omega_R \otimes_R R'$, and $\omega_{S'} = \omega_S \otimes_S S' = \omega_S \otimes_R R'$.

Since h is flat and Gorenstein of relative dimension 0, for any quasi-coherent $\mathscr{O}_Y$-module $\mathscr{F}$ we have $\mathrm{H}^k(h^!\mathscr{F}) = 0$ for $k \neq 0$, and so we identify $h^!\mathscr{F}$ with $\mathrm{H}^0(h^!\mathscr{F})$. Similarly, we identify $h'^!\mathscr{G}$ with $\mathrm{H}^0(h'^!\mathscr{G})$ for every quasi-coherent $\mathscr{O}_{Y'}$-module $\mathscr{G}$. For an R-module M, $h^!M$ is defined to be $\Gamma(X, h^!\widetilde{M})$. Similarly, for an R'-module N, $(h')^!N$ will denote $\Gamma(X', (h')^!\widetilde{N})$.

Let

$$\varsigma\colon h^!\omega_R \xrightarrow{\sim} \omega_S \tag{14.2.5}$$

denote the isomorphism obtained from $h^!g^!\mathscr{O}_Y \xrightarrow{\sim} f^!\mathscr{O}_Y$ and the Verdier isomorphisms $\mathbf{v}_g$ and $\mathbf{v}_f$. By (flat) base change, we have

$$\varsigma'\colon (h')^!\omega_{R'} \xrightarrow{\sim} \omega_{S'}. \tag{14.2.6}$$

In particular we have a trace map (for h' is finite)

$$\mathrm{tr}_{S'}\colon \omega_{S'} \to \omega_{R'} \tag{14.2.7}$$

corresponding to

$$h'_*\widetilde{\omega}_{S'} \underset{(*)_{R'}}{\xleftarrow{\sim}} h'_*(h')^!\widetilde{\omega}_{R'} \xrightarrow{\mathrm{Tr}_{h'}} \widetilde{\omega}_S$$

Our interest is in making $\mathrm{tr}_{S'}$ explicit.We point out that to define it, it was not necessary to assume that x is the only point of X lying over y. However, by shrinking X around x, we can be in the situation we are in.

Now suppose $h\colon X \to Y$ factors as in the following commutative diagram

$$\begin{tikzcd} X \arrow[r, hook, "i"] \arrow[dr, "h"'] & P \arrow[d, "p"] \\ & Y \end{tikzcd}$$

with $P = \operatorname{Spec} E$, $p\colon P \to Y$ *smooth of relative dimension* d, and i a *closed immersion*. We have a commutative diagram with each square cartesian

$$\begin{tikzcd} X' \arrow[r, "v"] \arrow[d, "j"'] \arrow[dr, phantom, "\square"] & X \arrow[d, "i"] \\ P' \arrow[r, "w"] \arrow[d, "p'"'] \arrow[dr, phantom, "\square"] & P \arrow[d, "p"] \\ Y' \arrow[r, "u"'] & Y \end{tikzcd} \tag{14.2.8}$$

with $h = p \circ i$ and $h' = p' \circ j$.

Let $E' = E \otimes_R R'$, $P' = \operatorname{Spec} E'$, $\pi = g \circ p$, and consistent with out notations above, let $\omega_E = \omega_{E/A}$, and $\omega_{E'} = \omega_E \otimes_R R'$.

We remark that $\omega_{R'}$ and $\omega_{S'}$ are the e-th graded pieces of the differenital graded algebras $\wedge^\bullet_{R'}(\Omega^1_{R/A} \otimes_R R')$ and $\wedge^\bullet_{S'}(\Omega^1_{S/A} \otimes_S S') = \wedge^\bullet_{S'}(\Omega^1_{S/A} \otimes_R R')$ respectively. Similarly, $\omega_{E'}$ is the $(n+e)$-th graded piece of $\wedge^\bullet_{E'}(\Omega^1_{E/A} \otimes_E E') = \wedge^\bullet_{E'}(\Omega^1_{E/A} \otimes_R R')$

Let

$$\phi\colon \omega_R \otimes_R \omega_{E/R} \xrightarrow{\sim} \omega_E$$

be the isomorphism $\phi = \Gamma(Y, \bar{\varphi}_{g,\,p})$, where $\bar{\varphi}_{g,\,p}$ is the map defined in (12.2.1). In other words $\phi(\nu \otimes \mu) = \mu \wedge p^*\nu$. Let

$$\phi'\colon \omega_{R'} \otimes_{R'} \omega_{E'/R'} \xrightarrow{\sim} \omega_{E'}$$

be the base change of ϕ. In greater detail, we have $\omega_{E'/R'} = \omega_{E/R} \otimes_R R'$, and therefore $\omega_{R'} \otimes_{R'} \omega_{E'/R'} = (\omega_R \otimes_R \omega_{E/R}) \otimes_R R'$. Set $\phi' = \phi \otimes \mathbf{1}$.

Next, let $I = \ker E \twoheadrightarrow S$, $J = \ker E' \twoheadrightarrow S'$. Write $N = (\wedge^d_S I/I^2)^*$ and $N' = (\wedge^d_{S'} J/J^2)^*$. Let

$$b\colon \omega_E \otimes_E N \xrightarrow{\sim} \omega_S$$

be the map given by (13.2.4). By base change, as in the definition of ϕ', we have a map $b' := b \otimes \mathbf{1}$:

$$b'\colon \omega_{E'} \otimes_{E'} N' \xrightarrow{\sim} \omega_{S'}.$$

Let $\varrho\colon \omega_R \otimes_R \omega_{E/R} \otimes_E N \to \omega_S$ and $\varrho'\colon \omega_{R'} \otimes_{R'} \omega_{E'/R'} \otimes_{E'} N' \to \omega_{S'}$ be the maps

$$\varrho = b \circ (\phi \otimes \mathbf{1}_N) \quad \text{and} \quad \varrho' = b' \circ (\phi' \otimes \mathbf{1}_{N'}) = \varrho \otimes \mathbf{1}_{R'}. \tag{14.2.9}$$

Finally, let

$$\psi\colon \omega_{E/R} \otimes_E N \xrightarrow{\sim} h^! R \quad \text{and} \quad \psi'\colon \omega_{E'/R'} \otimes_{E'} N' \xrightarrow{\sim} h'^!(R') \tag{14.2.10}$$

be the maps defined as in (13.2.8.3).

Proposition 14.2.11. *In the above situation, assume* $I = \ker(E \twoheadrightarrow S)$ *is generated by* $\mathbf{u} = (u_1, \dots, u_d)$ *and set* $f_k = u_k \otimes 1 \in E'$, *so that* $\mathbf{f} = (f_1, \dots, f_d)$ *generates* $J = \ker(E' \twoheadrightarrow S')$. *Set* $\mathscr{N} = \mathscr{N}_i^d (= \widetilde{N})$ *and* $\mathscr{N}' = \mathscr{N}_j^d (= \widetilde{N'})$. *Let* $\omega \in \omega_{S'}$.

(i) *The following diagram commutes:*

$$\begin{tikzcd}
(\omega_{R'} \otimes_{R'} \omega_{E'/R'}) \otimes_{S'} N' \arrow[r, "\varrho'"] \arrow[d, equal] & \omega_{S'} \\
\omega_{R'} \otimes_{R'} (\omega_{E'/R'} \otimes_{S'} N') \arrow[d, "\mathbf{1}\otimes\psi'"', "\wr"] & \\
\omega_{R'} \otimes_{R'} {h'}^{!}(R') \arrow[r, "\sim", "\chi^{h'}"'] & {h'}^{!}(\omega_{R'}) \arrow[uu, "\wr"', "\varsigma'"]
\end{tikzcd}$$

where $\chi^{h'}$ is the transitivity map defined in (13.2.8.2).

(ii) *If $\omega = s \cdot (h')^*(\nu)$, where $\nu \in \omega_{R'}$ and $s \in S'$, then*

$$\operatorname{tr}_{S'}(\omega) = \operatorname{Trc}_{S'/R'}(s) \cdot \nu.$$

(iii) *Let $\eta \in \Omega^n_{E/A} \otimes_R R'$ be any element such that $j^*\eta = \omega$. Then*

$$\operatorname{tr}_{S'}(\omega) = \nu \cdot \mathbf{res}_{X',p'} \begin{bmatrix} x \cdot \mu \\ f_1, \dots, f_d \end{bmatrix}$$

where $x \in E'$, $\mu \in \omega_{E'/R'}$, and $\nu \in \omega_{R'}$ are related via the formula

$$\mathrm{d}f_1 \wedge \cdots \wedge \mathrm{d}f_d \wedge \eta = x \cdot \mu \wedge {p'}^*\nu.$$

(iv) *Let μ, ν, η and x be as in (ii). Suppose $E = R[T_1, \dots, T_d]_{q(\mathbf{T})}$, where $q(\mathbf{T}) \in R[\mathbf{T}]$. Let $g_i \in E' \otimes_{R'} S'$ be the elements $g_i = T_i \otimes 1 - 1 \otimes \gamma_i$, where the $\gamma_i \in S'$ is the images of T_i, $i = 1, \dots, d$, and let*

$$\lambda \colon S' \to R'$$

be the map $\lambda = \lambda(\mathbf{f}, \mathbf{g})$ of Theorem 14.1.7. Then

$$\operatorname{tr}_{S'}(\omega) = \lambda(x|_{X'})\nu.$$

Proof. We point out that $\mathbf{u}$ and $\mathbf{f}$ are necessarily quasi-regular. We first prove (i). Consider the following diagram.

$$\begin{tikzcd}
v^*(h^*\omega_g \otimes i^*\omega_p \otimes \mathscr{N}) \arrow[r, "v^*(\varrho)", "\sim"'] \arrow[d, equal] \arrow[ddd, bend right=60, "v^*(\mathbf{1}\otimes\psi)", "\simeq"'] \arrow[dr, phantom, "\blacklozenge"] & v^*\omega_f \arrow[d, equal] \\
{h'}^*\widetilde{\omega}_{R'} \otimes j^*\omega_{p'} \otimes \mathscr{N}' \arrow[r, "\sim", "\varrho'"'] \arrow[d, "\mathbf{1}\otimes\psi'"', "\wr"] & \widetilde{\omega}_{S'} \\
{h'}^*\widetilde{\omega}_{R'} \otimes {h'}^{!}\mathscr{O}_{Y'} \arrow[r, "\sim", "\chi^{h'}"'] \arrow[dr, phantom, "\lozenge"] & {h'}^{!}\widetilde{\omega}_{R'} \arrow[u, "\wr"', "\varsigma'"] \\
v^*(h^*\omega_g \otimes h^{!}\mathscr{O}_Y) \arrow[r, "\sim", "v^*(\chi^h)"'] \arrow[u, "\mathbf{1}\otimes\theta^h_u", "\wr"'] & v^*h^{!}\omega_g \arrow[u, "\wr"', "\theta^h_u"] \arrow[uuu, bend right=60, "v^*(\varsigma)"', "\simeq"]
\end{tikzcd}$$

The rectangle ⧫ on the top commutes by definition of ϱ'. The sub-diagram on the right, the one labelled □, squeezed between the curved arrow and the vertical column, commutes by definition of ς'. The rectangle labelled ◊ at the bottom commutes by Proposition 5.2.10.

We now show that the sub-diagram on the left, labelled ■, squeezed between the curved arrow and the vertical column on the left, commutes. First, the composite of isomorphisms, with the middle arrow the base change isomorphism

$$v^*\mathscr{N}[-d] \xrightarrow{v^*\eta_i'} v^*i^!\mathscr{O}_P \xrightarrow{\ \sim\ } j^!\mathscr{O}_{P'} \xrightarrow{\eta_j'^{\,-1}} \mathscr{N}'[-d]$$

is the identity map on $\mathscr{N}'[-d]$ by Remark 4.2.5. Next, the composite of isomorphisms

$$w^*\omega_p[d] \xrightarrow{w^*\mathbf{v}_p} w^*p^!\mathscr{O}_Y \xrightarrow{\ \sim\ } p'^!\mathscr{O}_{Y'} \xrightarrow{\mathbf{v}_{p'}^{-1}} \omega_{p'}[d]$$

is the identity map on $\omega_{p'}[d]$ [**S2**, p. 740, Prop. 2.3.5 (b)]. Finally, the transitivity property of base change proved in Proposition A.1.1 (ii) (see also [**L4**, p. 183, Prop. 4.6.8]) tells us that the base change of the composite $p\circ i$ with respect to u is compatible with the base change for p and i with respect to u and $w\colon P'\to P$ respectively. Putting these together, we see that ■ also commutes.

The outer border commutes by Proposition 13.2.8.4, after using Theorem 13.2.6 to realise b as a concrete representation of the map $a_{X/P}$

It follows that the rectangle in the middle also commutes. This proves (i).

Next note that the following diagram commutes, by definition of the various isomorphisms involved.

$$\begin{tikzcd}
\omega_{R'}\otimes_{R'} h'^!(R') \arrow[r, "\sim", "\chi^{h'}"'] \arrow[d, "\mathbf{1}\otimes\mathrm{Tr}_{S'/R'}"'] & h'^!(R') \arrow[r, "\sim", "\varsigma'"'] \arrow[d, "\mathrm{Tr}_{S'/R'}"'] & \omega_{S'} \arrow[d, "\mathrm{tr}_{S'}"] \\
\omega_{R'} \arrow[r, equal] & \omega_{R'} \arrow[r, equal] & \omega_{R'}
\end{tikzcd}$$

Let $\boldsymbol{\tau}^{\#}_{h'} = \boldsymbol{\tau}^{\#}_{h',p',j}\colon h'_*(j^*\omega^{\#}_{p'}\otimes\mathscr{N}') \xrightarrow{\ \sim\ } \mathscr{O}_{Y'}$ be the map (3.4.2). Define

$$\boldsymbol{\tau}_{h'}\colon h'_*(j^*\omega_{p'}\otimes\mathscr{N}') \xrightarrow{\ \sim\ } \mathscr{O}_{Y'}$$

in the obvious way, namely by substituting $\omega^{\#}_{p'}$ in the definition of $\boldsymbol{\tau}^{\#}_{h'}$ by $\omega_{p'}$ via the Verdier isomorphism $\mathbf{v}_{p'}$. Write $\boldsymbol{\tau}_{S'/R'}$ for the global sections of $\boldsymbol{\tau}_{h'}$. From part (i) and the above commutative diagram, we see that

$$(*)\qquad \mathrm{tr}_{S'}\circ\varrho' = \mathbf{1}\otimes\boldsymbol{\tau}_{S'/R'}.$$

Now suppose $\mu\in\omega_{E'/R'}$ and $\nu\in\omega_{R'}$, By Proposition 3.5.4 and $(*)$ we get

$$(\dagger)\qquad \mathrm{tr}_{S'}(\rho'(\nu\otimes\mu\otimes\mathbf{1}/\mathbf{f})) = \mathbf{res}_{X',p'}\begin{bmatrix}\mu\\ f_1,\dots,f_d\end{bmatrix}\cdot\nu$$

Now if $\omega = s\cdot h'^*(\nu)$, then by definition of ϱ', if $x\in E'$ is a lift of s, we have

$$\varrho'(x\cdot(\nu\otimes \mathrm{d}f_1\wedge\cdots\wedge\mathrm{d}f_d\otimes\mathbf{1}/\mathbf{f}) = \omega,$$

whence by $(\dagger)$

$$\mathrm{tr}_{S'}(\omega) = \mathbf{res}_{X',p'}\begin{bmatrix}x\cdot\mathrm{d}f_1\wedge\cdots\wedge\mathrm{d}f_d\\ f_1,\dots,f_d\end{bmatrix}\cdot\nu.$$

The right side is equal to $\mathrm{Trc}_{S'/R'}(s)\cdot\nu$ by Theorem 14.1.8. This proves (ii). Part (iii) is a re-statement of $(\dagger)$. Indeed

$$\begin{aligned}\mathrm{tr}_{S'}(\omega) = \mathrm{tr}_{S'}(j^*\eta) &= \mathrm{tr}_{S'}(b'(\mathrm{d}f_1\wedge\cdots\wedge\mathrm{d}f_d\wedge\eta\otimes\mathbf{1}/\mathbf{f}))\\ &= \mathrm{tr}_{S'}(\varrho'(x\cdot(\nu\otimes\mu\otimes\mathbf{1}/\mathbf{f})))\\ &= \nu\cdot\mathbf{res}_{X',p'}\begin{bmatrix}x\cdot\mu\\ f_1,\dots,f_d\end{bmatrix}\end{aligned}$$

Part (iv) follows from (iii) and Proposition 14.1.10. □

REMARK 14.2.12. We have already observed that it was not necessary to assume $h^{-1}(y)$ consisted of exactly one point, namely x, in order to define $\mathrm{tr}_{S'}$. In fact more can be be said. Suppose $u\colon X \to \overline{X}$ is an open immersion of (ordinary) Y-schemes such that the structure map $\bar{h}\colon \overline{X} \to Y$ is quasi-finite, and $\bar{h}^{-1}(y) = \{x_1, \dots, x_m\}$, with $x_1 = x$. In this case, the fibre dimension of $\bar{f} = g \circ \bar{h}$ is n. As before, set $\omega^{\#}_{\bar{f}} = \mathrm{H}^{-n}(\bar{f}^!\mathscr{O}_Z)$. Now, $\overline{X}' = \overline{X} \times_Y Y'$ is finite over Y, since Y' is the spectrum of a complete local ring. Let $\bar{h}'\colon \overline{X}' \to Y'$ be the base change of $\bar{h}$. Now $\overline{X}' = \operatorname{Spec} \prod_{i=1}^m S'_i$, where S'_i is the completion of the local ring $S_i = \mathscr{O}_{\overline{X}, x_i}$. Let $\overline{S}' = \prod_i S'_i$, and let $X'_i = \operatorname{Spec} S'_i$, so that X'_i is open and closed in $\overline{X}'$, and $\overline{X}' = \coprod_i X'_i$. Let $h'_i\colon X'_i \to Y'$ be the restriction of $\bar{h}'$ to X'_i. Note $X'_1 = X'$, $S'_1 = S'$, and $h'_1 = h'$. If $\omega^{\#}_{S'_i} = \omega^{\#}_{\bar{f},\, x_i} \otimes_{S_i} S'_i$, and $\omega^{\#}_{\overline{S}'} = \oplus_i \omega^{\#}_{S'_i}$ (the direct sum thought of as an $\overline{S}'$- module, then, as in the argument used in (14.2.6), we have, analogous to ς', isomorphisms $\mathrm{H}^0(h'^!_i\omega_{R'}) \xrightarrow{\sim} \omega^{\#}_{S'_i}$ and $\mathrm{H}^0((\bar{h}')^!\omega_{R'}) \xrightarrow{\sim} \omega^{\#}_{\overline{S}'}$,[1] whence abstract trace maps

$$\mathrm{tr}^{\#}_{S'_i}\colon \omega^{\#}_{S'_i} \longrightarrow \omega_{R'} \qquad (i = 1, \dots m)$$

and

$$\mathrm{tr}^{\#}_{\overline{S}'}\colon \omega^{\#}_{\overline{S}'} \longrightarrow \omega_{R'}.$$

Clearly $\mathrm{tr}^{\#}_{\overline{S}'} = \sum_i \mathrm{tr}^{\#}_{S'_i}$. The Verdier isomorphism $\mathbf{v}_f\colon \omega_f \xrightarrow{\sim} \omega^{\#}_f$ base changes to $\mathbf{v}_{S'}\colon \omega_{S'} \xrightarrow{\sim} \omega^{\#}_{S'}$ and it is clear that $\mathrm{tr}_{S'} = \mathrm{tr}^{\#}_{S'} \circ \mathbf{v}_{S'}$. Finally, *if* $\bar{h}\colon \overline{X} \to Y$ *is finite*, say $\overline{X} = \operatorname{Spec} \overline{S}$, then we have a map $\mathrm{tr}^{\#}_{\bar{h}}\colon \bar{h}_*\omega^{\#}_{\bar{f}} \to \omega^{\#}_g$ defined in (14.2.4.1). Consistent with the above notations, set $\omega^{\#}_{\overline{S}} = \Gamma(\overline{X},\, \omega^{\#}_{\bar{f}})$. Let $\mathrm{tr}^{\#}_{\overline{S}}\colon \omega^{\#}_{\overline{S}} \to \omega_R$ be the map $\Gamma(Y,\, \mathbf{v}_g^{-1} \circ \mathrm{tr}^{\#}_{\bar{h}})$. Then clearly

$$\mathrm{tr}^{\#}_{\overline{S}} \otimes_R R' = \mathrm{tr}^{\#}_{\overline{S}'} = \sum_k \mathrm{tr}^{\#}_{S'_k},$$

where $\mathrm{tr}^{\#}_{\overline{S}}$ is the global sections of $\mathrm{tr}^{\#}_{\bar{h}}$ defined in (14.2.4.1). In particular, if $\bar{f}\colon \overline{X} \to Z$ is smooth, then with $\mathrm{tr}_{\overline{S}} = \Gamma(\overline{X},\, \mathrm{tr}_f)$ we have

$$\mathrm{tr}_{\overline{S}} \otimes_R R' = \sum_k \mathrm{tr}_{S'_k}. \tag{14.2.12.1}$$

14.2.13. The Kunz-Lipman trace. Suppose, as we have for most of this section, $X = \operatorname{Spec} S$, $Y = \operatorname{Spec} R$, and $Z = \operatorname{Spec} A$, and as before suppose $f\colon X \to Z$ and $g\colon Y \to Z$ are smooth of relative dimension n, $f = g \circ h$, and now assume h is *finite*, and not merely separated and quasi-finite. In this case (and in more general situations) we have a trace map

$$\sigma_{S/R}\colon \omega_S \longrightarrow \omega_R$$

or, in sheaf-theoretic terms,

$$\sigma_h\colon h_*\omega_f \longrightarrow \omega_g$$

due to Lipman and Kunz, defined in Kunz's book [**Ku**, p. 254, 16.4]. The idea is attributed by Kunz to Lipman (see footnote in *loc.cit.*)

The *Kunz-Lipman trace* $\sigma_{S/R}$ can be understood punctually. In greater detail, the Tate trace $\lambda(\mathbf{f}, \mathbf{g})$ of Theorem 14.1.7 is denoted τ^x_f in [**Ku**, p. 370, (F.20)] (and

[1] Regarding $(\bar{h}')^!\omega_{R'}$ as a complex of $\overline{S}'$-modules associated to $(\bar{h}')^!\widetilde{\omega}_{R'}$ etc.

studied in some detail in F.18–F.28 of *ibid*). Now suppose y is a point in Y. Fix $x \in h^{-1}(y)$ and pick an affine open subscheme $U = \operatorname{Spec} S_U$ of X such that $h^{-1}(y) \cap U = \{x\}$, and a presentation

$$R[T_1, \dots, T_d]_{q(\mathbf{T})}/(u_1, \dots, u_d) = S_U.$$

Such a U and presentation always exists. Let R' be as before, the completion of the local ring $\mathscr{O}_{Y,y}$, and let S' be the completion of the local ring $\mathscr{O}_{X,x}$. Let $E = R[\mathbf{T}]_{q(\mathbf{T})}$, and $E' = E \otimes_R R'$. Let $f_1, \dots, f_d$ be the images of $u_1, \dots, u_d$ in E'. We continue to denote the image of the variables T_k in E' as T_k. Let ω_R, $\omega_{R'}$, ω_{S_U}, $\omega_{S'}$, $\mathrm{tr}_{S'}$ etc., be as before. Let $\gamma_k \in S'$ be the image of f_k, and set $g_k = X_k \otimes 1 - 1 \otimes \gamma_k \in E' \otimes_{R'} S'$. Finally let

$$\lambda \colon S' \to R'$$

be the Tate trace $\lambda(\mathbf{f}, \mathbf{g})$ of Theorem 14.1.7. Since $\omega_{S'}$ is a direct summand of $\omega_S \otimes_R R'$, the map $\sigma_{S/R}$ restricts to a map

$$\sigma_{S'} \colon \omega_{S'} \to R'.$$

For $\omega \in \omega_{S'}$ and $\eta \in \omega_{E'}$ a pre-image of ω under the natural surjective map $\omega_{E'} \to \omega_{S'}$, suppose $x \in E'$, $\nu \in \omega_{R'}$ are such that

$$\mathrm{d}f_1 \wedge \cdots \wedge \mathrm{d}f_d \wedge \eta = x \cdot \mathrm{d}T_1 \wedge \cdots \wedge \mathrm{d}T_d \wedge \nu.$$

Using properties Tr 3) and Tr 4) of [**Ku**, pp. 245-246, § 16], proved in [*ibid*, p. 254, Thm. 16.1], the definition of the Kunz-Kipman trace in [*ibid*, p. 254, 16.4] gives

$$\sigma_{S'} = \lambda(\bar{x}) \cdot \nu,$$

where $\bar{x} \in S'$ is the image of $x \in E'$. This means, by the formula in Proposition 14.2.11 (iv),

$$\sigma_{S'} = \mathrm{tr}_{S'}.$$

Once again by the above mentioned properties Tr 3) and Tr 4) of $\sigma_{S/R}$, and by (14.2.12.1), this gives $\widehat{\sigma_{h,y}} = \widehat{\mathrm{tr}_{h,y}}$, where $\widehat{(-)}$ denote completion of an $\mathscr{O}_{Y,y}$-module with respect to the maximal ideal. Since y is arbitrary in Y, we have

$$\sigma_h = \mathrm{tr}_h. \tag{14.2.13.1}$$

Clearly, we don't need X, Y and Z to be affine for the argument to go through.

THEOREM 14.2.14. *Let $f \colon X \to Z$ and $g \colon Y \to Z$ be smooth separated maps of ordinary schemes of relative dimension n, and suppose $f = g \circ h$, where $h \colon X \to Y$ is a finite map.*

(i) *Suppose $Z = \operatorname{Spec} A$, $Y = \operatorname{Spec} R$ and $X = \operatorname{Spec} S$. Let $\omega = s(h^*(\nu))$ where $s \in S$ and $\nu \in \Omega^n_{R/A}$. Then*

$$\mathrm{tr}_h(\omega) = \mathrm{Trc}_{S/R}(s)\, \nu.$$

(ii) *If $\sigma \colon h_*\omega_f \to \omega_g$ is the Kunz-Lipman trace then*

$$\sigma_h = \mathrm{tr}_h.$$

PROOF. Part (ii) of Proposition 14.2.11 gives (i). Part (ii) is simply (14.2.13.1). ∎

14.3. Regular Differentials again

The fact that tr_h agrees with the Kunz-Lipman trace σ_h allows us to prove Theorem 11.4.2 in a different way. Suppose f, g, h are as above, with the caveat that we no longer assume that f is smooth, but assume f is of finite type, the smooth locus of f, X^{sm} contains all the associated points of X. The map g remains smooth, and h finite. Assume further that:

(i) X, Y and Z are excellent have no embedded points;
(ii) $X = \operatorname{Spec} S$, $Y = \operatorname{Spec} R$, and $Z = \operatorname{Spec} A$;
(iii) $R \to S$ is injective.

We use the notations of Chapter 11. Thus $f^K = \mathrm{H}^{-n}(f^!)$ is as in §11.1, and $\omega^{\mathrm{reg}}_{X/Z}$ is the sheaf of regular differential n-forms discussed in §11.4. Since we are in the affine situation, we work with modules and algebras over A, R, and S, and choose appropriate notations. To that end, let $k(R)$ and $k(S)$ be the total ring of fractions of R and S respectively. Set $\omega_R = \Gamma(X, \omega_g)$, $\omega_{k(S)} = \Gamma(X, \Omega^n_{k(X)/k(Z)})$, $\Omega_{k(R)} = \Gamma(X, \Omega^n_{k(Y)/k(Z)})$.

Standard arguments show that there is an scheme theoretically dense open subscheme U of Y, such that $h^{-1}(U)$ is in X^{sm} and is scheme theoretically dense in X (e.g., $U = Y \smallsetminus h(X \smallsetminus X^{\mathrm{sm}})$). We have the trace map $\mathrm{tr}_{h_U} : (h|_U)_*\omega_{f|_{h^{-1}U}} \to \omega_{g|_U}$, where $h_U \colon h^{-1}(U) \to U$ is the restriction of h. By taking stalks at generic points (we have no embedded points!) we get a map

$$\mathrm{tr}_{k(S)} \colon \omega_{k(S)} \longrightarrow \omega_{k(R)}.$$

We point out that $\omega_R \subset \omega_{k(R)}$. The content of the next result is that $\overline{\omega}_S$ is a "complementary module" in the sense of Kunz and Waldi [**KW**, § 4]. It is equivalent to Theorem 11.4.2, via Theorem 14.2.14, but we give a direct proof along the lines of the proof given of a related statement in [**L2**].

THEOREM 14.3.1. *Let $\overline{\omega}_S \subset \omega_{k(S)}$ be the image of the injective map $\omega^{\#}_S \to \omega_{k(S)}$ defined in* (11.4.1). *Then*

$$\overline{\omega}_S = \{\nu \in \omega_{k(S)} \mid \mathrm{tr}_{k(S)}(s\nu) \in \omega_R, \forall s \in S\}.$$

PROOF. The proof follows, *mutatis mutandis*, the one given in [**L2**, p. 34, Lemma (2.2)]. We give it here, with the necessary changes, for completeness. We have a natural isomorphism $\mathrm{Hom}_R(S, \omega_R) \xrightarrow{\sim} \omega^{\#}_S$ obtained by applying H^{-n} to $h^{\flat}\omega_g[n] \xrightarrow{\sim} f^!\mathscr{O}_Z$, whence an isomorphism

$$\overline{\omega}_S \xrightarrow{\sim} \mathrm{Hom}_R(S, \omega_R).$$

One checks (by using the open set $U = Y \smallsetminus h(X \smallsetminus X^{\mathrm{sm}})$ as an intermediary if necessary) that the following diagram commutes

$$\begin{tikzcd} \mathrm{Hom}_R(S, \omega_R) \arrow[r, hook] & \mathrm{Hom}_{k(R)}(k(S), \omega_{k(R)}) \\ \overline{\omega}_S \arrow[u, "\wr"] \arrow[r, hook] & \omega_{k(S)} \arrow[u, "\wr"'] \end{tikzcd}$$

where the isomorphism on the right is $\nu \mapsto (x \mapsto \mathrm{tr}_{k(S)}(x\nu))$, for $\nu \in \omega_{k(S)}$ and $x \in k(S)$. The result follows since the image of $\mathrm{Hom}_R(S, \omega_R)$ in $\mathrm{Hom}_{k(R)}(k(S), \omega_{k(R)})$ consists of $k(R)$-linear maps $\psi \colon k(S) \to \omega_{k(R)}$ such that $\psi(s) \in \omega_R$ for every $s \in S$.

In other words, such ψ are characterised by the property that $\mathbf{e}(s\psi) \in \omega_R$ for every $s \in S$, where

$$\mathbf{e}\colon \mathrm{Hom}_{k(R)}(k(S),\, \omega_{k(R)}) \to \omega_{k(R)}$$

is "evaluation at 1". Since $\mathbf{e}$ corresponds to $\mathrm{tr}_{k(S)}$ under the upward arrow on the right in the above diagram, we are done. $\square$

The next statement is a re-statement of Theorem 11.4.2, but the point is that it is also a consequence of Theorem 14.3.1.

COROLLARY 14.3.2. *Let ω_S^{reg} be the S-module whose associated quasi-coherent sheaf is $\omega_{X/Y}^{\mathrm{reg}}$. Then $\omega_S^{\mathrm{reg}} = \overline{\omega}_S$.*

PROOF. Let $U = Y \smallsetminus h(X \smallsetminus X^{\mathrm{sm}})$ and $h_U\colon h^{-1}(U) \to U$ the restriction of h. From Theorem 14.2.14 (ii), $\mathrm{tr}_{h_U} = \sigma_{h_U}$. The result follows from the characterisation of $\overline{\omega}_S$ as a complementary module in Theorem 14.3.1 and the definition of regular differentials in [**HK1**, p.58]. $\square$

14.3.3. We would like draw out the differences between the approach in Chapter 11 and that of this subsection. In the former, we treat the theory of regular differential forms as a settled theory, and freely use the results in [**HK1**], [**HK2**], and [**HS**] to arrive at a proof of Theorem 11.4.2 using our characterisation of the Verdier isomorphism in terms of standard residues along sections. In the "settled theory" mentioned above, $\omega_{X/Z}^{\mathrm{reg}}$ is defined via local quasi-normalisations, i.e. via quasi-finite maps from open subschemes of X to $\mathbb{A}^n_Z$, their compactifications by Zariski's Main Theorem and complementary modules [**KW**]. The theory of residues and traces used there make no reference to Verdier's isomorphism, and are developed *ab initio* for the purpose at hand. In Chapter 11, we mapped our theory on to all of that.

In contrast, from the results in this subsection, if $\omega_{X/Z}^{\mathrm{reg}}$ is *defined* as the image of the injective map $f^K\mathscr{O}_Y \to \Omega^n_{k(X)/k(Z)}$ as in (11.4.1), then we show that every time one has a finite dominant Z-map $h\colon X \to Y$ of schemes, such that $Y \to Z$ is smooth, then $\omega_{X/Z}^{\mathrm{reg}}$ is necessarily the complementary module on the right side of Theorem 14.3.1 (which can clearly be defined even when Y is not affine). Using [**EGA**, IV_3, (13.3.2)] and Zariski's main theorem, as in the first two paragraphs proof of Proposition 14.2.4, we see that locally we can always arrange matters so that X is covered by affine open subschemes, each of which is finite over an affine smooth Z-scheme (in fact an affine open subscheme of $\mathbb{P}^n_Z$), and hence $\omega_{X/Z}^{\mathrm{reg}}$ has a local description via complementary modules. This gives a different proof, than that given in [**KW**], that these complementary modules glue, and do not depend on the choice of the various finite maps of the sort just discussed. Finally our theory of residues and traces ensures that all the important results in [**HK1**], [**HK2**], and [**HS**] can be recovered.

Our approach (in this subsection) is closer in spirit to the approach to these matters in [**L2**], though even here it is necessarily different, since we use, consistently, Verdier's isomorphism, and we work over an arbitrary (noetherian) base rather than over a perfect field. It should be said that in [**KW**] and [**Ku**], the theory is for general differential algebras, and that in [**KW**], generic complete intersection algebras $A \to S$ are considered.

CHAPTER 15

The Residue Symbol

15.1. The definition of the symbol

Let $f\colon X \to Y$ be a separated smooth map of relative dimension r, and let $t_1, \dots, t_r \in \Gamma(X, \mathscr{O}_X)$ be such that if $\mathscr{I}$ is the quasi-coherent ideal sheaf generated by $\mathbf{t} = (t_1, \dots, t_r)$, then $Z := \mathbf{Spec}\,(\mathscr{O}_X/\mathscr{I})$ is finite over Y. Let $i\colon Z \to X$ be the closed immersion and $h\colon Z \to Y$ the finite map. In this case it is well-known that h is flat and $\mathbf{t}$ is a regular $\mathscr{O}_{X,z}$-sequence for every $z \in Z$ ([**EGA**, IV$_3$, Théorème (11.3.8)] or [**M**, p. 177, Corollary to Thm. 22.5]). In particular $h_*\mathscr{O}_Z$ locally free over $\mathscr{O}_Y$.

In this situation, according to (3.4.2), we have a map

$$\boldsymbol{\tau}^{\#}_{\boldsymbol{h}} = \boldsymbol{\tau}^{\#}_{\boldsymbol{h},\boldsymbol{f},\boldsymbol{i}}\colon h_*i^*\omega_f^{\#} \otimes_{\mathscr{O}_Z} \mathscr{N}_i^{\,r} \longrightarrow \mathscr{O}_Y,$$

allowing us to define

$$\boldsymbol{\tau}_{\boldsymbol{h}} = \boldsymbol{\tau}_{\boldsymbol{h},\boldsymbol{f},\boldsymbol{i}}\colon h_*i^*\omega_f \otimes_{\mathscr{O}_Z} \mathscr{N}_i^{\,r} \longrightarrow \mathscr{O}_Y \tag{15.1.1}$$

as the composite

$$h_*i^*\omega_f \otimes_{\mathscr{O}_Z} \mathscr{N}_i^{\,r} \xrightarrow[\text{via } \bar{\mathbf{v}}]{\sim} h_*i^*\omega_f^{\#} \otimes_{\mathscr{O}_Z} \mathscr{N}_i^{\,r} \xrightarrow{\boldsymbol{\tau}^{\#}_{\boldsymbol{h}}} \mathscr{O}_Y.$$

If $\bar{t_i} \in \Gamma(Z\, \mathscr{I}/\mathscr{I}^2)$ is the section generated by the image of t_i, then $\bar{t_1} \wedge \dots \wedge \bar{t_i}$ is a generator of the free rank one $\mathscr{O}_Z$-module $\wedge^r_{\mathscr{O}_Z}\mathscr{I}/\mathscr{I}^2$. As before, let $\mathbf{1}/\mathbf{t} \in \Gamma(Z, \mathscr{N}_i^{\,r}) = \mathrm{Hom}_Z(\mathscr{I}/\mathscr{I}^2, \mathscr{O}_Z)$ be the dual generator. For $\omega \in \Gamma(X, \omega_f)$ let $\omega/\mathbf{t} \in \Gamma(Z, i^*\omega_f \otimes_{\mathscr{O}_Z} \mathscr{N}_i^{\,r})$ be the image of $\omega \otimes \mathbf{1}/\mathbf{t} \in \Gamma(Z, i^*\omega_f) \otimes \Gamma(Z, \mathscr{N}_i^{\,r})$. With these notations, we folow [**RD**] and [**C1**] and define the *residue symbol* as

$$\mathrm{Res}_{X/Y}\begin{bmatrix} \omega \\ t_1, \dots, t_r \end{bmatrix} := \Gamma(Z, \boldsymbol{\tau}_{\boldsymbol{h}})(\omega/\mathbf{t}) \in \Gamma(Z, \mathscr{O}_Z). \tag{15.1.2}$$

In [**RD**, III, §9] a list of statements about the residue symbol are made without proof. The statements (with minor corrections to the statements in [**RD**]) have been proved by Conrad in [**C1**, A.2, Appendix A]. Since our approach to residues and the residue symbol follows a different route (via Verdier's isomorphism) we provide independent proofs of these statements in §15.2 below. Here are the statements (R1)–(R10), as in [**C1**], with modifications to take care of our conventions. In the statements, ω, $f\colon X \to Y$, Z, $t_1, \dots, t_r$ are as above, except in (R4).

(R1). Let $s_i = \sum_j u_{ij}t_j$ where $u_{ij} \in \Gamma(X, \mathscr{O}_X)$, $1 \le i, j \le r$, and suppose the closed subscheme of X cut out by the s_i's is finite over Y. Then

$$\mathrm{Res}_{X/Y}\begin{bmatrix} \omega \\ t_1, \dots, t_r \end{bmatrix} = \mathrm{Res}_{X/Y}\begin{bmatrix} \det(u_{ij})\,\omega \\ s_1, \dots, s_r \end{bmatrix}$$

(R2). (*Localisation*) We give the version in [**C1**, p. 239]. Suppose $g\colon X' \to X$ is separated and étale, $Z' = g^{-1}(Z)$, and the map $g'\colon Z' \to Z$ is finite, where g' is induced from g. We have a commutative diagram of schemes

$$\begin{tikzcd} Z' \arrow[r, "i'"] \arrow[d, "g'"'] \arrow[dr, phantom, "\square"] & X' \arrow[d, "g"] \\ Z \arrow[r, "i"] \arrow[dr, "h"'] & X \arrow[d, "f"] \\ & Y \end{tikzcd}$$

where, as indicated in the diagram, the square on the top is cartesian. Assume that the function on Z given by $z \mapsto \operatorname{rank}_{\mathscr{O}_{Z,z}} {g'}_*(\mathscr{O}_{Z'})_z$ extends to a locally constant function $\mathrm{rk}_{Z'/Z}$ in a Zariski open neighbourhood V of Z in X. Then, for every $\omega \in \Gamma(X, \omega_f)$, we have,

$$\operatorname{Res}_{V/Y}\begin{bmatrix}\omega\cdot \mathrm{rk}_{Z'/Z}\\ t_1,\dots,t_r\end{bmatrix} = \operatorname{Res}_{X/Y}\begin{bmatrix}\omega'\\ t'_1,\dots,t'_r\end{bmatrix},$$

where $t'_i = g^*(t_i) \in \Gamma(X', \mathscr{O}_{X'})$ and $\omega' = g^*(\omega) \in \Gamma(X', \omega_{fg})$.

(R3). (*Restriction*) Suppose we have a commutative diagram of schemes

$$\begin{tikzcd} X \arrow[r, hook, "i"] \arrow[dr, "f"'] & P \arrow[d, "\pi"] \\ & Y \end{tikzcd}$$

with f smooth and separated of relative dimension r, π smooth and separated of relative dimension $n = d + r$, i a closed immersion, with X cut out by $s_1, \dots, s_d \in \Gamma(P, \mathscr{O}_P)$, and suppose $t'_1, \dots, t'_r \in \Gamma(P, \mathscr{O}_P)$ are such that $s_1, \dots, s_d, t'_1, \dots, t'_r$ cut out a scheme Z which is finite over Y, and finally suppose t_j is the restriction of t'_j to X for $j = 1, \dots, r$. Then for every $\nu \in \Gamma(P, \Omega^r_{P/Y})$,

$$\operatorname{Res}_{P/Y}\begin{bmatrix}\mathrm{d}s_1\wedge\cdots\wedge \mathrm{d}s_d\wedge\nu\\ s_1,\dots,s_d,\, t'_1,\dots,t'_r\end{bmatrix} = \operatorname{Res}_{X/Y}\begin{bmatrix}i^*\nu\\ t_1,\dots,t_r\end{bmatrix}.$$

(R4). (*Transitivity*) Suppose we have a pair of separated maps $X \xrightarrow{f} Y \xrightarrow{g} Z$ are a pair of separated smooth maps, f of relative dimension e and g of relative dimension d. Suppose $s_1, \dots, s_d \in \Gamma(Y, \mathscr{O}_Y)$ cuts out a scheme W' in Y which is finite over Z, and with $s'_j = f^*(s_j)$, suppose we have $t_1, \dots, t_e \in \Gamma(X, \mathscr{O}_X)$ such that $s'_1, \dots, s'_d, t_1, \dots, t_e$ cut out a scheme W in X which is finite over Z. For $\mu \in \Gamma(\mathscr{O}_Y, \omega_f)$ and $\nu \in \Gamma(\mathscr{O}_X, \omega_g)$ we have:

$$\operatorname{Res}_{Y/Z}\begin{bmatrix}\operatorname{Res}_{X/Y}\begin{bmatrix}\mu\\ t_1,\dots,t_e\end{bmatrix}\nu\\ s_1,\dots,s_d\end{bmatrix} = \operatorname{Res}_{X/Z}\begin{bmatrix}\mu\wedge f^*\nu\\ t_1,\dots,t_e,\, s'_1,\dots,s'_d\end{bmatrix}.$$

(R5). (*Base Change*) Formation of the residue symbol commutes with base change.

(R6). (*Trace Formula*) For any $\varphi \in \Gamma(X,\, \mathscr{O}_X)$

$$\operatorname{Res}_{X/Y}\begin{bmatrix}\varphi\cdot \mathrm{d}t_1\wedge\cdots\wedge \mathrm{d}t_r\\ t_1,\dots,t_r\end{bmatrix} = \operatorname{Trc}_{Z/Y}(\varphi|_Z).$$

(R7). (*Intersection Formula*) For any collection of positive integers $k_1,\dots,k_r$ not all equal to 1,

$$\operatorname{Res}_{X/Y}\begin{bmatrix}\mathrm{d}t_1\wedge\cdots\wedge \mathrm{d}t_r\\ t_1^{k_1},\dots,t_r^{k_r}\end{bmatrix} = 0.$$

(R8). (*Duality*) (See [**C1**, p. 240, (R8)].) If $\omega|_Z = 0$, then

$$\operatorname{Res}_{X/Y}\begin{bmatrix}\omega\\ t_1,\dots,t_r\end{bmatrix} = 0.$$

Conversely, let $\{Y_j\}$ be an étale covering of Y such that Y_j is affine, $Z_j = Z\times_Y Y_j$ decomposes into a finite disjoint union of Z_{jk}'s with each Z_{jk} contained in an open subscheme X_{jk} of $X_j := X\times_Y Y_j$, with $X_{jk}\cap Z_{jm} = \emptyset$ for $m\neq k$. Also assume that $\Gamma(X_{jk},\, \mathscr{O}_{X_{jk}}) \to \Gamma(Z_{jk},\, \mathscr{O}_{Z_{jk}})$ is surjective[1]. If

$$\operatorname{Res}_{X_{jk}/Y_j}\begin{bmatrix}f\omega\\ t_1,\dots,t_r\end{bmatrix} = 0$$

for all $f\in\Gamma(X_{jk},\, \mathscr{O}_{X_{jk}})$, then $\omega|_Z = 0$.

(R9). (*Exterior Differentiation*) For $\nu \in \Gamma(X,\, \Omega^{r-1}_{X/Y})$ and positive integers $k_1,\dots,k_r$,

$$\operatorname{Res}_{X/Y}\begin{bmatrix}\mathrm{d}\nu\\ t_1^{k_1},\dots,t_n^{k_r}\end{bmatrix} = \sum_{i=1}^{r} k_i\cdot \operatorname{Res}_{X/Y}\begin{bmatrix}\mathrm{d}t_i\wedge\nu\\ t_1^{k_1},\dots,t_i^{k_i+1},\dots,t_n^{k_r}\end{bmatrix}.$$

(R10). (*Residue Formula*) Let $h\colon X'\to X$ be a finite map, with X' smooth over Y of relative dimension r. Let $t'_j = h^*(t_j)\in\Gamma(X',\, \mathscr{O}_{X'})$. Then

$$\operatorname{Res}_{X'/Y}\begin{bmatrix}\nu\\ t'_1,\dots,t'_r\end{bmatrix} = \operatorname{Res}_{X/Y}\begin{bmatrix}\operatorname{tr}_h(\nu)\\ t_1,\dots,t_r\end{bmatrix},$$

for every $\nu\in\Gamma(X',\, \omega_{fh})$, where $\operatorname{tr}_h\colon h_*\omega_{fh}\to\omega_f$ is the map in (14.2.2)[2].

15.2. Proofs

For a quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$, let

$$\psi = \psi(\mathscr{F})\colon h_*(i^*\mathscr{F}\otimes_{\mathscr{O}_Z}(\wedge^r_{\mathscr{O}_Z}\mathscr{I}/\mathscr{I}^2)^*) \longrightarrow \mathrm{R}^r_Z f_*\mathscr{F} \tag{15.2.1}$$

be defined by applying H^0 to the composite

$$h_* i^{\blacktriangle}\mathscr{F}[r] \xrightarrow{\ \sim\ } \mathbf{R}f_* i_* i^{\blacktriangle}\mathscr{F}[r] \underset{\eta_i}{\xrightarrow{\ \sim\ }} \mathbf{R}f_* i_* i^{\flat}\mathscr{F}[r] \longrightarrow \mathbf{R}f_*\mathbf{R}\varGamma_Z\mathscr{F}[r]. \tag{15.2.2}$$

[1] Such Y_j's, Z_{jk}'s, and X_{jk}'s always exist, using direct limit arguments.

[2] We point out that tr_h has an explicit description (in terms of the Kunz-Lipman trace) given in Theorem 14.2.14 (ii).

where $\eta_i\colon i^\blacktriangle \xrightarrow{\sim} i^\flat$ is the isomorphism in (C.2.11). (See (3.4.3) and (3.4.4).) According to Theorem 3.4.8, the following diagram commutes

$$\begin{tikzcd} h_*(i^*\omega_f \otimes_{\mathscr{O}_Z} (\wedge^r_{\mathscr{O}_Z} \mathscr{I}/\mathscr{I}^2)^*) \arrow[r, "\boldsymbol{\tau_h}"] \arrow[d, "\psi(\omega_f)"'] & \mathscr{O}_Z \arrow[d, equal] \\ \mathrm{R}^r_Z f_*\omega_f \arrow[r, "\mathbf{res}_Z"] & \mathscr{O}_Z \end{tikzcd} \tag{15.2.3}$$

A few things are worth pointing out. First, $\mathrm{Res}_{X/Y}\begin{bmatrix}\omega\\ t_1,\dots,t_r\end{bmatrix}$ is linear in ω, and since $\boldsymbol{\tau_h}$ is a map of sheaves, the residue symbol is local over Y. Moreover, according to Remark 3.4.9, if U is an open subscheme of X containing Z, then

$$\mathrm{Res}_{X/Y}\begin{bmatrix}\omega\\ t_1,\dots,t_r\end{bmatrix} = \mathrm{Res}_{U/Y}\begin{bmatrix}\omega\\ t_1,\dots,t_r\end{bmatrix}. \tag{15.2.4}$$

From (15.2.3) we see easily that if Z is a disjoint union of $Z_1,\dots,Z_m$ and X_i is open in X with $X_i \cap Z = Z_i$, then as in [**C1**, p. 239, (A.1.5)], we have,

$$\mathrm{Res}_{X/Y}\begin{bmatrix}\omega\\ t_1,\dots,t_r\end{bmatrix} = \sum_{i=1}^{m}\mathrm{Res}_{X_i/Y}\begin{bmatrix}\omega\\ t_1,\dots,t_r\end{bmatrix}. \tag{15.2.5}$$

We also note that by Theorem 4.3.1, and [**S2**, p. 740, Thm. 2.3.5 (b)], the residue symbol (15.1.2) is stable under arbitrary (noetherian) base change. This proves (R5).

If $Y = \mathrm{Spec}\,A$ and there is an open affine subscheme $U = \mathrm{Spec}\,R$ of X containing Z, then by Proposition 3.5.4 and (15.2.4), we see that

$$\mathrm{Res}_{X/Y}\begin{bmatrix}\omega\\ t_1,\dots,t_r\end{bmatrix} = \mathbf{res}_Z\begin{bmatrix}\omega\\ t_1,\dots,t_r\end{bmatrix}. \tag{15.2.6}$$

Since the formation of the residue symbol is compatible with arbitrary noetherian base change, i.e., since (R5) is true, we can prove a number of things by assuming Y is the spectrum of an artin local ring, or of a complete local ring. In greater detail, many of the formulas we have to prove are of the form $\alpha = \beta$ where $\alpha, \beta \in \Gamma(Y, \mathscr{O}_Y)$. It is clearly enough to prove that the germs α_y and β_y are equal at every $y \in Y$. So suppose $y \in Y$ and $A = \mathscr{O}_y$, the local ring at y, and $\mathfrak{m}$ is the maximal ideal of A. To show $\alpha_y = \beta_y$ it is clearly enough to show $\alpha_y \otimes_A A/\mathfrak{m}^n = \beta_y \otimes_A A/\mathfrak{m}^n$ are equal for every $n \in \mathbb{N}$, and by (R5), to prove this for a given positive integer n, it is enough to assume $Y = \mathrm{Spec} A/\mathfrak{m}^n$. Once we are in this situation, using (15.2.4), (15.2.5), we are in a situation where (15.2.6) applies. Note that we are in the situation where (15.2.6) applies even when pass to the completion of A with respect to $\mathfrak{m}$. Occasionally, by a further faithful flat base change on Y, we may assume Y is a strictly henselian local ring, or even a strictly henselian artin local ring.

With this in mind, (R1) follows from Theorem 3.5.5, (R3) from Corollary 13.2.7, (R4) from Theorem 13.1.1, (R6) from Theorem 14.1.8. For (R10), first note that tr_h is compatible with arbitrary noetherian base change by Proposition 14.2.4. So once again, the problem is stable under base change, and we may assume we are in a situation where (15.2.6) applies. This gives us (R.10) via Proposition 14.2.3. We have already seen that (R5) is true.

It remains to prove (R2), (R7), (R8) and (R9).

For (R2), we may assume, as in the proof of (R2) in [**C1**], that Y is the spectrum of a strictly henselian artin local ring. We are immediately reduced, via (15.2.5), to the case where Z and Z' consist of a single component each, $Z' = Z$, and g is the identity map. In this case the completion of X' along Z' is the same as the completion of X along Z, whence, since $\mathbf{res}_Z$ and $\mathbf{res}'_Z$ are really only dependent on the formal schemes, we are done.

To prove (R7) we assume without loss of generality that $Y = \operatorname{Spec} A$, where A is an artin local ring, that Z is supported at one point, say z_0. By shrinking X around Z (via (15.2.4)) if necessary, we may assume that the map $\pi\colon X \to \mathbb{A}^r_A = \operatorname{Spec} A[T_1, \dots, T_r]$ defined by $\mathbf{t}$ is a quasi-finite and that $\pi^{-1}(W) = Z$, where W is the closed subscheme of $\mathbb{A}^r_A$ cut out by $T_1, \dots, T_r$. By Zariski's Main Theorem, we have a finite map $\bar{\pi}\colon \overline{X} \to \mathbb{A}^r_A$ which is a compactification of π, in the sense that there exists an open immersion $u\colon X \to \overline{X}$ such that $\bar{\pi} \circ u = \pi$. Let $P = \mathbb{A}^r_A \smallsetminus \bar{\pi}(\overline{X} \smallsetminus X)$. Then P is open in $\mathbb{A}^r_A$, $W \subset P$, and $\bar{\pi}^{-1}(P) \subset X$. Replacing X by $\bar{\pi}^{-1}(P)$ if necessary, we may assume $\pi\colon X \to P$ is finite. Shrinking P around W, we may assume P and X, are affine, say $P = \operatorname{Spec} D$ and $X = \operatorname{Spec} E$. The map π is flat by [**M**, p. 174, Thm. 22.3 (3′)], since Z is flat over Y, and $\operatorname{Tor}^D_1(A, E) = 0$ (the latter by noting that $K^\bullet(\mathbf{T}) \otimes_D E = K^\bullet(\mathbf{t})$). By (R10) and Theorem 14.2.14 (i) we have

$$\begin{aligned}\operatorname{Res}_{X/Y}\begin{bmatrix}\mathrm{d}t_1 \wedge \cdots \wedge \mathrm{d}t_r\\ t_1^{k_1}, \dots, t_r^{k_r}\end{bmatrix} &= \operatorname{Res}_{P/Y}\begin{bmatrix}\operatorname{Trc}_{Z/W}(1) \cdot \mathrm{d}T_1 \wedge \cdots \wedge \mathrm{d}T_r\\ T_1^{k_1}, \dots, T_r^{k_r}\end{bmatrix}\\ &= \operatorname{rk}_{B/A} \cdot \operatorname{Res}_{P/Y}\begin{bmatrix}\mathrm{d}T_1 \wedge \cdots \wedge \mathrm{d}T_r\\ T_1^{k_1}, \dots, T_r^{k_r}\end{bmatrix}\end{aligned}$$

where $B = \mathscr{O}_{Z,z_0}$. The last expression is zero if $k_1, \dots, k_r$ are not all equal to 1, since $W \to Y$ is an isomorphism. This proves (R7)

For (R8), one direction is obvious, namely if $\omega|_Z = 0$ then $\operatorname{Res}_{X/Y}\left[\begin{smallmatrix}\omega\\ t_1, \dots, t_r\end{smallmatrix}\right] = 0$, for in this case $i^*\omega \otimes \mathbf{1/t} = 0$. For the "converse", by faithful flat descent we may assume $j = 1$, $Y = Y_j$, i.e., we may assume $Y = \operatorname{Spec} A$. Moreover, via (15.2.4) and (15.2.5), we may replace X by X_{ij} if necessary, and assume that $\Gamma(X, \mathscr{O}_X) \to \Gamma(Z, \mathscr{O}_Z)$ is surjective. Since $h\colon Z \to Y$ is finite, Z is affine, say $Z = \operatorname{Spec} B$. Write $\omega_{B/A}$ for $\Gamma(Z, i^*\omega_f \otimes \mathscr{N}_i^r)$. The map $\boldsymbol{\tau}_h$ induces a natural isomorphism $\omega_{B/A} \xrightarrow{\sim} \operatorname{Hom}_A(B, A)$, which for any $\nu \in \Gamma(X, \omega_f)$, sends $\nu/\mathbf{t} \in \omega_{B/A}$ to $\varphi_\nu \in \operatorname{Hom}_A(B, A)$ where

$$\varphi_\nu(g) = \operatorname{Res}_{X/Y}\begin{bmatrix}\widetilde{g} \cdot \nu\\ t_1, \dots, t_r\end{bmatrix} \qquad (g \in B)$$

where $\widetilde{g} \in \Gamma(X, \mathscr{O}_X)$ is any pre-image of g under the surjective map $\Gamma(X, \mathscr{O}_X) \to \Gamma(Z, \mathscr{O}_Z)$. It is clear that under our hypotheses, $\varphi_\omega = 0$, whence the section $\omega/\mathbf{t} = 0$. This means $\omega|_Z = 0$.

It remains to prove (R9)

Proof of (R9). As before, we reduce to the case where $Y = \operatorname{Spec} A$, A an artin local ring, $X = \operatorname{Spec} R$ and $Z_{\mathrm{red}} = \{z_0\}$, where z_0 is a closed point of X lying over the closed point of Y. We may assume A has an algebraically closed residue field [**EGA**, 0_{III}, 10.3.1]. Recall from §C.5, especially (C.5.1), that for an R-module M, we have the notion of a *stable Koszul complex* $K^\bullet_\infty(\mathbf{t}, M)$. We need this notion for arbitrary sheaves $\mathscr{F}$ of abelian groups on X (which need not even be $\mathscr{O}_X$-modules). To that end, let $U_i = \{t_i \neq 0\} = \operatorname{Spec} R_{t_i}$, $i = 1, \dots, r$, $\mathfrak{U} = \{U_i\}$, and for a sheaf of abelian groups $\mathscr{F}$ on X, and $\mathrm{C}^\bullet(\mathfrak{U},\,\mathscr{F})$ the *ordered* Čech complex associated with $\mathfrak{U}$. Since $\check{\mathrm{H}}^0(\mathfrak{U},\,\mathscr{F}) = \mathscr{F}(U)$, the natural restriction map $\mathscr{F}(X) \to \mathscr{F}(U)$ gives us a complex $K^\bullet_\infty(\mathbf{t},\,\mathscr{F})$ defined as

$$0 \longrightarrow \mathscr{F}(X) \longrightarrow \mathrm{C}^0(\mathfrak{U},\,\mathscr{F}) \longrightarrow \mathrm{C}^1(\mathfrak{U},\,\mathscr{F}) \longrightarrow \dots \longrightarrow \mathrm{C}^{r-1}(\mathfrak{U},\,\mathscr{F}) \longrightarrow 0$$

with $K^0_\infty(\mathbf{t},\,\mathscr{F}) = \mathscr{F}(X)$, $K^{i+1}_\infty(\mathbf{t},\,\mathscr{F}) = \mathrm{C}^i(\mathfrak{U},\,\mathscr{F})$ for $i \geq 0$, the first map being the composite

$$\mathscr{F}(X) \longrightarrow \mathscr{F}(U) \longrightarrow \mathrm{C}^0(\mathfrak{U},\,\mathscr{F})$$

and the remaining maps the usual coboundary maps on Čech cohomology. If M is an R-module, clearly $K^\bullet_\infty(\mathbf{t},\,\widetilde{M}) = K^\bullet_\infty(\mathbf{t},\,M)$. Note that $K^\bullet_\infty(\mathbf{t},\,\mathscr{F})$ is functorial in $\mathscr{F}$, as $\mathscr{F}$ varies over sheaves of abelian groups on X. In what follows, following standard conventions, we write $U_{i_1\dots i_p} := U_{i_1} \cap \dots \cap U_{i_p}$ for $1 \leq i_1 < \dots < i_p \leq r$

Since $\mathrm{H}^0(K^\bullet_\infty(\mathbf{t},\,\mathscr{F})) = \Gamma_Z(X,\,\mathscr{F})$, we have a functorial map of complexes

$$\Gamma_Z(X,\,\mathscr{F})[0] \longrightarrow K^\bullet_\infty(\mathbf{t},\,\mathscr{F}) \tag{15.2.7}$$

which is one readily checks is a quasi-isomorphism when $\mathscr{F}$ is flasque. If $\mathscr{G}^\bullet$ is a complex of *flasque* sheaves of abelian groups on X, and $\mathbf{D}(|X|)$ denotes the derived category of sheaves of abelian groups on X, then (15.2.7) gives us a pair of isomorphisms in $\mathbf{D}(|X|)$

$$\mathbf{R}\Gamma_Z(X,\,\mathscr{G}^\bullet) \xrightarrow{\;\sim\;} \Gamma_Z(X,\,\mathscr{G}^\bullet) \xrightarrow[\text{(15.2.7)}]{\;\sim\;} \mathrm{Tot}(\mathrm{C}^\bullet(\mathfrak{U},\,\mathscr{G}^\bullet)). \tag{15.2.8}$$

The first isomorphism is from general principles (since flasque sheaves have no higher cohomologies with support), and the second is from the fact that (15.2.7) is a quasi-isomorphism on flasque sheaves.

Now suppose $\mathscr{F}$ is a sheaf of abelian groups on X and $\mathscr{F} \to \mathscr{G}^\bullet$ a *flasque resolution* of $\mathscr{F}$. Since $\mathbf{R}\Gamma_Z(X\,\mathscr{F}) \xrightarrow{\;\sim\;} \Gamma_Z(X,\,\mathscr{G}^\bullet)$, (15.2.8) gives us an isomorphism

$$\mathbf{R}\Gamma_Z(X\,\mathscr{F}) \xrightarrow{\;\sim\;} \mathrm{Tot}(\mathrm{C}^\bullet(\mathfrak{U},\,\mathscr{G}^\bullet)), \tag{15.2.9}$$

where the right side is the total complex of the double complex $\mathrm{C}^\bullet(\mathfrak{U},\,\mathscr{G}^\bullet)$

By examining the "columns" of the double complex $\mathrm{C}^\bullet(\mathfrak{U},\,\mathscr{G}^\bullet)$ one obtains a map of complexes

$$K^\bullet_\infty(\mathbf{t},\,\mathscr{F}) \longrightarrow \mathrm{Tot}(\mathrm{C}^\bullet(\mathfrak{U},\,\mathscr{G}^\bullet)). \tag{15.2.10}$$

We therefore have a map in $\mathbf{D}(|X|)$, which is functorial in $\mathscr{F}$ varying over sheaves of abelian groups,

$$K^\bullet_\infty(\mathbf{t},\,\mathscr{F}) \longrightarrow \mathbf{R}\Gamma_Z(X,\,\mathscr{F}) \tag{15.2.11}$$

given by $(15.2.11) = (15.2.9)^{-1} \circ (15.2.10)$.

If $\mathscr{F}$ is *quasi-coherent* then (15.2.10) is a quasi-isomorphism, since $U_{i_1\dots i_p}$ and X are affine, whence $\mathscr{F}(U_{i_1\dots i_p})[0] \to \mathscr{G}^\bullet(U_{i_1\dots i_p})$ and $\mathscr{F}(X)[0] \to \mathscr{G}^\bullet(X)$ are quasi-isomorphisms. This means, (15.2.11) *is an isomorphism when* $\mathscr{F}$ *is quasi-coherent*. In fact, in this case, by definition it agrees with (C.5.2).

Let $\mathrm{d}^{r-1}_{X/Y} \colon \Omega^{r-1}_{X/Y} \to \Omega^r_{X/Y}$ be the standard exterior derivative map. Note that $\mathrm{d}^{r-1}_{X/Y}$ is not $\mathscr{O}_X$-linear. Nevertheless our discussion above gives us a commutative diagram:

$$
\begin{tikzcd}
K^{\bullet}_{\infty}(\mathbf{t}, \Omega^{r-1}_{X/Y}) \arrow[r, "\mathrm{d}^{r-1}_{X/Y}"] \arrow[d, "(15.2.11)"', "\wr"] & K^{\bullet}_{\infty}(\mathbf{t}, \Omega^{r}_{X/Y}) \arrow[d, "\wr"', "(15.2.11)"] \\
\mathbf{R}\Gamma_Z(X,\, \Omega^{r-1}_{X/Y}) \arrow[r, "\mathrm{d}^{r-1}_{X/Y}"'] & \mathbf{R}\Gamma_Z(X,\, \Omega^{r}_{X/Y})
\end{tikzcd}
\tag{15.2.12}
$$

Using the generalized fraction notation in (C.5.3) and the fact that (15.2.11) is described for quasi-coherent sheaves by (C.5.2), the commutativity of (15.2.12) gives:

$$
\begin{aligned}
\mathrm{H}^r_Z(\mathrm{d}^{r-1}_{X/Y}) \begin{bmatrix} \eta \\ t_1^{k_1}, \dots, t_r^{k_r} \end{bmatrix} = & \begin{bmatrix} \mathrm{d}\eta \\ t_1^{k_1}, \dots, t_r^{k_r} \end{bmatrix} \\
& - \sum_{j=1}^{r} k_j \begin{bmatrix} \mathrm{d}t_j \wedge \eta \\ t_1^{k_1}, \dots, t_j^{k_j+1}, \dots, t_r^{k_r} \end{bmatrix}.
\end{aligned}
\tag{15.2.13}
$$

Thus (R9) is equivalent to:

$$
\mathbf{res}_Z \circ \mathrm{H}^r_Z(\mathrm{d}^{r-1}_{X/Y}) = 0. \tag{15.2.14}
$$

If I is the ideal of R generated by $\mathbf{t}$, R^* the completion of I in the I-adic topology, $I^* = \mathbf{t}R^* = IR^*$, $\mathscr{X} = \operatorname{Spf}(R^*, I^*)$, then (15.2.14) is equivalent to

$$
\operatorname{tr}_{\mathscr{X}/Y} \circ \mathrm{H}^r_{I^*}(\mathrm{d}^{r-1}_{\mathscr{X}/Y}) = 0, \tag{15.2.15}
$$

where $\mathrm{d}^{r-1}_{\mathscr{X}/Y} \colon \Omega^{r-1}_{\mathscr{X}/Y} \to \Omega^{r}_{\mathscr{X}/Y}$ is the exterior differentiation on the exterior algebra of universally finite differential forms on $\mathscr{X}/Y$.

Since the residue field of A is algebraically closed, the formal Y-scheme $\mathscr{X}$ is isomorphic as a Y-scheme to $\operatorname{Spf} A[|T_1, \dots, T_r|]$, where $A[|T_1, \dots, T_r|]$ is given the $\mathbf{T}$-adic topology. And using the equivalence of (15.2.14) and (15.2.15) the other way, we are done if we prove (15.2.14) for $R = A[\mathbf{T}]$ and Z the scheme cut out by $\mathbf{T}$. In this case, η is a finite sum of $(r-1)$-forms of the kind

$$
\eta_{j,a_1,\dots,a_r} = T_1^{a_1} \dots T_r^{a_r} \cdot \mathrm{d}T_1 \wedge \dots \wedge \widehat{\mathrm{d}T_j} \wedge \dots \wedge \mathrm{d}T_r,
$$

where a_i are non-negative integers. Since $Z \to Y$ is an isomorphism in this case, $\mathbf{res}_Z$ is the standard residue which we know explicitly, the right side of (15.2.13) $\eta = \eta_{j,a_1,\dots,a_r}$ and $t_i = T_i$, is trivially seen to vanish.

APPENDIX A

Base change and completions

A.1. Basic properties of flat-base-change isomorphism for $-^{\#}$

We gather a few basic properties of the flat base-change map of (2.2.1). By default, we work with complexes in $\widetilde{\mathbf{D}}_{\mathrm{qc}}^{+}$.

Consider a cartesian square $\mathfrak{s}$ of noetherian formal schemes

$$\begin{tikzcd} \mathscr{V} \arrow[r, "v"] \arrow[d, "g"'] \arrow[dr, phantom, "\square"] & \mathscr{X} \arrow[d, "f"] \\ \mathscr{W} \arrow[r, "u"'] & \mathscr{Y} \end{tikzcd}$$

with f in $\mathbb{G}$ and u flat so that we have a flat-base-change isomorphism

$$\beta_{\mathfrak{s}}^{\#}\colon \mathbf{\Lambda}_{\mathscr{V}} v^* f^{\#} \xrightarrow{\ \sim\ } g^{\#} u^*$$

as in (2.2.1). If f (and hence g) is pseudoproper, then another description of $\beta_{\mathfrak{s}}^{\#}$ is that it is the map adjoint to the composite of the following natural maps (cf. [**AJL2**, Theorem 8.1, p. 86]).

$$\mathbf{R}g_*\mathbf{R}\varGamma'_{\mathscr{V}}\mathbf{\Lambda}_{\mathscr{V}} v^* f^{\#} \xrightarrow{\ \sim\ } \mathbf{R}g_*\mathbf{R}\varGamma'_{\mathscr{V}} v^* f^{\#} \to \mathbf{R}g_* v^*\mathbf{R}\varGamma'_{\mathscr{X}} f^{\#} \xrightarrow{\ \sim\ } u^*\mathbf{R}f_*\mathbf{R}\varGamma'_{\mathscr{X}} f^{\#} \xrightarrow{\mathrm{Tr}_f} u^*$$

If f, g are formally étale, then we have natural isomorphisms $f^{\#} \xrightarrow{\ \sim\ } \mathbf{\Lambda}_{\mathscr{X}} f^*$ and $g^{\#} \xrightarrow{\ \sim\ } \mathbf{\Lambda}_{\mathscr{V}} g^*$ induced by the corresponding ones for $(-)^!$, $f^! \xrightarrow{\ \sim\ } \mathbf{R}\varGamma'_{\mathscr{X}} f^*$ and $g^! \xrightarrow{\ \sim\ } \mathbf{R}\varGamma'_{\mathscr{V}} g^*$ respectively. In this case, the base-change map $\beta_{\mathfrak{s}}^!$ for $(-)^!$ is induced by the composite of the canonical isomorphisms

$$\mathbf{R}\varGamma'_{\mathscr{V}} v^* f^! \xrightarrow{\ \sim\ } \mathbf{R}\varGamma'_{\mathscr{V}} v^*\mathbf{R}\varGamma'_{\mathscr{X}} f^* \xrightarrow{\ \sim\ } \mathbf{R}\varGamma'_{\mathscr{V}} v^* f^* \xrightarrow{\ \sim\ } \mathbf{R}\varGamma'_{\mathscr{V}} g^* u^* \xrightarrow{\ \sim\ } g^! u^*.$$

Hence another description of $\beta_{\mathfrak{s}}^{\#}$ is that it is given by the composite of the following isomorphisms

$$\mathbf{\Lambda}_{\mathscr{V}} v^* f^{\#} \xrightarrow{\ \sim\ } \mathbf{\Lambda}_{\mathscr{V}} v^*\mathbf{\Lambda}_{\mathscr{X}} f^* \xrightarrow{\ \sim\ } \mathbf{\Lambda}_{\mathscr{V}} v^* f^* \xrightarrow{\ \sim\ } \mathbf{\Lambda}_{\mathscr{V}} g^* u^* \xrightarrow{\ \sim\ } g^{\#} u^*.$$

In particular, if $\mathscr{F} \in \mathbf{D}_{\mathrm{c}}^{+}(\mathscr{Y})$, or if u is open or if $\mathscr{V}$ is an ordinary scheme, then $\beta_{\mathfrak{s}}^{\#}(\mathscr{F})$ is given by the natural composite

$$v^* f^{\#}\mathscr{F} \xrightarrow{\ \sim\ } v^* f^*\mathscr{F} \xrightarrow{\ \sim\ } g^* u^*\mathscr{F} \xrightarrow{\ \sim\ } g^{\#} u^*\mathscr{F}.$$

Next we look at transitivity properties of $\beta^{\#}$ vis-á-vis extension of the square $\mathfrak{s}$ horizontally or vertically. These are also proved by reducing to the corresponding property for $\beta^!$ (see [**Nay**, Theorem 2.3.2(i)]).

Proposition A.1.1. *Let the notations be as above.*

(i) *Consider cartesian squares $\mathfrak{s}_1$, $\mathfrak{s}_2$ as follows*

$$\begin{tikzcd} \mathscr{V}_2 \arrow[r, "v_2"] \arrow[d, "h"'] \arrow[dr, phantom, "\mathfrak{s}_2"] & \mathscr{V}_1 \arrow[r, "v_1"] \arrow[d, "g"'] \arrow[dr, phantom, "\mathfrak{s}_1"] & \mathscr{X} \arrow[d, "f"] \\ \mathscr{W}_2 \arrow[r, "u_2"'] & \mathscr{W}_1 \arrow[r, "u_1"'] & \mathscr{Y} \end{tikzcd}$$

where f, g, h are in $\mathbb{G}$ and u_i, v_i are flat. Let $u = u_1u_2$ and $v = v_1v_2$ and let $\mathfrak{s}$ denote the composite cartesian diagram. Then the following diagram of isomorphisms commutes.

$$\begin{tikzcd} \mathbf{\Lambda}_{\mathscr{V}_2} v_2^* v_1^* f^{\#} \arrow[r] \arrow[d] & \mathbf{\Lambda}_{\mathscr{V}_2} v_2^* \mathbf{\Lambda}_{\mathscr{V}_1} v_1^* f^{\#} \arrow[r, "via"', "\beta^{\#}_{\mathfrak{s}_1}"] & \mathbf{\Lambda}_{\mathscr{V}_2} v_2^* g^{\#} u_1^* \arrow[r, "via"', "\beta^{\#}_{\mathfrak{s}_2}"] & h^{\#} u_2^* u_1^* \arrow[d] \\ \mathbf{\Lambda}_{\mathscr{V}_2} v^* f^{\#} \arrow[rrr, "via\ \beta^{\#}_{\mathfrak{s}}"'] & & & h^{\#} u^* \end{tikzcd}$$

(ii) *Consider cartesian squares $\mathfrak{s}_1$, $\mathfrak{s}_2$ as follows*

$$\begin{tikzcd} \mathscr{V}_2 \arrow[r, "w"] \arrow[d, "g_2"'] \arrow[dr, phantom, "\mathfrak{s}_2"] & \mathscr{X}_2 \arrow[d, "f_2"] \\ \mathscr{V}_1 \arrow[r, "v"] \arrow[d, "g_1"'] \arrow[dr, phantom, "\mathfrak{s}_1"] & \mathscr{X}_1 \arrow[d, "f_1"] \\ \mathscr{W} \arrow[r, "u"'] & \mathscr{Y} \end{tikzcd}$$

where f_i, g_i are in $\mathbb{G}$ and u, v, w are flat. Let $f = f_1f_2$ and $g = g_1g_2$ and let $\mathfrak{s}$ denote the composite cartesian diagram. Then the following diagram of isomorphisms commutes.

$$\begin{tikzcd} \mathbf{\Lambda}_{\mathscr{V}_2} w^* f_2^{\#} f_1^{\#} \arrow[r, "via"', "\beta^{\#}_{\mathfrak{s}_2}"] \arrow[d] & g_2^{\#} v^* f_1^{\#} \arrow[r] & g_2^{\#} \mathbf{\Lambda}_{\mathscr{V}_1} v^* f_1^{\#} \arrow[r, "via"', "\beta^{\#}_{\mathfrak{s}_1}"] & g_2^{\#} g_1^{\#} u^* \arrow[d] \\ \mathbf{\Lambda}_{\mathscr{V}_2} w^* f^{\#} \arrow[rrr, "via\ \beta^{\#}_{\mathfrak{s}}"'] & & & g^{\#} u^* \end{tikzcd}$$

Proof. (i). For convenience we shall consider the transposed version of the diagram in question. Using the definitions $f^{\#} = \mathbf{\Lambda}_{\mathscr{X}} f^!$, $g^{\#} = \mathbf{\Lambda}_{\mathscr{V}_1} g^!$, $h^{\#} = \mathbf{\Lambda}_{\mathscr{V}_2} h^!$ and the isomorphisms in (1.3.1) we reduce to checking that the outer border of the following diagram of isomorphisms commutes where to reduce clutter we have dropped the **R**'s.

$$
\begin{tikzcd}
\mathbf{\Lambda}_{\mathscr{Y}_2} v_2^* v_1^* \mathbf{\Lambda}_{\mathscr{X}} f^! \arrow[rrr] \arrow[d] & & & \mathbf{\Lambda}_{\mathscr{Y}_2} v^* \mathbf{\Lambda}_{\mathscr{X}} f^! \arrow[d] \\
\mathbf{\Lambda}_{\mathscr{Y}_2} v_2^* v_1^* f^! \arrow[r] \arrow[d] & \mathbf{\Lambda}_{\mathscr{Y}_2} \varGamma_{\mathscr{Y}_2} v_2^* v_1^* f^! \arrow[rr] \arrow[d] & & \mathbf{\Lambda}_{\mathscr{Y}_2} \varGamma_{\mathscr{Y}_2} v^* f^! \arrow[dd, "\text{via } \beta_{\mathsf{s}}^!"] \\
\mathbf{\Lambda}_{\mathscr{Y}_2} v_2^* \mathbf{\Lambda}_{\mathscr{Y}_1} \varGamma_{\mathscr{Y}_1} v_1^* f^! \arrow[r] \arrow[d, "\text{via } \beta_{\mathsf{s}_1}^!"'] & \mathbf{\Lambda}_{\mathscr{Y}_2} \varGamma_{\mathscr{Y}_2} v_2^* \varGamma_{\mathscr{Y}_1} v_1^* f^! \arrow[d, "\text{via } \beta_{\mathsf{s}_1}^!"'] \arrow[r, phantom, "\boxplus"] & {} & \\
\mathbf{\Lambda}_{\mathscr{Y}_2} v_2^* \mathbf{\Lambda}_{\mathscr{Y}_1} g^! u_1^* \arrow[r] & \mathbf{\Lambda}_{\mathscr{Y}_2} \varGamma_{\mathscr{Y}_2} v_2^* g^! u_1^* \arrow[r, "\text{via } \beta_{\mathsf{s}_2}^!"'] & \mathbf{\Lambda}_{\mathscr{Y}_2} h^! u_2^* u_1^* \arrow[r] & \mathbf{\Lambda}_{\mathscr{Y}_2} h^! u^*
\end{tikzcd}
$$

The unlabelled arrows are obvious natural maps. The rectangle $\boxplus$ commutes by the transitivity of base-change for $(-)^!$. Commutativity of the remaining parts is obvious.

(ii). Once again we consider the transpose of the diagram under consideration. Using the isomorphisms in (1.3.1) we reduce to checking that the outer border of the following diagram of isomorphisms commutes.

$$
\begin{tikzcd}
\mathbf{\Lambda}_{\mathscr{Y}_2} w^* \mathbf{\Lambda}_{\mathscr{X}_2} f_2^! \mathbf{\Lambda}_{\mathscr{X}_1} f_1^! \arrow[r] \arrow[d] & \mathbf{\Lambda}_{\mathscr{Y}_2} w^* \mathbf{\Lambda}_{\mathscr{X}_2} f_2^! f_1^! \arrow[r] \arrow[d] & \mathbf{\Lambda}_{\mathscr{Y}_2} w^* \mathbf{\Lambda}_{\mathscr{X}_2} f^! \arrow[d] \\
\mathbf{\Lambda}_{\mathscr{Y}_2} \varGamma_{\mathscr{Y}_2} w^* f_2^! \mathbf{\Lambda}_{\mathscr{X}_1} f_1^! \arrow[r] \arrow[d, "\text{via } \beta_{\mathsf{s}_2}^!"'] & \mathbf{\Lambda}_{\mathscr{Y}_2} \varGamma_{\mathscr{Y}_2} w^* f_2^! f_1^! \arrow[r] \arrow[d, "\text{via } \beta_{\mathsf{s}_2}^!"'] & \mathbf{\Lambda}_{\mathscr{Y}_2} \varGamma_{\mathscr{Y}_2} w^* f^! \arrow[dd, "\text{via } \beta_{\mathsf{s}}^!"] \\
\mathbf{\Lambda}_{\mathscr{Y}_2} g_2^! v^* \mathbf{\Lambda}_{\mathscr{X}_1} f_1^! \arrow[r] \arrow[d] & \mathbf{\Lambda}_{\mathscr{Y}_2} g_2^! v^* f_1^! \arrow[d] \arrow[dr, phantom, "\boxplus"] & \\
\mathbf{\Lambda}_{\mathscr{Y}_2} g_2^! \mathbf{\Lambda}_{\mathscr{Y}_1} \varGamma'_{\mathscr{Y}_1} v^* f_1^! \arrow[r] \arrow[d, "\text{via } \beta_{\mathsf{s}_1}^!"'] & \mathbf{\Lambda}_{\mathscr{Y}_2} g_2^! \varGamma'_{\mathscr{Y}_1} v^* f_1^! \arrow[d, "\text{via } \beta_{\mathsf{s}_1}^!"'] & \mathbf{\Lambda}_{\mathscr{Y}_2} g^! u^* \arrow[d] \\
\mathbf{\Lambda}_{\mathscr{Y}_2} g_2^! \mathbf{\Lambda}_{\mathscr{Y}_1} g_1^! u^* \arrow[r] & \mathbf{\Lambda}_{\mathscr{Y}_2} g_2^! g_1^! u^* \arrow[r] & \mathbf{\Lambda}_{\mathscr{Y}_2} g^! u^*
\end{tikzcd}
$$

The rectangle $\boxplus$ commutes by transitivity of base-change for $(-)^!$ while the other rectangles commute for obvious reasons. $\square$

Completion maps, being pseudo-proper, formally étale, and flat, give rise to additional compatibility issues. Now we consider some special situations involving completion maps.

Let $\mathscr{X}$ be a formal scheme and $\mathscr{I} \subset \mathscr{O}_{\mathscr{X}}$ an open coherent ideal. Let $\mathscr{W} := \widehat{\mathscr{X}}$ be the completion of $\mathscr{X}$ along $\mathscr{I}$ and $\kappa\colon \mathscr{W} \to \mathscr{X}$ the corresponding completion map. Then there are canonical isomorphisms (see proof of [**AJL3**, Lemma 4.1], and of [**AJL2**, Proposition 5.2.4])

$$
\kappa_* \mathbf{R}\varGamma'_{\mathscr{W}} \kappa^* \xrightarrow{\ \sim\ } \kappa_* \kappa^* \mathbf{R}\varGamma_{\mathscr{I}} \xrightarrow{\ \sim\ } \mathbf{R}\varGamma_{\mathscr{I}} \kappa_* \kappa^* \xleftarrow{\ \sim\ } \mathbf{R}\varGamma_{\mathscr{I}}. \tag{A.1.2}
$$

For the next two results regarding $(-)^{\#}$ for completion maps, we will first need to look at the corresponding results for $(-)^!$. For that purpose we recall that in [**Nay**], $(-)^!$ is obtained by gluing the pseudofunctor $(-)^\times$ over pseudoproper maps in $\mathbb{G}$ given by

$$f^\times = \text{ right adjoint to } \mathbf{R}f_*, \qquad (f \text{ pseudoproper}),$$

with the pseudofunctor $(-)^*_t$ over étale maps in $\mathbb{G}$ given by

$$f \mapsto \mathbf{R}\mathit{\Gamma}'_{\mathscr{X}} f_*, \qquad (f\colon \mathscr{X} \to \mathscr{Y} \text{ étale}),$$

and this gluing utilizes, among other things, the étale base-change isomorphisms associated to cartesian squares involving étale base change of a pseudoproper map (see [**Nay**, Theorem 7.1.6, §7.2.7]).

LEMMA A.1.3. *For a completion map $\kappa\colon \mathscr{W} \to \mathscr{X}$ by an open coherent ideal $\mathscr{I} \subset \mathscr{O}_{\mathscr{X}}$ as above and for $\mathscr{F} \in \mathbf{D}^+_{\mathrm{c}}(\mathscr{X})$, the isomorphism $\kappa^*\mathscr{F} \to \kappa^{\#}\mathscr{F}$ of (2.1.3) is also the map adjoint to the composite ψ given by*

$$\kappa_*\mathbf{R}\mathit{\Gamma}'_{\mathscr{W}}\kappa^* \xrightarrow{\ \sim\ } \mathbf{R}\mathit{\Gamma}_{\mathscr{I}} \to \mathbf{1}.$$

SKETCH PROOF. It suffices to prove that the corresponding property for $\kappa^!$ holds, i.e., the canonical isomorphism $\phi = \phi_\kappa\colon \mathbf{R}\mathit{\Gamma}'_{\mathscr{W}}\kappa^* \xrightarrow{\ \sim\ } \kappa^!$ is the map adjoint to ψ. Indeed, as per the proof of [**Nay**, Theorem 7.1.6], the isomorphism ϕ equals $(\beta^!)^{-1}$ where $\beta^!\colon \mathbf{1}^*\kappa^! \xrightarrow{\ \sim\ } \mathbf{1}^!\kappa^* = \mathbf{R}\mathit{\Gamma}'_{\mathscr{W}}\kappa^*$ is the base-change isomorphism associated to the cartesian square in the following diagram.

$$\begin{tikzcd}
\mathscr{W} \arrow[r, "\mathbf{1}"] & \mathscr{W} \arrow[r, "\mathbf{1}"] \arrow[d, "\mathbf{1}"'] & \mathscr{W} \arrow[d, "\kappa"] \\
& \mathscr{W} \arrow[r, "\kappa"'] & \mathscr{X}
\end{tikzcd}$$

Therefore, $\phi = \alpha_1\alpha_2^{-1}$ for α_i as given in the commutative diagram below, where α_1 is the canonical map $\kappa^*\kappa_* \to \mathbf{1}$, (which is an isomorphism over $\mathbf{D}^+_{\mathrm{qct}}(\mathscr{W})$) while α_2 results from the fact that the trace $\mathrm{Tr}^!_\kappa\colon \kappa_*\kappa^! \to \mathbf{1}$ factors through $\mathbf{R}\mathit{\Gamma}'_{\mathscr{I}} \to \mathbf{1}$.

$$\begin{tikzcd}
\kappa^! & \kappa^*\kappa_*\kappa^! \arrow[l, "\sim", "\alpha_1"'] \arrow[r, "\sim", "\alpha_2"'] \arrow[rr, bend left, "\kappa^*\mathrm{Tr}^!_\kappa"] & \mathbf{R}\mathit{\Gamma}'_{\mathscr{W}}\kappa^* \arrow[r] & \kappa^*
\end{tikzcd}$$

The adjointness of ϕ and ψ amounts to proving that $\mathrm{Tr}^!_\kappa\kappa_*(\phi) = \psi$, which results from the commutativity of the following.

$$\begin{tikzcd}
\kappa_*\kappa^! \arrow[dr, equal] & \kappa_*\kappa^*\kappa_*\kappa^! \arrow[l, "\sim", "\kappa_*\alpha_1"'] \arrow[rr, "\sim", "\kappa_*\alpha_2"'] & & \kappa_*\mathbf{R}\mathit{\Gamma}'_{\mathscr{W}}\kappa^* \arrow[r] & \kappa_*\kappa^* \\
& \kappa_*\kappa^! \arrow[u, "\mathbf{1}\to\kappa_*\kappa^*"'] \arrow[rr, "\mathrm{Tr}^!_\kappa"'] & & \mathbf{R}\mathit{\Gamma}_{\mathscr{I}} \arrow[u, "\cong"'] \arrow[r] & \mathbf{1} \arrow[u]
\end{tikzcd}$$

$\square$

LEMMA A.1.4. *Consider a cartesian diagram in $\mathbb{G}$ as follows*

$$\begin{tikzcd} \mathscr{V} \arrow[r, "\bar\kappa"] \arrow[d, "g"'] \arrow[dr, phantom, "\square"] & \mathscr{X} \arrow[d, "f"] \\ \mathscr{W} \arrow[r, "\kappa"'] & \mathscr{Y} \end{tikzcd}$$

where $\kappa, \bar\kappa$ are completion maps by open coherent ideal sheaves. Let $\mathscr{F} \in \mathbf{D}_{\mathrm{c}}^{+}(\mathscr{Y})$.

(i) *The following diagram of obvious natural isomorphisms commutes.*

$$\begin{tikzcd} \bar\kappa^* f^{\#}\mathscr{F} \arrow[r, "\sim", "(2.2.2)"'] \arrow[d, "(2.1.3)"', "\wr"] & g^{\#}\kappa^*\mathscr{F} \arrow[d, "(2.1.3)", "\wr"'] \\ \bar\kappa^{\#} f^{\#}\mathscr{F} \arrow[r, "\sim"] & g^{\#}\kappa^{\#}\mathscr{F} \end{tikzcd}$$

(ii) *If f is flat then the following diagram of obvious natural isomorphisms commutes.*

$$\begin{tikzcd} g^*\kappa^*\mathscr{F} \arrow[r, "\sim"] \arrow[d, "(2.1.3)"', "\wr"] & \bar\kappa^* f^*\mathscr{F} \arrow[d, "(2.1.3)", "\wr"'] \\ g^*\kappa^{\#}\mathscr{F} \arrow[r, "\sim", "(2.2.2)"'] & \bar\kappa^{\#} f^*\mathscr{F} \end{tikzcd}$$

SKETCH PROOF. As a consequence of the gluing result in [**Nay**, Theorem 7.1.6], via the canonical isomorphisms $\phi_{\bar\kappa}\colon \bar\kappa^! \xrightarrow{\sim} \mathbf{R}\varGamma'_{\mathscr{V}}\bar\kappa^*$ and $\phi_{\kappa}\colon \kappa^! \xrightarrow{\sim} \mathbf{R}\varGamma'_{\mathscr{W}}\kappa^*$, for the situation in (i), we have a commutative diagram of isomorphisms

$$\begin{tikzcd} \mathbf{R}\varGamma'_{\mathscr{V}}\bar\kappa^* f^! \arrow[r, "\beta^!"] \arrow[d] & g^!\mathbf{R}\varGamma'_{\mathscr{W}}\kappa^* \arrow[d] \\ \bar\kappa^! f^! \arrow[r] & g^!\kappa^! \end{tikzcd}$$

reflecting the compatibility of β with the pseudofunctorial structure of $(-)^!$, while for the one in (ii), we have a commutative diagram of isomorphisms as follows,

$$\begin{tikzcd} \mathbf{R}\varGamma'_{\mathscr{V}} g^*\mathbf{R}\varGamma'_{\mathscr{W}}\kappa^* \arrow[r] \arrow[d] & \mathbf{R}\varGamma'_{\mathscr{V}}\bar\kappa^* f^* \arrow[d] \\ \mathbf{R}\varGamma'_{\mathscr{V}} g^*\kappa^! \arrow[r, "\beta^!"'] & \bar\kappa^! f^* \end{tikzcd}$$

reflecting the compatibility of β with the pseudofunctorial structure of $(-)^*_t$ over étale maps. The result now follows by applying $\mathbf{\Lambda}$'s appropriately in each diagram and using the pre-pseudofunctorial properties of $(-)^{\#}$. □

A.2. Compatibility with completions

Let $f\colon \mathscr{X} \to \mathscr{Y}$ be a pseudo-proper map and let $\mathscr{J}$ be an ideal of definition of $\mathscr{X}$. Suppose $\mathscr{I}$ is an open coherent ideal in $\mathscr{O}_{\mathscr{Y}}$ and $\kappa\colon \mathscr{U} \to \mathscr{Y}$ is the completion of $\mathscr{Y}$ with respect to $\mathscr{I}$. Let $\mathscr{V} = \mathscr{X} \times_{\mathscr{Y}} \mathscr{U}$ and $\kappa'\colon \mathscr{V} \to \mathscr{X}$, $g\colon \mathscr{V} \to \mathscr{U}$ the projection maps. Note that $\mathscr{V}$ is the completion of $\mathscr{X}$ with respect to the $\mathscr{O}_{\mathscr{X}}$-ideal $\mathscr{I}\mathscr{O}_{\mathscr{X}} + \mathscr{J}$, and κ' is the completion map. We thus have a cartesian square:

$$\begin{tikzcd}
\mathscr{V} \arrow[r, "\kappa'"] \arrow[d, "f'"'] \arrow[dr, phantom, "\square"] & \mathscr{X} \arrow[d, "f"] \\
\mathscr{U} \arrow[r, "\kappa"'] & \mathscr{Y}
\end{tikzcd}$$

By (A.1.2), we have $\kappa_*\mathbf{R}\varGamma'_{\mathscr{U}}\kappa^* \xrightarrow{\sim} \mathbf{R}\varGamma_{\mathscr{J}}$ and $\kappa'_*\mathbf{R}\varGamma'_{\mathscr{V}}\kappa'^* \xrightarrow{\sim} \mathbf{R}\varGamma_{\mathscr{I}\mathscr{O}_{\mathscr{X}}+\mathscr{J}}$.

Proposition A.2.1. *The following diagram commutes:*

$$\begin{tikzcd}
\mathbf{R}f_*\kappa'_*\mathbf{R}\varGamma'_{\mathscr{V}}\kappa'^*f^{\#} \arrow[rr, "\sim", "\gamma"'] \arrow[d, "\wr"'] & & \kappa_*\mathbf{R}f'_*\mathbf{R}\varGamma'_{\mathscr{V}}(f')^{\#}\kappa^* \arrow[dd, "\kappa_*\mathrm{Tr}_{f'}"] \\
\mathbf{R}f_*\mathbf{R}\varGamma_{\mathscr{I}\mathscr{O}_{\mathscr{X}}+\mathscr{J}}f^{\#} \arrow[d, "\wr"'] & & \\
\mathbf{R}f_*\mathbf{R}\varGamma_{\mathscr{I}\mathscr{O}_{\mathscr{X}}}\mathbf{R}\varGamma'_{\mathscr{X}}f^{\#} \arrow[d, "\wr"'] & & \kappa_*\mathbf{R}\varGamma'_{\mathscr{U}}\kappa^* \arrow[d, "\wr"'] \\
\mathbf{R}\varGamma_{\mathscr{J}}\mathbf{R}f_*\mathbf{R}\varGamma'_{\mathscr{X}}f^{\#} \arrow[rr, "\mathrm{Tr}_f"'] & & \mathbf{R}\varGamma_{\mathscr{J}}
\end{tikzcd}$$

where γ is induced by applying the functor $\mathbf{R}f_*\kappa'_*\mathbf{R}\varGamma'_{\mathscr{V}}\kappa'^*$ to the natural isomorphism $\kappa'^*f^{\#} \xrightarrow{\sim} f'^{\#}\kappa^*$ and the upward pointing arrow on the southwest corner is the isomorphism of [**AJL2**, Proposition 5.2.8 (d)].

Proof. In the diagram of the proposition, let $\alpha\colon \mathbf{R}f_*\kappa'_*\mathbf{R}\varGamma'_{\mathscr{V}}\kappa'^*f^{\#} \to \mathbf{R}\varGamma_{\mathscr{J}}$ be the map obtained by composing maps along the route in the diagram which starts at the northwest corner, travels south and then east. Let $\beta\colon \mathbf{R}f_*\kappa'_*\mathbf{R}\varGamma'_{\mathscr{V}}\kappa'^*f^{\#} \to \mathbf{R}\varGamma_{\mathscr{J}}$ be the composition which starts in the easterly direction and then moves south. Let $\psi\colon \mathbf{R}\varGamma_{\mathscr{J}} \to \mathbf{1}_{\mathbf{D}(\mathscr{Y})}$ be the natural map. We have to show that $\alpha = \beta$. This is equivalent to showing

$$\psi\circ\alpha = \psi\circ\beta. \tag{A.2.2}$$

We now proceed to prove (A.2.2). In what follows we identify κ'^* with $\kappa'^{\#}$ and κ^* with $\kappa^{\#}$. Recall that the isomorphism $\kappa'^*\colon f^{\#} \xrightarrow{\sim} f'^{\#}\kappa^*$ mentioned in the theorem can be interpreted in two ways, and the two interpretations agree: (a) as a base change isomorphism, and (b) as the composite

$$\kappa'^*f^{\#} = \kappa'^{\#}f^{\#} \xrightarrow{\sim} (f\kappa')^{\#} = (\kappa f')^{\#} \xrightarrow{\sim} f'^{\#}\kappa^{\#} = f'^{\#}\kappa^*. \tag{A.2.3}$$

We point out the trace map $\mathrm{Tr}_\kappa\colon \kappa_*\mathbf{R}\varGamma'_{\mathscr{U}}\kappa^* \to {}_{\mathbf{D}(\mathscr{Y})}$ under the identification $\kappa^* = \kappa^{\#}$ is the composite $\kappa_*\mathbf{R}\varGamma'_{\mathscr{U}}\kappa^* \xrightarrow{\sim} \mathbf{R}\varGamma_{\mathscr{J}} \to \mathbf{1}_{\mathbf{D}(\mathscr{Y})}$. Similarly, $\mathrm{Tr}_{\kappa'}\colon \kappa'_*\mathbf{R}\varGamma'_{\mathscr{V}}\kappa'^* \to R\varGamma'_{\mathscr{X}}$ is the composite $\kappa'_*\mathbf{R}\varGamma'_{\mathscr{V}}\kappa'^* \xrightarrow{\sim} \mathbf{R}\varGamma_{\mathscr{I}\mathscr{O}_{\mathscr{X}}+\mathscr{J}} \to \mathbf{R}f_*\mathbf{R}\varGamma_{\mathscr{J}} = \mathbf{R}\varGamma'_{\mathscr{X}}$.

From the definition of the isomorphism in (A.2.3) it follows that the following diagram commutes:

$$\begin{tikzcd}
\mathbf{R}f_*\kappa'_*\mathbf{R}\varGamma'_{\mathscr{V}}\kappa'^*f^{\#} \arrow[r, "\sim", "\text{via (A.2.3)}"'] \arrow[d, "\wr"'] & \kappa_*\mathbf{R}f'_*\mathbf{R}\varGamma'_{\mathscr{V}}(f')^{\#}\kappa^* \arrow[d, "\kappa_*\mathrm{Tr}_{f'}"] \\
\mathbf{R}f_*\mathbf{R}\varGamma_{\mathscr{I}\mathscr{O}_{\mathscr{X}}+\mathscr{J}}f^{\#} \arrow[d] & \kappa_*\mathbf{R}\varGamma'_{\mathscr{U}}\kappa^* \arrow[d, "\wr"] \\
\mathbf{R}f_*\mathbf{R}\varGamma_{\mathscr{J}}f^{\#} \arrow[d, equal] & \mathbf{R}\varGamma_{\mathscr{I}} \arrow[d, "\psi"] \\
\mathbf{R}f_*\mathbf{R}\varGamma'_{\mathscr{X}}f^{\#} \arrow[r, "\mathrm{Tr}_f"'] & \mathbf{1}_{\mathbf{D}(\mathscr{Y})}
\end{tikzcd}$$

Let $\theta\colon \mathbf{R}f_*\kappa'_*\mathbf{R}\varGamma'_{\mathscr{V}}\kappa'^*f^{\#} \to \mathbf{1}_{\mathbf{D}(\mathscr{Y})}$ be the map obtained from taking any route from the top left corner to the bottom right corner in the above commutative diagram. Note that $\theta = \psi\circ\beta$. It is therefore enough to show that $\theta = \psi\circ\alpha$. Consider the following diagram where the arrow in the top row and the second map in the second row arise from the natural maps $\mathbf{R}\varGamma_{\mathscr{I}\mathscr{O}_{\mathscr{X}}+\mathscr{J}} \to \mathbf{R}\varGamma_{\mathscr{J}}$ and $\mathbf{R}\varGamma_{\mathscr{I}\mathscr{O}_{\mathscr{X}}} \to \mathbf{1}_{\mathbf{D}(\mathscr{X})}$ respectively:

$$\begin{tikzcd}
 & \mathbf{R}f_*\mathbf{R}\varGamma_{\mathscr{I}\mathscr{O}_{\mathscr{X}}+\mathscr{J}}f^{\#} \arrow[r] \arrow[d, "\wr"'] & \mathbf{R}f_*\mathbf{R}\varGamma_{\mathscr{J}}f^{\#} \arrow[d, equal] \\
\mathbf{R}\varGamma_{\mathscr{I}}\mathbf{R}f_*\mathbf{R}\varGamma'_{\mathscr{X}}f^{\#} \arrow[r, "\sim"] \arrow[d, "\mathbf{R}\varGamma_{\mathscr{I}}(\mathrm{Tr}_f)"'] & \mathbf{R}f_*\mathbf{R}\varGamma_{\mathscr{I}\mathscr{O}_{\mathscr{X}}}\mathbf{R}\varGamma'_{\mathscr{X}}f^{\#} \arrow[r] & \mathbf{R}f_*\mathbf{R}\varGamma'_{\mathscr{X}}f^{\#} \arrow[d, "\mathrm{Tr}_f"] \\
\mathbf{R}\varGamma_{\mathscr{I}} \arrow[rr, "\psi"'] & & \mathbf{1}_{\mathbf{D}(\mathscr{Y})}
\end{tikzcd}$$

We claim this diagram commutes. The sub-rectangle on the top clearly commutes. According to [**AJL2**, Proposition 5.2.8 (d)], the composite of the two arrows in the second row is the natural map arising from $\psi\colon \mathbf{R}\varGamma_{\mathscr{I}} \to \mathbf{1}_{\mathbf{D}(\mathscr{Y})}$. It follows that the rectangle at the bottom also commutes, whence the whole diagram commutes. This proves that $\theta = \psi\circ\alpha$. Thus $\psi\circ\alpha = \theta = \psi\circ\beta$, establishing (A.2.2). $\square$

A.3. Completions and compactifications

Suppose $f\colon X \to Y$ is a map of ordinary schemes in $\mathbb{G}$ and $Z \hookrightarrow X$ is a closed subscheme such that $Z \to Y$ is *proper*. Let $\kappa\colon \mathscr{X} = X_{/Z} \to X$ be the formal completion of X along Z and $\widehat{f}\colon \mathscr{X} \to Y$ the composition $\widehat{f} = f\circ\kappa$. Then $\widehat{f}$ is *pseudo-proper*. The isomorphism $\kappa^* \xrightarrow{\sim} \kappa^{\#}$ of (2.1.3) gives us an isomorphism $\kappa^*f^{\#} \xrightarrow{\sim} \widehat{f}^{\#}$, and hence an isomorphism $\alpha\colon \mathbf{R}f_*\kappa_*\mathbf{R}\varGamma'_{\mathscr{X}}\kappa^*f^{\#} \xrightarrow{\sim} \mathbf{R}\widehat{f}_*\mathbf{R}\varGamma'_{\mathscr{X}}\widehat{f}^{\#}$. On the other hand we have $\beta\colon \mathbf{R}f_*\kappa_*\mathbf{R}\varGamma'_{\mathscr{X}}\kappa^*f^{\#} \xrightarrow{\sim} \mathbf{R}f_*\mathbf{R}\varGamma_Z f^{\#}$ induced by (A.1.2). We thus have an isomorphism

$$\alpha\circ\beta^{-1}\colon \mathbf{R}f_*\mathbf{R}\varGamma_Z f^{\#} \xrightarrow{\sim} \mathbf{R}\widehat{f}_*\mathbf{R}\varGamma'_{\mathscr{X}}\widehat{f}^{\#}. \tag{A.3.1}$$

If $u\colon X \to X'$ is an open immersion of finite type Y-schemes, with $g\colon X' \to Y$ the structure map, then the natural isomorphism

$$\mathbf{R}f_*\mathbf{R}\varGamma_Z f^{\#} \xrightarrow{\ \sim\ } \mathbf{R}f_*\mathbf{R}\varGamma_Z u^*g^{\#} = \mathbf{R}g_*\mathbf{R}\varGamma_{u(Z)}g^{\#}$$

fits into a commutative diagram

$$\text{(A.3.2)}\qquad \begin{tikzcd} \mathbf{R}f_*\mathbf{R}\varGamma_Z f^{\#} \arrow[d, "\wr"'] \arrow[dr, "(\text{A.3.1})"] & \\ \mathbf{R}g_*\mathbf{R}\varGamma_{u(Z)}g^{\#} \arrow[r, "(\text{A.3.1})"'] & \mathbf{R}\widehat{f}_*\mathbf{R}\varGamma'_{\mathscr{X}}\widehat{f}^{\#} \end{tikzcd}$$

If f is *proper*, then the isomorphism $\kappa^{\#}f^{\#} \xrightarrow{\ \sim\ } \widehat{f}^{\#}$ is the one adjoint to the composite

$$\mathbf{R}f_*\kappa_*\mathbf{R}\varGamma'_{\mathscr{X}}\kappa^{\#}f^{\#} \xrightarrow{\mathbf{R}f_*(\mathrm{Tr}_\kappa)} \mathbf{R}f_*f^{\#} \to \mathbf{1}$$

and so the isomorphism $\kappa^*f^{\#} \xrightarrow{\ \sim\ } \widehat{f}^{\#}$ is characterised by the commutativity of the following diagram.

$$\text{(A.3.3)}\qquad \begin{tikzcd} \mathbf{R}f_*\mathbf{R}\varGamma_Z f^{\#} \arrow[d, "\text{natural}"'] & \mathbf{R}f_*\kappa_*\mathbf{R}\varGamma'_{\mathscr{X}}\kappa^*f^{\#} \arrow[l, "\sim"', "\beta"] \arrow[r, "\sim", "\alpha"'] & \mathbf{R}\widehat{f}_*\mathbf{R}\varGamma'_{\mathscr{X}}\widehat{f}^{\#} \arrow[d, "\mathrm{Tr}_{\widehat{f}}"] \\ \mathbf{R}f_*f^{\#} \arrow[rr, "\mathrm{Tr}_f"'] & & \mathbf{1} \end{tikzcd}$$

In general, when f is not necessarily proper, it is still separated (being in $\mathbb{G}$) and hence we do have a *compactification* of f, i.e., an open immersion of Y-schemes $u\colon X \to \bar{X}$, such that the structure map $\bar{f}\colon \bar{X} \to Y$ is *proper*. We have a commutative diagram:

$$\text{(A.3.4)}\qquad \begin{tikzcd} \mathscr{X} \arrow[r, "\kappa"] \arrow[dr, "\widehat{f}"'] & X \arrow[r, hook, "u"] \arrow[d, "f"] & \bar{X} \arrow[dl, "\bar{f}"] \\ & Y & \end{tikzcd}$$

We then have the following lemma.

Lemma A.3.5. *Under the assumptions and notation of* (A.3.4), *the following diagram commutes:*

$$\begin{tikzcd} \mathbf{R}f_*\mathbf{R}\varGamma_Z f^{\#} \arrow[rr, "\sim", "(\text{A.3.1})"'] \arrow[d, "\wr"'] & & \mathbf{R}\widehat{f}_*\varGamma'_{\mathscr{X}}\widehat{f}^{\#} \arrow[d, "\mathrm{Tr}_{\widehat{f}}"] \\ \mathbf{R}\bar{f}_*\mathbf{R}\varGamma_{u(Z)}\bar{f}^{\#} \arrow[r] & \mathbf{R}\bar{f}_*\bar{f}^{\#} \arrow[r, "\mathrm{Tr}_{\bar{f}}"'] & \mathbf{1} \end{tikzcd}$$

In particular, the composite

$$\mathbf{R}f_*\mathbf{R}\varGamma_Z f^{\#} \xrightarrow{\ \sim\ } \mathbf{R}\bar{f}_*\mathbf{R}\varGamma_{u(Z)}\bar{f}^{\#} \to \mathbf{R}\bar{f}_*\bar{f}^{\#} \xrightarrow{\mathrm{Tr}_{\bar{f}}} \mathbf{1}$$

is independent of the compactification $(u,\,\bar{f})$ *of* f.

PROOF. We expand the diagram to

$$
\begin{tikzcd}
\mathbf{R}f_*\mathbf{R}\varGamma_Z f^{\#} \arrow[r, "\sim", "(\text{A.3.1})"'] \arrow[d, "\wr"'] & \mathbf{R}\widehat{f}_*\varGamma'_{\mathscr{X}}\widehat{f}^{\#} \arrow[r, "\mathrm{Tr}_{\widehat{f}}"] & \mathbf{1} \\
\mathbf{R}\bar{f}_*\mathbf{R}\varGamma_{u(Z)}\bar{f}^{\#} \arrow[r] \arrow[ur, "\sim", "(\text{A.3.1})"'] & \mathbf{R}\bar{f}_*\bar{f}^{\#} \arrow[ur, "\mathrm{Tr}_{\bar{f}}"'] &
\end{tikzcd}
$$

The triangle on the left commutes by (A.3.2). The parallelogram is simply (A.3.3), for $\alpha\circ\beta^{-1} =$ (A.3.1). $\square$

APPENDIX B

Closed immersions and completions

B.1. The variance theory $-^\flat$

Let $i\colon \mathscr{Z} \to \mathscr{X}$ be a closed immersion of noetherian formal schemes. We use $\overline{i}$ to denote the flat map of ringed spaces $(\mathscr{Z}, \mathscr{O}_{\mathscr{Z}}) \to (\mathscr{X}, i_*\mathscr{O}_{\mathscr{Z}})$. We define the functor $i^\flat\colon \mathbf{D}(\mathscr{X}) \to \mathbf{D}(\mathscr{Z})$ by

$$i^\flat := \overline{i}^*\mathbf{R}\mathscr{H}om^\bullet_{\mathscr{X}}(i_*\mathscr{O}_{\mathscr{Z}}, -).$$

The functor $i^\flat$ enjoys the following properties (see [**AJL2**, Examples 6.1.3(4)]).

1) $i^\flat(\mathbf{D}^+_{\mathrm{qc}}(\mathscr{X})) \subset \mathbf{D}^+_{\mathrm{qc}}(\mathscr{Z})$ and $i^\flat(\mathbf{D}^+_{\mathrm{c}}(\mathscr{X})) \subset \mathbf{D}^+_{\mathrm{c}}(\mathscr{Z})$. This follows from the fact that $i_*\mathscr{O}_{\mathscr{Z}}$ is coherent $\mathscr{O}_{\mathscr{X}}$-module.

2) There is a natural isomorphism $i^\flat\mathbf{R}\varGamma'_{\mathscr{X}} \xrightarrow{\ \sim\ } \mathbf{R}\varGamma_{\mathscr{Z}} i^\flat$ whose composition with the natural map $\mathbf{R}\varGamma_{\mathscr{Z}} i^\flat \to i^\flat$ is the natural map $i^\flat\mathbf{R}\varGamma'_{\mathscr{X}} \to i^\flat$.

3) Using 2) we also obtain that $i^\flat(\mathbf{D}^+_{\mathrm{qct}}(\mathscr{X})) \subset \mathbf{D}^+_{\mathrm{qct}}(\mathscr{Z})$. Hence we also deduce that $i^\flat(\widetilde{\mathbf{D}}^+_{\mathrm{qc}}(\mathscr{X})) \subset \widetilde{\mathbf{D}}^+_{\mathrm{qc}}(\mathscr{Z})$.

4) There is a canonical trace map on $\mathbf{D}(\mathscr{X})$, namely

$$\mathrm{Tr}^\flat_i\colon i_*i^\flat = \mathbf{R}\mathscr{H}om^\bullet_{\mathscr{X}}(i_*\mathscr{O}_{\mathscr{Z}}, -) \longrightarrow \mathbf{1}, \tag{B.1.1}$$

which is given by "evaluation at 1", and which induces a natural map of functors from $i^\flat\colon \widetilde{\mathbf{D}}^+_{\mathrm{qc}}(\mathscr{X}) \to \widetilde{\mathbf{D}}^+_{\mathrm{qc}}(\mathscr{Z})$ to the right adjoint $i^\times$ of $i_*\colon \widetilde{\mathbf{D}}^+_{\mathrm{qc}}(\mathscr{Z}) \to \widetilde{\mathbf{D}}^+_{\mathrm{qc}}(\mathscr{X})$. Moreover, this induced map $i^\flat \to i^\times$ is an isomorphism. Keeping in mind that the values of $(-)^!$ range in $\mathbf{D}^+_{\mathrm{qct}}$, we deduce that for any $\mathscr{F} \in \widetilde{\mathbf{D}}^+_{\mathrm{qc}}(\mathscr{X})$, there is a natural isomorphism

$$i^\flat\mathbf{R}\varGamma'_{\mathscr{X}}\mathscr{F} \xrightarrow{\ \sim\ } \mathbf{R}\varGamma_{\mathscr{Z}} i^\flat\mathscr{F} \xrightarrow[\text{via } \mathrm{Tr}_i]{\ \sim\ } i^!\mathscr{F}$$

and hence for $\mathscr{F} \in \widetilde{\mathbf{D}}^+_{\mathrm{qc}}(\mathscr{X})$, there is also a natural isomorphism

$$\mathbf{\Lambda}_{\mathscr{Z}} i^\flat\mathscr{F} \xrightarrow{\ \sim\ } i^\#\mathscr{F}$$

where the corresponding trace map Tr_i is the natural composite

$$i_*\mathbf{R}\varGamma'_{\mathscr{Z}}\mathbf{\Lambda}_{\mathscr{Z}} i^\flat \xrightarrow{\ \sim\ } i_*\mathbf{R}\varGamma'_{\mathscr{Z}} i^\flat \to i_*i^\flat \xrightarrow{\mathrm{Tr}^\flat_i} 1.$$

In particular, if $\mathscr{F} \in \mathbf{D}^+_{\mathrm{c}}(\mathscr{X})$, or if $\mathscr{Z}$ is an ordinary scheme, then we have a canonical isomorphism

$$i^\flat\mathscr{F} \xrightarrow{\ \sim\ } i^\#\mathscr{F}. \tag{B.1.2}$$

B.2. Completion and $-^\flat$

Suppose X is an ordinary scheme, $\mathscr{I}$ a coherent ideal sheaf on X, Z the closed subscheme of X defined by $\mathscr{I}$, and $\kappa\colon \mathscr{X} = X_{/Z} \to X$ the completion of X along Z. We then have a commutative diagram with i and j closed immersions:

$$\begin{tikzcd} Z \arrow[r, hook, "j"] \arrow[dr, hook, "i"'] & \mathscr{X} \arrow[d, "\kappa"] \\ & X \end{tikzcd}$$

We define $\bar{i}$ and $\bar{j}$ as in B.1 above, and it follows that if $\mathscr{F}$ is a $j_*\mathscr{O}_Z$-module, then $\bar{i}^*\kappa_*\mathscr{F} = \bar{j}^*\mathscr{F}$. We also define $i^\flat, j^\flat$ as in B.1 and in what follows we will drop the symbols i_*, j_* occurring in the definition of $i^\flat, j^\flat$ respectively. Finally note that, since Z is an ordinary scheme so that $\varGamma'_Z$ is the identity functor, $i^\#$ and $j^\#$ are right adjoint to i_* and j_* respectively.

The natural map

$$\mathbf{R}\mathscr{H}om^\bullet_X(\mathscr{O}_Z,\, -) \longrightarrow \kappa_*\mathbf{R}\mathscr{H}om^\bullet_{\mathscr{X}}(\mathscr{O}_Z,\, \kappa^*-) \tag{B.2.1}$$

is an isomorphism, whence we have an isomorphism

$$i^\flat \xrightarrow{\;\sim\;} j^\flat\kappa^* \tag{B.2.2}$$

given by

$$\bar{i}^*\mathbf{R}\mathscr{H}om^\bullet_X(\mathscr{O}_Z,\, -) \xrightarrow[\bar{i}^*(\text{B.2.1})]{\;\sim\;} \bar{i}^*\kappa_*\mathbf{R}\mathscr{H}om^\bullet_{\mathscr{X}}(\mathscr{O}_Z,\, \kappa^*-) = \bar{j}^*\mathbf{R}\mathscr{H}om^\bullet_{\mathscr{X}}(\mathscr{O}_Z,\, \kappa^*-).$$

The essential content of the following lemma is that (B.2.2) is, up to canonical identifications, the inverse of the canonical isomorphism $j^\#\kappa^\# \xrightarrow{\;\sim\;} (\kappa j)^\# = i^\#$.

Lemma B.2.3. *The following diagram commutes*

$$\begin{tikzcd} j^\flat\kappa^* \arrow[r, "\sim", "(\text{B.1.2})"'] & j^\#\kappa^* \arrow[r, "\sim", "(2.1.3)"'] & j^\#\kappa^\# \arrow[d, "\wr"] \\ i^\flat \arrow[u, "(\text{B.2.2})", "\wr"'] \arrow[rr, "\sim", "(\text{B.1.2})"'] & & i^\# \end{tikzcd}$$

where the unlabelled isomorphism $j^\#\kappa^\# \xrightarrow{\;\sim\;} i^\#$ is the canonical one.

Proof. Keeping in mind that the canonical maps $j_*j^\flat \to 1$ and $j_*j^\# \to 1$ factor through $\mathbf{R}\varGamma'_{\mathscr{X}} \to 1$ and that the canonical map $i_*i^\flat \to 1$ factors through $\mathbf{R}\varGamma_Z \to 1$ we see that the diagram of the lemma corresponds, via adjointness of $i^\#$ to i_*, to the outer border of the following commutative diagram of obvious natural maps.

$$\begin{tikzcd}
i_*j^{\flat}\kappa^* \arrow[rr] \arrow[dr] & & i_*j^{\sharp}\kappa^* \arrow[r] \arrow[d] & i_*j^{\sharp}\kappa^{\sharp} \arrow[d] \\
& \kappa_*j_*j^{\flat}\kappa^* \arrow[r] \arrow[d, equal] & \kappa_*j_*j^{\sharp}\kappa^* \arrow[r] \arrow[d] & \kappa_*j_*j^{\sharp}\kappa^{\sharp} \arrow[d] \\
& \kappa_*\mathbf{R}\mathscr{H}om^{\bullet}_{\mathscr{X}}(\mathscr{O}_Z,\, \kappa^*(-)) \arrow[r] \arrow[d] & \kappa_*\mathbf{R}\varGamma'_{\mathscr{X}}\kappa^* \arrow[r] \arrow[d] & \kappa_*\mathbf{R}\varGamma'_{\mathscr{X}}\kappa^{\sharp} \arrow[d] \\
i_*i^{\flat} \arrow[uuu] \arrow[r, equal] & \mathbf{R}\mathscr{H}om^{\bullet}_{X}(\mathscr{O}_Z,\, -) \arrow[r] & \mathbf{R}\varGamma_Z \arrow[r] & 1
\end{tikzcd}$$

$\square$

APPENDIX C

Koszul complexes

Since our goal is to understand Verdier's isomorphism explicitly, we have to lay out our conventions for maps between complexes, especially the fundamental local isomorphism which is at the heart of explicit formulas for residues, and hence integrals (i.e., traces).

C.1. Our version of Koszul complexes

Let R be a noetherian ring. For $t \in R$, we write $K_\bullet(t)$ for the homology complex

$$0 \longrightarrow K_1(t) \longrightarrow K_0(t) \longrightarrow 0$$

where $K_1(t) = K_0(t) = R$ and the arrow between them is multiplication by t. For a sequence of elements $\mathbf{t} = (t_1, \ldots, t_r)$ in R, we set $K_\bullet(\mathbf{t})$ to be the complex:

$$K_\bullet(\mathbf{t}) = K_\bullet(t_1) \otimes_R \cdots \otimes_R K_\bullet(t_r).$$

For an R-module M and an integer i, we write $K^i(\mathbf{t}, M) = \mathrm{Hom}_R(K_i(\mathbf{t}, M)$ and define $\partial^i\colon K^i(\mathbf{t}, M) \to K^{i+1}(\mathbf{t}, M)$ to be the transpose of the differential $K_{i+1}(\mathbf{t}) \to K_i(\mathbf{t})$, without the intervention of any signs. Then $K^\bullet(\mathbf{t}, M)$ together with $\partial^\bullet$ is a cohomology complex, and this is what we will call the *Koszul (cohomology) complex* on M and $\mathbf{t}$. We write $K^\bullet(\mathbf{t})$ for $K^\bullet(\mathbf{t}, R)$. We refer the reader to [**C1**, pp. 17–18] for a discussion of various versions of Koszul complexes and the relationship between them. Here are three basic properties:

1) $K^\bullet(\mathbf{t}, M)$ is bounded by degrees 0 and r, with $K^0(\mathbf{t}, M) = K^r(\mathbf{t}, M) = M$.
2) $K^\bullet(\mathbf{t}, M) = M \otimes_R K^\bullet(\mathbf{t})$.
3) $K^i(\mathbf{t}, M)$ is the direct sum of $\binom{n}{i}$ copies of M.

The reason we use this version is the relationship with a certain Čech complex associated to an affine open cover of $\operatorname{Spec} R \smallsetminus Z$, where Z is the closed subscheme defined by the vanishing of the t_i's (see §C.5). The homology complex $K_\bullet(\mathbf{t})$ is also called a Koszul complex, and to distinguish it from $K^\bullet(\mathbf{t})$, we will call it the *Koszul homology complex* on $\mathbf{t}$.

There is a well known way in which these Koszul complexes vary with respect to $\mathbf{t}$. Let I be the ideal generated by $\mathbf{t}$. Let J be an ideal in R such that $I \subset J$, and such that J is generated by $\mathbf{g} = (g_1, \ldots, g_r)$. Since $I \subset J$ we have $u_{ij} \in R$ such that

$$t_i = \sum_{j=1}^{r} u_{ij} g_j \qquad (i = 1, \ldots, r).$$

As is well-known, one has a map of homology Koszul complexes

$$U_\bullet\colon K_\bullet(\mathbf{t}) \longrightarrow K_\bullet(\mathbf{g})$$

such that

– $H_0(U_\bullet)\colon R/I \to R/J$ is the natural surjection.

– $R = K_0(\mathbf{t}) \xrightarrow{U_0} K_0(\mathbf{g}) = R$ is the identity map on R.
– $R = K_n(\mathbf{t}) \xrightarrow{U_n} K_n(\mathbf{g}) = R$ is the map $x \mapsto \det(u_{ij}) \cdot x$.

Taking transposes and tensoring with M we get a map on (cohomology) Koszul complexes:

$$U^\bullet = U^\bullet_M \colon K^\bullet(\mathbf{g},\, M) \longrightarrow K^\bullet(\mathbf{t},\, M) \tag{C.1.1}$$

such that U^0 is the identity map on M and

$$U^n \colon M \to M \tag{C.1.2}$$

is the map $m \mapsto \det(u_{ij}) \cdot m$.

C.2. The Fundamental Local Isomorphism

With R as above, suppose $\mathbf{t} = (t_1, \dots, t_r)$ is an R-sequence, I the ideal generated by $\{t_1, \dots, t_r\}$, and $A = R/I$. Then

1) The ideal I is the image of the coboundary map from $K^{r-1}(\mathbf{t})$ to $K^r(\mathbf{t}) = R$, and the resulting map of complexes $K^\bullet(\mathbf{t}) \to A[-r]$ is a quasi-isomorphism. Thus we have an isomorphism in $\mathbf{D}(\mathrm{Mod}_R)$:

$$K^\bullet(\mathbf{t}) \xrightarrow{\sim} A[-r]. \tag{C.2.1}$$

Since $K^\bullet(\mathbf{t})$ is a (bounded) complex of free modules, for every complex $M^\bullet$ we have an isomorphism in $\mathbf{D}(\mathrm{Mod}_R)$

$$M^\bullet \otimes_R K^\bullet(\mathbf{t}) \xrightarrow{\sim} M^\bullet \overset{\mathbf{L}}{\otimes}_R (A[-r]) = \overline{M}^\bullet \overset{\mathbf{L}}{\otimes}_A (A[-r]) \tag{C.2.2}$$

where $\overline{M}^\bullet = M^\bullet \otimes_R A$.

2) For any R-module M, since $M = K^0(\mathbf{t},\, M)$, we have $\mathrm{Hom}_R(A,\, M) = \ker\left(K^0(\mathbf{t},\, M) \to K^1(\mathbf{t},\, M)\right)$ as $\mathrm{Hom}_R(A,\, M)$ identifies with the submodule of I-torsion elements of M namely $(0 \underset{M}{:} I)$ in the usual way (i.e., by "evaluation at 1"). We thus have a map of complexes $\mathrm{Hom}_R(A,\, M)[0] \to K^\bullet(\mathbf{t},\, M)$ which is a quasi-isomorphism if M is an *injective* R-module. It follows that, more generally, if $M^\bullet$ is a bounded-below complex, and $M^\bullet \to E^\bullet$ is an injective resolution with $E^\bullet$ bounded-below, then we have quasi-isomorphisms $M^\bullet \otimes_R K^\bullet(\mathbf{t}) \to E^\bullet \otimes_R K^\bullet(\mathbf{t})$ and $\mathrm{Hom}_R(A,\, E^\bullet) \to E^\bullet \otimes_R K^\bullet(\mathbf{t})$ so that in $\mathbf{D}(\mathrm{Mod}_R)$ we have an isomorphism

$$M^\bullet \otimes_R K^\bullet(\mathbf{t}) \xrightarrow{\sim} \mathbf{R}\mathrm{Hom}^\bullet_R(A,\, M^\bullet) \tag{C.2.3}$$

fitting into a commutative diagram in $\mathbf{D}(\mathrm{Mod}_R)$ as follows.

$$\begin{tikzcd} M^\bullet \otimes_R K^\bullet(\mathbf{t}) \arrow[r, "\sim"] \arrow[d, "\text{(C.2.3)}"', "\wr"] & E^\bullet \otimes_R K^\bullet(\mathbf{t}) \\ \mathbf{R}\mathrm{Hom}^\bullet_R(A,\, M^\bullet) \arrow[r, equal] & \mathrm{Hom}_R(A,\, E^\bullet) \arrow[u, "\wr"'] \end{tikzcd}$$

In particular we have an isomorphism

$$\psi_{\mathbf{t}} \colon M^\bullet \overset{\mathbf{L}}{\otimes}_R (A[-r]) \xrightarrow{\sim} \mathbf{R}\mathrm{Hom}^\bullet_R(A,\, M^\bullet)$$

where $\psi_{\mathbf{t}} = \text{(C.2.3)} \circ \text{(C.2.2)}^{-1}$.

3) Let $\frac{\mathbf{1}}{\mathbf{t}}$ (or $\mathbf{1}/\mathbf{t}$ for typographical convenience) be the element of $(\wedge_A^r I/I^2)^*$ defined in (3.5.3). Then $(\wedge_A^r I/I^2)^*$ is a free A module of rank one, with $\frac{\mathbf{1}}{\mathbf{t}}$ as a generator. One therefore has an isomorphism:

$$\lambda_{\mathbf{t}} \colon A \xrightarrow{\ \sim\ } (\wedge_A^r I/I^2)^*, \tag{C.2.4}$$

given by $1 \mapsto (-1)^r \mathbf{1}/\mathbf{t}$. The reason for the sign $(-1)^r$ will be clear later. We thus get an isomorphism,

$$\eta_{R,A}(M^\bullet) \colon M^\bullet \overset{\mathbf{L}}{\otimes}_R ((\wedge_A^r I/I^2)^*[-r]) \xrightarrow{\ \sim\ } \mathbf{R}\mathrm{Hom}_R^\bullet(A,\, M^\bullet) \tag{C.2.5}$$

with $\eta_{R,A} = \psi_{\mathbf{t}} \circ (\lambda_{\mathbf{t}}[-r])^{-1}$. The crucial property here is that $\eta_{R,A}$ *does not depend on* $\mathbf{t}$*, even though* $\psi_{\mathbf{t}}$ *and* $\lambda_{\mathbf{t}}$ *do.*

The data above fits into the following commutative diagram

$$\begin{tikzcd}
M^\bullet \overset{\mathbf{L}}{\otimes}_R (A[-r]) \arrow[d, "\wr", "\text{via } \lambda_{\mathbf{t}}"'] \arrow[dr, "\psi_{\mathbf{t}}"] & M^\bullet \otimes_R K^\bullet(\mathbf{t}) \arrow[l, "\sim"', "\text{(C.2.2)}"] \arrow[r, "\sim"] \arrow[d, "\wr", "\text{(C.2.3)}"'] & E^\bullet \otimes_R K^\bullet(\mathbf{t}) \\
M^\bullet \overset{\mathbf{L}}{\otimes}_R ((\wedge_A^r I/I^2)^*[-r]) \arrow[r, "\sim", "\eta_{R,A}"'] & \mathbf{R}\mathrm{Hom}_R^\bullet(A,\, M^\bullet) \arrow[r, equal] & \mathrm{Hom}_R(A,\, E^\bullet) \arrow[u, "\wr"']
\end{tikzcd} \tag{C.2.6}$$

Let M be an R-module. Our version of the *fundamental local isomorphism* is the isomorphism

$$\phi_{R,A}(M) \colon M \otimes_R (\wedge_A^r I/I^2)^* \xrightarrow{\ \sim\ } \mathrm{Ext}_R^r(A,\, M) \tag{C.2.7}$$

given by

$$\phi_{R,A}(M) = \mathrm{H}^0(\eta_{R,A}(M[r])).$$

Let us globalize this construction. Let $\mathscr{X}$ be a formal scheme, and $\mathscr{I}$ a coherent ideal sheaf such that the resulting closed immersion $i : \mathscr{Z} \hookrightarrow \mathscr{X}$ is a regular immersion of codimension r, i.e., it is given locally by a regular sequence of length r. Let us write $\mathscr{N}_i$ for the normal bundle of $\mathscr{Z}$ in $\mathscr{X}$, i.e. $\mathscr{N}_i = (\mathscr{I}/\mathscr{I}^2)^*$ and set

$$\mathscr{N}_i^r := \wedge^r \mathscr{N}_i = (\wedge_{\mathscr{O}_{\mathscr{Z}}}^r \mathscr{I}/\mathscr{I}^2)^*. \tag{C.2.8}$$

There is a natural isomorphism

$$\mathscr{N}_i^r = (\wedge_{\mathscr{O}_{\mathscr{Z}}}^r \mathscr{I}/\mathscr{I}^2)^* \xrightarrow{\ \sim\ } \bar{i}^* \mathscr{E}xt^r_{\mathscr{O}_{\mathscr{X}}}(\mathscr{O}_{\mathscr{Z}},\, \mathscr{O}_{\mathscr{X}}) = H^r i^\flat \mathscr{O}_{\mathscr{X}}$$

obtained by locally gluing the isomorphisms coming from (C.2.7) in view of the fact that $\eta_{R,A}$ is independent of the choice of $\mathbf{t}$. Since $i^\flat \mathscr{O}_{\mathscr{X}}$ has homology only in degree r as is obvious locally from (C.2.5), we obtain a natural isomorphism

$$\mathscr{N}_i^r[-r] \xrightarrow{\ \sim\ } i^\flat \mathscr{O}_{\mathscr{X}} = \bar{i}^* \mathbf{R}\mathscr{H}om^\bullet_{\mathscr{X}}(\mathscr{O}_{\mathscr{Z}},\, \mathscr{O}_{\mathscr{X}}). \tag{C.2.9}$$

Set

$$i^\blacktriangle := \mathbf{L}i^*(-) \overset{\mathbf{L}}{\otimes}_{\mathscr{O}_{\mathscr{Z}}} (\mathscr{N}_i^r[-r]). \tag{C.2.10}$$

Then for $\mathscr{F} \in \mathbf{D}_{\mathrm{qc}}(\mathscr{X})$ we have an isomorphism

$$\eta_i(\mathscr{F}) \colon i^\blacktriangle \mathscr{F} \xrightarrow{\ \sim\ } i^\flat \mathscr{F} \tag{C.2.11}$$

given by the composite

$$
\begin{aligned}
i^{\blacktriangle}\mathscr{F} = \mathbf{L}i^{*}(\mathscr{F}) \overset{\mathbf{L}}{\otimes}_{\mathscr{O}_{\mathscr{Z}}} (\mathscr{N}_i^{r}[-r]) &\xrightarrow{\ \sim\ } \mathbf{L}i^{*}\mathscr{F} \overset{\mathbf{L}}{\otimes}_{\mathscr{O}_{\mathscr{Z}}} \bar{i}^{*}\mathbf{R}\mathscr{H}om^{\bullet}_{\mathscr{X}}(\mathscr{O}_{\mathscr{Z}},\, \mathscr{O}_{\mathscr{X}}) \\
&\xrightarrow{\ \sim\ } \bar{i}^{*}(\mathscr{F} \overset{\mathbf{L}}{\otimes}_{\mathscr{O}_{\mathscr{X}}} \mathbf{R}\mathscr{H}om^{\bullet}_{\mathscr{X}}(\mathscr{O}_{\mathscr{Z}},\, \mathscr{O}_{\mathscr{X}})) \\
&\xrightarrow{\ \sim\ } \bar{i}^{*}\mathbf{R}\mathscr{H}om^{\bullet}_{\mathscr{X}}(\mathscr{O}_{\mathscr{Z}},\, \mathscr{F}) = i^{\flat}\mathscr{F}
\end{aligned} \tag{C.2.12}
$$

where the first isomorphism is given by (C.2.9) while the third one results from the fact that $i_*\mathscr{O}_{\mathscr{Z}}$ is coherent and has finite tor dimension over $\mathscr{O}_{\mathscr{X}}$.

For $\mathscr{F} \in \mathbf{D}^{+}_{\mathrm{c}}(\mathscr{X})$, let

$$
\eta_i'(\mathscr{F})\colon i^{\blacktriangle}\mathscr{F} \xrightarrow{\ \sim\ } i^{\#}\mathscr{F} \tag{C.2.13}
$$

be the composite $\eta_i' = (\text{B.1.2}) \circ \eta_i$.

Remark C.2.14. In the above, the isomorphism $i^{\blacktriangle}\mathscr{O}_{\mathscr{X}} \xrightarrow{\ \sim\ } i^{\flat}\mathscr{O}_{\mathscr{X}}$ in (C.2.9) is what drives the isomorphism (C.2.11). In slightly greater detail, for $\mathscr{F} \in \mathbf{D}^{+}_{\mathrm{c}}(\mathscr{X})$, we have (by definition of $i^{\blacktriangle}$):

$$
i^{\blacktriangle}\mathscr{F} = \mathbf{L}i^{*}(\mathscr{F}) \otimes_{\mathscr{O}_{\mathscr{Z}}} i^{\blacktriangle}(\mathscr{O}_{\mathscr{X}}).
$$

We also have an isomorphism (whose inverse is the composite of the last two maps in (C.2.12))

$$
\mathbf{L}i^{*}(\mathscr{F}) \otimes_{\mathscr{O}_{\mathscr{Z}}} i^{\flat}(\mathscr{O}_{\mathscr{X}}) \xrightarrow{\ \sim\ } i^{\flat}(\mathscr{F}).
$$

Applying $i^{\blacktriangle}\mathscr{O}_{\mathscr{X}} \xrightarrow{\ \sim\ } i^{\flat}\mathscr{O}_{\mathscr{X}}$ (from (C.2.9)) to the two isomorphisms above, we get $\eta_i(\mathscr{F})$.

The isomorphism $\mathbf{L}i^{*}(\mathscr{F}) \otimes_{\mathscr{O}_{\mathscr{Z}}} i^{\flat}(\mathscr{O}_{\mathscr{X}}) \xrightarrow{\ \sim\ } i^{\flat}(\mathscr{F})$ above is such that “evaluation at 1” is respected. In greater detail if $\mathrm{Tr}^{\flat}_i\colon i_*i^{\flat} \to \mathbf{1}$ is as in (B.1.1), then the composite $\mathscr{F} \otimes_{\mathscr{O}_{\mathscr{X}}} i_*i^{\flat}\mathscr{O}_{\mathscr{X}} \xrightarrow{\ \sim\ } i_*(\mathbf{L}i^{*}(\mathscr{F}) \otimes_{\mathscr{O}_{\mathscr{Z}}} i^{\flat}(\mathscr{O}_{\mathscr{X}})) \xrightarrow{\ \sim\ } i_*i^{\flat}(\mathscr{F}) \xrightarrow{\mathrm{Tr}^{\flat}_i} (\mathscr{F})$ is equal to $1 \otimes \mathrm{Tr}^{\flat}_i(\mathscr{O}_{\mathscr{X}})$. This means that if $\mathrm{Tr}^{\blacktriangle}_i\colon i_*i^{\blacktriangle} \to \mathbf{1}$ is defined by the formula

$$
\mathrm{Tr}^{\blacktriangle}_i = \mathrm{Tr}^{\flat}_i \circ i_*\eta_i,
$$

then the following diagram commutes

$$
\begin{tikzcd}
i_*(\mathbf{L}i^{*}\mathscr{F} \otimes_{\mathscr{O}_{\mathscr{Z}}} i^{\blacktriangle}\mathscr{O}_{\mathscr{X}}) \arrow[r, equal] & i_*i^{\blacktriangle} \arrow[dd, "\mathrm{Tr}^{\blacktriangle}_i(\mathscr{F})"] \\
\mathscr{F} \otimes_{\mathscr{O}_{\mathscr{X}}} i_*i^{\blacktriangle}\mathscr{O}_{\mathscr{X}} \arrow[u, "\text{projection formula}", "\wr"'] \arrow[d, "\mathbf{1}\otimes\mathrm{Tr}^{\blacktriangle}_i(\mathscr{O}_{\mathscr{X}})"'] & \\
\mathscr{F} \otimes_{\mathscr{O}_{\mathscr{X}}} \mathscr{O}_{\mathscr{X}} \arrow[r, equal] & \mathscr{F}
\end{tikzcd} \tag{C.2.14.1}
$$

C.2.15. If $\mathscr{X} = X$ is an ordinary scheme, so that $i^{\#} = i^{!}$, then the maps $\eta_i(\mathscr{F})$ and $\eta_i'(\mathscr{F})$ above can be extended to isomorphisms for $\mathscr{F} \in \mathbf{D}_{\mathrm{qc}}(X)$, without any boundedness hypotheses on $\mathscr{F}$. In greater detail, recall that a complex $\mathscr{F}$ of $\mathscr{O}_X$-modules is called *perfect* if there exist $a, b \in \mathbb{Z}$, $a \le b$, and locally $\mathscr{F}$ is $\mathbf{D}(X)$-isomorphic to a complex E of finite rank free $\mathscr{O}_X$-modules with $E^n = 0$ for $n \notin [a, b]$. The map i_* takes perfect complexes to perfect complexes (locally use appropriate Koszul complexes!). In other words i is a *quasi-perfect map* (see [**L4**, p. 192, Definition 4.7.2]). According to a result of Neeman in [**Ne1**] and Bondal and van den Bergh in [**BB**], since i_* takes perfect complexes to perfect complexes, one has a unique isomorphism (with $Z = \mathscr{X}$)

$$\mathbf{L}i^*(\mathscr{F}) \overset{\mathbf{L}}{\otimes}_{\mathscr{O}_Z} i^!\mathscr{O}_X \xrightarrow{\sim} i^!\mathscr{F}$$

such that Diagram (C.2.14.1) commutes with $i^\blacktriangle$ replaced by $i^!$, $\mathrm{Tr}_i^\blacktriangle$ by Tr_i, the equality on the top row by i_* of the isomorphism displayed above, and allowing $\mathscr{F}$ to vary $\mathbf{D}_{\mathrm{qc}}(X)$ rather than in $\mathbf{D}_{\mathrm{c}}^+(X)$. It is now clear that one can extend η_i' to an isomorphism of functors on $\mathbf{D}_{\mathrm{qc}}(X)$. As for η_i, see [**C1**, p. 53, (2.5.3)], keeping in mind the differing sign conventions for $K^\bullet(\mathbf{t})$ as well as the order of the tensor product. In fact the isomorphism $\bar{i}^*(\mathscr{F} \overset{\mathbf{L}}{\otimes}_{\mathscr{O}_X} \mathbf{R}\mathscr{H}om^\bullet_X(\mathscr{O}_Z,\, \mathscr{O}_X)) \xrightarrow{\sim} \bar{i}^*\mathbf{R}\mathscr{H}om^\bullet_X(\mathscr{O}_Z,\, \mathscr{F})$ in (C.2.12) works for $\mathscr{F} \in \mathbf{D}_{\mathrm{qc}}(X)$ when X is an ordinary scheme.

In view of (C.2.14.1), in order to understand $\mathrm{Tr}_i^\blacktriangle$ it is enough to understand $\mathrm{Tr}_i^\blacktriangle(\mathscr{O}_X)$. We give an explicit representation of $\mathrm{Tr}_i^\blacktriangle$ when $X = \operatorname{Spec} R$, $Z = \operatorname{Spec} A$, and the $I = \ker R \twoheadrightarrow A$ is generated by a quasi-regular sequence $\mathbf{t} = (t_1, \dots, t_r)$i, i.e., the situation we have been with for most of this section. Let $N = \Gamma(X,\, \mathscr{N}_i^r)$. In this case, the quasi-isomorphism of complexes of R-modules $K^\bullet(\mathbf{t}) \to A[-r]$ in (C.2.1), is the map defined by $K^r(\mathbf{t}) = R \xrightarrow{\text{natural}} R/I = (A[-r])^r$.

Using the isomorphism $A \xrightarrow{\sim} N$ given by $1 \mapsto \mathbf{1/t}$ we get a quasi-isomorphism

$$\varphi_{\mathbf{t}} \colon K^\bullet(\mathbf{t}) \longrightarrow N[-r],$$

where $\varphi_{\mathbf{t}}$ is defined by $\varphi_{\mathbf{t}}^r \colon K^r(\mathbf{t}) = R \to N = (N[-r])^r$, the arrow $R \to N$ being $1 \mapsto \mathbf{1/t}$. For a complex of R-modules $M^\bullet$, let $\mathrm{Tr}^\blacktriangle_{A/R}(M^\bullet) \colon M^\bullet \otimes_R N[-r] \to M^\bullet$ and $\mathrm{Tr}^\flat_{A/R}(M^\bullet) \colon \mathbf{R}\mathrm{Hom}^\bullet_R(A, M^\bullet) \to M^\bullet$ be the maps corresponding to $\mathrm{Tr}_i^\blacktriangle(\widetilde{M^\bullet})$ and $\mathrm{Tr}_i^\flat(\widetilde{M^\bullet})$. By definition of (C.2.11), we have a commutative diagram in the category $\mathbf{D}(\mathrm{Mod}_R)$ with *isomorphisms* bordering the triangle on the right:

$$\begin{tikzcd}
N[-r] \arrow[d, "\mathrm{Tr}^\blacktriangle_{A/R}(R)"'] \arrow[dr, "\eta_i"] & K^\bullet(\mathbf{t}) \arrow[l, "\varphi_{\mathbf{t}}"'] \arrow[d, "(\mathrm{C.2.3})"] \\
R & \mathbf{R}\mathrm{Hom}^\bullet_R(A,\, R) \arrow[l, "\mathrm{Tr}^\flat_{A/R}(R)"]
\end{tikzcd}$$

The composite $\mathrm{Tr}^\flat_{A/R}(R) \circ (\mathrm{C.2.3})$ is the natural projection

$$\pi_{\mathbf{t}} \colon K^\bullet(\mathbf{t}) \twoheadrightarrow K^0(\mathbf{t}) = R$$

which is a map of complexes, since $K^\bullet(\mathbf{t})$ has no negative terms. Thus

$$\mathrm{Tr}^\blacktriangle_{A/R}(R) = \pi_{\mathbf{t}} \circ \varphi_{\mathbf{t}}^{-1}. \tag{C.2.15.1}$$

C.3. Compatibility with completions

In view of the above, Lemma B.2.3 has a useful re-interpretation in the special case where the two closed immersions of Z into X and $\mathscr{X}$ are regular immersions of codimension r. In greater detail, suppose as in Section B.2, we have a commutative diagram

$$\begin{tikzcd}
Z \arrow[r, hook, "j"] \arrow[dr, hook, "i"'] & \mathscr{X} \arrow[d, "\kappa"] \\
& X
\end{tikzcd}$$

with X an ordinary scheme, but with i, j *regular closed immersions*, $\mathscr{X} = X_{/Z}$ the completion of X along Z, κ the completion map, and let $\mathscr{I}$ and $\mathscr{J} = \mathscr{I}\mathscr{O}_{\mathscr{X}}$ be the ideal sheaves for Z in X and $\mathscr{X}$ respectively. Now regarding $\mathscr{I}/\mathscr{I}^2$ and $\mathscr{J}/\mathscr{J}^2$ as invertible sheaves on Z, we have an obvious identification $\mathscr{I}/\mathscr{I}^2 = \mathscr{J}/\mathscr{J}^2$, whence the identification $j^{\blacktriangle}\kappa^* = i^{\blacktriangle}$. Then the following is an easy corollary to Lemma B.2.3.

Lemma C.3.1. *The following diagram commutes.*

$$\begin{tikzcd} j^{\blacktriangle}\kappa^* \arrow[rr, "\sim", "\eta'_j"'] \arrow[d, equal] & & j^{\#}\kappa^* \arrow[r, "\sim", "(2.1.3)"'] & j^{\#}\kappa^{\#} \arrow[d, "\wr"] \\ i^{\blacktriangle} \arrow[rrr, "\sim", "\eta'_i"'] & & & i^{\#} \end{tikzcd}$$

where the unlabelled isomorphism $j^{\#}\kappa^{\#} \xrightarrow{\sim} i^{\#}$ is the canonical one.

C.4. Flat base change of $-^{\blacktriangle}$ and of $-^{\#}$

Suppose we have a cartesian diagram $\mathfrak{s}$ of formal schemes

(C.4.1)
$$\begin{tikzcd} \mathscr{W}' \arrow[r, "j"] \arrow[d, "\kappa"'] \arrow[dr, phantom, "\square"] & \mathscr{W} \arrow[d, "\kappa_0"] \\ \mathscr{X}' \arrow[r, "i"'] & \mathscr{X} \end{tikzcd}$$

such that i is a regular immersion (i.e., given locally by the vanishing of a regular sequence) and κ_0 is the completion of $\mathscr{X}$ with respect to a closed subscheme given by a coherent ideal. By (C.2.13), for any $\mathscr{F} \in \mathbf{D}_{\mathrm{c}}^{+}(\mathscr{X})$ and $\mathscr{G} \in \mathbf{D}_{\mathrm{c}}^{+}(\mathscr{W})$ there are natural isomorphisms

$$i^{\blacktriangle}\mathscr{F} \xrightarrow{\sim} i^{\#}\mathscr{F}, \qquad j^{\blacktriangle}\mathscr{G} \xrightarrow{\sim} j^{\#}\mathscr{G}.$$

Now, on one hand we have the flat base-change isomorphism

$$\beta^{\#}_{\mathfrak{s}}\colon \kappa^* i^{\#} \xrightarrow{\sim} j^{\#}\kappa_0^*$$

of (2.2.2) while on the other we have an isomorphism

(C.4.2)
$$\kappa^* i^{\blacktriangle} \xrightarrow{\sim} j^{\blacktriangle}\kappa_0^*$$

given by the composite

$$\begin{aligned} \kappa^*((\mathbf{L}i^*(-) \overset{\mathbf{L}}{\otimes} \mathscr{N}_i^r[-r]) &\xrightarrow{\sim} (\mathbf{L}j^*\mathbf{L}\kappa_0^*(-)) \overset{\mathbf{L}}{\otimes} \kappa^*\mathscr{N}_i^r[-r] \\ &\xrightarrow{\sim} (\mathbf{L}j^*\kappa_0^*(-)) \overset{\mathbf{L}}{\otimes} \mathscr{N}_j^r[-r] \end{aligned}$$

where the second isomorphism is the one that arises from the canonical isomorphism $\kappa^*\mathscr{N}_i \xrightarrow{\sim} \mathscr{N}_j$. Fortunately these two flat-base-change isomorphisms are compatible:

Proposition C.4.3. *For the diagram $\mathfrak{s}$ in (C.4.1), for any $\mathscr{F} \in \mathbf{D}_{\mathrm{c}}^{+}(\mathscr{X})$ the following diagram commutes.*

(C.4.3.1)
$$\begin{tikzcd} \kappa^* i^{\blacktriangle}\mathscr{F} \arrow[r, "\sim", "(C.4.2)"'] \arrow[d, "\eta'_i"', "\wr"] & j^{\blacktriangle}\kappa_0^*\mathscr{F} \arrow[d, "\wr"', "\eta'_j"] \\ \kappa^* i^{\#}\mathscr{F} \arrow[r, "\beta^{\#}_{\mathfrak{s}}"'] & j^{\#}\kappa_0^*\mathscr{F} \end{tikzcd}$$

Proof. As per the definition of η' in (C.2.13), the diagram in (C.4.3.1) expands as follows

$$
\begin{tikzcd}
\kappa^* i^{\blacktriangle}\mathscr{F} \arrow[r, "\sim", "\text{(C.4.2)}"'] \arrow[d, "\eta_i"', "\wr"] & j^{\blacktriangle}\kappa_0^*\mathscr{F} \arrow[d, "\eta_j", "\wr"'] \\
\kappa^* i^{\flat}\mathscr{F} \arrow[r, "\beta^\flat"'] \arrow[d, "\text{(B.1.2)}"', "\wr"] & j^{\flat}\kappa_0^*\mathscr{F} \arrow[d, "\text{(B.1.2)}", "\wr"'] \\
\kappa^* i^{\#}\mathscr{F} \arrow[r, "\beta^\#_{\mathbf{s}}"'] & j^{\#}\kappa_0^*\mathscr{F}
\end{tikzcd}
$$

where $\beta^\flat$ is induced by the natural isomorphism

$$\kappa_0^*\mathbf{R}\mathscr{H}om^\bullet_{\mathscr{O}_{\mathscr{X}}}(i_*\mathscr{O}_{\mathscr{X}'}, \mathscr{F}) \xrightarrow{\ \sim\ } \mathbf{R}\mathscr{H}om^\bullet_{\mathscr{O}_{\mathscr{W}}}(j_*\mathscr{O}_{\mathscr{W}'}, \kappa_0^*\mathscr{F}). \tag{C.4.4}$$

It is straightforward to check that the top rectangle commutes. For the bottom one, using the adjointness property of $j^\#$, it suffices to check that the outer border of the following diagram commutes where, as before, $\varGamma'_{\mathscr{X}} = \mathbf{R}\varGamma'_{\mathscr{X}}$ etc..

$$
\begin{tikzcd}
j_*\varGamma'_{\mathscr{W}'}\kappa^* i^\flat \arrow[rrr, "\text{via } \beta^\flat"] \arrow[drr] \arrow[ddr, "a_1"] \arrow[ddd, "i^\flat \cong i^\#"'] & & & j_*\varGamma'_{\mathscr{W}} j^\flat \kappa_0^* \arrow[dd] \\
& & j_*\kappa^* i^\flat \arrow[d] \arrow[dr, "\text{via } \beta^\flat"] & \\
& \kappa_0^* i_*\varGamma'_{\mathscr{X}'} i^\flat \arrow[r] \arrow[d, "i^\flat \cong i^\#"'] \arrow[dr, phantom, "\boxminus"] & \kappa_0^* i_* i^\flat \arrow[r, "\text{(C.4.4)}"'] \arrow[dr, "\text{via } \mathrm{Tr}_i^\flat"'] & j_* j^\flat \kappa_0^* \arrow[d, "\text{via } \mathrm{Tr}_j^\flat"] \\
j_*\varGamma'_{\mathscr{W}'}\kappa^* i^\# \arrow[r, "a_2"'] & \kappa_0^* i_*\varGamma'_{\mathscr{X}'} i^\# \arrow[rr, "\text{via } \mathrm{Tr}_i"'] & & \kappa_0^*
\end{tikzcd}
$$

Here the maps a_i are induced by the composite of natural maps

$$j_*\mathbf{R}\varGamma'_{\mathscr{W}'}\kappa^* \to j_*\kappa^*\mathbf{R}\varGamma'_{\mathscr{X}'} \xrightarrow{\ \sim\ } \kappa_0^* i_*\varGamma'_{\mathscr{X}'}.$$

The unlabelled maps are the obvious natural ones. The diagram $\boxminus$ commutes by definition of the map $i^\flat \to i^\#$ in (B.1.2). Commutativity of the remaining parts is easy to check. $\square$

C.5. Stable Koszul complexes and generalized fractions

Let R, I, A be as above, and let $\mathbf{t} = (t_1, \dots, t_d)$ be generators for I. *Note that, for now, we are not requiring* $\mathbf{t}$ *to be a quasi-regular sequence.* We now recall the relationship between $K^\bullet(\mathbf{t}, M)$ and the local cohomology of M and relate the above discussion to generalized fractions leading to the explicit formula in Lemma C.5.4 below. For an r-tuple of positive integers $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_r)$, let $\mathbf{t}^{\boldsymbol{\alpha}} = (t_1^{\alpha_1}, \dots, t_r^{\alpha_r})$.

Let

$$
\begin{aligned}
K^{\bullet}_{\infty}(\mathbf{t}) &:= \varinjlim_{\boldsymbol{\alpha}} K^{\bullet}(\mathbf{t}^{\boldsymbol{\alpha}}) \\
K^{\bullet}_{\infty}(\mathbf{t},\, M) &:= \varinjlim_{\boldsymbol{\alpha}} K^{\bullet}(\mathbf{t}^{\boldsymbol{\alpha}},\, M) = M \otimes_R K^{\bullet}_{\infty}(\mathbf{t}).
\end{aligned}
\tag{C.5.1}
$$

The complex $K^{\bullet}_{\infty}(\mathbf{t},\, M)$ is called the *stable Koszul complex of M associated to* $\mathbf{t}$ and it has a well known relationship with the Čech complex $\mathrm{C}^{\bullet} = \mathrm{C}^{\bullet}(\mathfrak{U},\, \widetilde{M})$ associated with the open cover $\mathfrak{U} = \{\{t_i \neq 0\} \mid i = 1, \ldots, r\}$ of the scheme $U := \operatorname{Spec} R \smallsetminus V(I)$. The relationship is that $\mathrm{C}^i = K^{i+1}_{\infty}(\mathbf{t},\, M)$ for $i \geq 0$ and in this range and the coboundary maps $\mathrm{C}^i \to \mathrm{C}^{i+1}$ and $K^{i+1}_{\infty}(\mathbf{t},\, M) \to K^{i+2}_{\infty}(\mathbf{t},\, M)$ are equal. We also note that the natural map $K^r(\mathbf{t},\, M) \to K^r_{\infty}(\mathbf{t}, M) = \mathrm{C}^{r-1}$, is the map $M \to M_{t_1 \ldots t_r}$ given by $m \mapsto m/t_1 \ldots t_r$.

We point out that there is an obvious commutative diagram

$$
\begin{tikzcd}
\varinjlim_{\boldsymbol{\alpha}} \operatorname{Hom}_R(R/\mathbf{t}^{\boldsymbol{\alpha}},\, M) \arrow[r] \arrow[d, equal] & K^0_{\infty}(\mathbf{t},\, M) \arrow[d, equal] \\
\Gamma_I(M) \arrow[r, hook] & M
\end{tikzcd}
$$

where the horizontal arrow in the top row is the one obtained by applying a direct limit to the map of direct systems $\operatorname{Hom}_R(R/\mathbf{t}^{\boldsymbol{\alpha}},\, M) \to K^0(\mathbf{t}^{\boldsymbol{\alpha}},\, M)$ and the horizontal arrow in the bottom row is the natural inclusion. If, as before, $M \to E^{\bullet}$ is an injective resolution of M, we have $\operatorname{Hom}_R(R/\mathbf{t}^{\boldsymbol{\alpha}}R,\, E^{\bullet}) \xrightarrow{\ \sim\ } E^{\bullet} \otimes_R K^{\bullet}(\mathbf{t}^{\boldsymbol{\alpha}})$ whence an isomorphism

$$
\varinjlim_{\boldsymbol{\alpha}} \operatorname{Hom}_R(R/\mathbf{t}^{\boldsymbol{\alpha}}R,\, E^{\bullet}) \xrightarrow{\ \sim\ } E^{\bullet} \otimes_R K^{\bullet}_{\infty}(\mathbf{t}).
$$

We then we have a diagram of isomorphisms in $\mathbf{D}(\mathrm{Mod}_R)$:

$$
\begin{tikzcd}
K^{\bullet}_{\infty}(\mathbf{t},\, M) \arrow[r, equal] \arrow[d, dotted, "\wr"] & M^{\bullet} \otimes_R K^{\bullet}_{\infty}(\mathbf{t}) \arrow[d, "\wr"] \\
\mathbf{R}\Gamma_I(M) \arrow[d, equal] & E^{\bullet} \otimes_R K^{\bullet}_{\infty}(\mathbf{t}) \\
\Gamma_I(E^{\bullet}) \arrow[r, equal] & \varinjlim_{\boldsymbol{\alpha}} \operatorname{Hom}_R(R/\mathbf{t}^{\boldsymbol{\alpha}}R,\, E^{\bullet}) \arrow[u, "\wr"']
\end{tikzcd}
$$

Since all solid arrows in this diagram are isomorphisms, we can fill the dotted arrow, i.e., we have a unique isomorphism

$$
K^{\bullet}_{\infty}(\mathbf{t},\, M) \xrightarrow{\ \sim\ } \mathbf{R}\Gamma_I(M)
\tag{C.5.2}
$$

which fills the dotted arrow to make the diagram commute. Since $K^j_{\infty}(\mathbf{t},\, M) = 0$ for $j > r$, we have a surjective map $M_{t_1 \ldots t_r} = K^r_{\infty}(\mathbf{t},\, M) \to \mathrm{H}^r_I(M)$. The image of $m/t_1^{\alpha_1} \ldots t_r^{\alpha_r} \in M_{t_1 \ldots t_r}$ is denoted by the generalized fraction $\left\{ \genfrac{}{}{0pt}{}{m}{t_1^{\alpha_1}, \ldots, t_r^{\alpha_r}} \right\}$.

Now, standard excision arguments give us a map $\mathrm{H}^{r-1}(U,\, \widetilde{M}) \to H^r_I(M)$ which is an isomorphism when $r \geq 2$ and surjective when $r = 1$. In the Čech complex $\mathrm{C}^{\bullet}$, we have $\mathrm{C}^j = 0$ for $j \geq r$. We thus have a composition of surjective maps

$$
M_{t_1 \ldots t_r} = \mathrm{C}^{r-1} \twoheadrightarrow \mathrm{H}^{r-1}(\mathfrak{U},\, \widetilde{M}) \xrightarrow{\ \sim\ } \mathrm{H}^{r-1}(U,\, \widetilde{M}) \twoheadrightarrow \mathrm{H}^r_I(M).
$$

The image of $\frac{m}{t_1^{\alpha_1}\dots t_r^{\alpha_r}} \in M_{t_1\dots t_r}$ is denoted by the generalized fraction $\left[\genfrac{}{}{0pt}{}{m}{t_1^{\alpha_1},\dots,t_r^{\alpha_r}}\right]$. The two generalized fractions are related by the formula

$$\left[\begin{matrix} m \\ t_1^{\alpha_1},\,\dots,\,t_r^{\alpha_r}\end{matrix}\right] = (-1)^r \left\{\begin{matrix} m \\ t_1^{\alpha_1},\,\dots,\,t_r^{\alpha_r}\end{matrix}\right\} \tag{C.5.3}$$

(see [**LNS**, p.47, Lemma 4.1.1]).

Lemma C.5.4. *Suppose the sequence* $\mathbf{t}$ *above is a quasi-regular sequence in* R. *Let* M *be an* R*-module. Then the composite map (with* $\phi_{R,A}(M)$ *as in* (C.2.7)*)*

$$M \otimes_R (\wedge_A^r I/I^2)^* \xrightarrow[\phi_{R,A}(M)]{\sim} \mathrm{Ext}_R^r(A,\,M) \longrightarrow \mathrm{H}_I^r(M) \tag{C.5.4.1}$$

is given by

$$m \otimes \frac{\mathbf{1}}{\mathbf{t}} \mapsto \left[\begin{matrix} m \\ t_1,\,\dots,\,t_r\end{matrix}\right] \qquad (m \in M)$$

Where $\frac{\mathbf{1}}{\mathbf{t}}$ *is as in* (3.5.3).

Proof. For an arbitrary bounded complex $M^\bullet$ consider the following commutative diagram

$$\begin{tikzcd}
M^\bullet \overset{\mathbf{L}}{\otimes}_R (A[-r]) \arrow[ddd, "\lambda_{\mathbf{t}}[-r]"', "\wr"] & M^\bullet \otimes_R K^\bullet(\mathbf{t}) \arrow[l, "\sim"'] \arrow[r] \arrow[d, "\wr"] & M^\bullet \otimes_R K_\infty^\bullet(\mathbf{t}) \arrow[d, "\wr"] \\
 & E^\bullet \otimes_R K^\bullet(\mathbf{t}) \arrow[r] & E^\bullet \otimes_R K_\infty^\bullet(\mathbf{t}) \\
 & \mathrm{Hom}_R(A,\,E^\bullet) \arrow[r] \arrow[u, "\wr"] \arrow[d, equal] & \Gamma_I(E^\bullet) \arrow[u, "\wr"] \arrow[d, equal] \\
M^\bullet \overset{\mathbf{L}}{\otimes}_R ((\wedge_A^r I/I^2)^*[-r]) \arrow[r, "\sim", "\eta_{R,A}"'] & \mathbf{R}\mathrm{Hom}_R^\bullet(A,\,M^\bullet) \arrow[r] & \mathbf{R}\Gamma_I(M^\bullet)
\end{tikzcd} \tag{C.5.4.2}$$

Set $M^\bullet = M[r]$ in the above and apply the cohomology functor $\mathrm{H}^0(-)$. We get a commutative diagram

$$\begin{tikzcd}
M \otimes_R A \arrow[r, equal] \arrow[d, "\lambda_{\mathbf{t}}"', "\wr"] & M/IM \arrow[r] & \mathrm{H}^r(M \otimes_R \mathbf{K}_\infty^\bullet(\mathbf{t})) \arrow[d, "\wr"'] \\
M \otimes_R (\wedge_A^r I/I^2)^* \arrow[r, "\sim", "\phi_{R,A}(M)"'] & \mathrm{Ext}_R^r(A,\,M) \arrow[r, "\text{natural}"] & \mathrm{H}_I^r(M)
\end{tikzcd}$$

Let us write $[x]$ for the image of $x \in M_{t_1\dots t_r} = M \otimes_R K_\infty^r(\mathbf{t})$ in the module $\mathrm{H}^r(M \otimes_R K_\infty^\bullet(\mathbf{t}))$. Then chasing an element $m \otimes (\mathbf{1}/\mathbf{t}) \in M \otimes_R (\wedge_A^r I/I^2)^*$ by first going north (via $\lambda_{\mathbf{t}}^{-1}$) and then east along the above rectangle, we arrive at the element $(-1)^r[m/t_1\dots t_r] \in \mathrm{H}^r(M \otimes_R K_\infty^\bullet(\mathbf{t}))$. The assertion follows from (C.5.3). $\square$

C.6. Duality for composite for closed immersions

Let R be a noetherian ring, $I \subset R$ an ideal, $A = R/I$ and $i\colon \operatorname{Spec} A \hookrightarrow \operatorname{Spec} R$ the closed immersion corresponding to the natural surjection $R \twoheadrightarrow R/I = A$. Let $M^\bullet$ be a bounded below complex of A-modules. Consider the "evaluation at 1" map:

$$\mathbf{ev}_I\colon \mathbf{R}\mathrm{Hom}^\bullet_R(A,\, M^\bullet) \longrightarrow M^\bullet.$$

As is well-known (and easy to verify from the definitions) the following diagram commutes

$$\begin{tikzcd} i_*i^\flat\widetilde{M^\bullet} \arrow[r, "\sim", "\text{(B.1.2)}"'] \arrow[dr, bend right, "\mathbf{ev}_I"'] & i_*i^!\widetilde{M^\bullet} \arrow[d, "\mathrm{Tr}_i"] \\ & \widetilde{M^\bullet} \end{tikzcd}$$

Now suppose $\bar{L} \subset A$ is an ideal, and $L \subset R$ the unique R-ideal such that $L \supset I$ and $L/I = \bar{L}$. We then have the standard isomorphism

$$\mathbf{R}\mathrm{Hom}^\bullet_A(B,\, \mathbf{R}\mathrm{Hom}^\bullet_R(A,\, M^\bullet)) \xrightarrow{\ \sim\ } \mathbf{R}\mathrm{Hom}^\bullet_R(B,\, M^\bullet) \tag{C.6.1}$$

which, after replacing $M^\bullet$ by a complex of injective modules if necessary, amounts to the observation that elements in an R-module which are killed by I and also by $\bar{L}$ are the exactly elements which are killed by L. The following diagram clearly commutes

$$\begin{tikzcd} \mathbf{R}\mathrm{Hom}^\bullet_A(B,\, \mathbf{R}\mathrm{Hom}^\bullet_R(A,\, M^\bullet)) \arrow[r, "\sim"] \arrow[d, "\mathbf{ev}_{\bar{L}}"'] & \mathbf{R}\mathrm{Hom}^\bullet_R(B,\, M^\bullet) \arrow[d, "\mathbf{ev}_L"] \\ \mathbf{R}\mathrm{Hom}^\bullet_R(A,\, M^\bullet) \arrow[r, "\mathbf{ev}_I"'] & M^\bullet \end{tikzcd}$$

This means that the following diagram commutes (with $j\colon \operatorname{Spec} B \to \operatorname{Spec} A$ the natural inclusion):

$$\begin{tikzcd} j^\flat i^\flat\widetilde{M^\bullet} \arrow[r, "\sim", "\text{(C.6.1)}"'] \arrow[d, "\text{(B.1.2)}"', "\wr"] & (ij)^\flat\widetilde{M^\bullet} \arrow[d, "\wr"', "\text{(B.1.2)}"] \\ j^!i^!\widetilde{M^\bullet} \arrow[r, "\sim", "\text{natural}"'] & (ij)^!\widetilde{M^\bullet} \end{tikzcd} \tag{C.6.2}$$

Suppose I is generated by $\mathbf{t} = (t_1, \dots, t_d)$, $\bar{L}$ is generated by $\bar{\mathbf{u}} = (\bar{u}_1,\, \dots,\, \bar{u}_e)$, and $u_i \in L$ are lifts of $\bar{u}_i$ for $i = 1, \dots, e$. Set $\mathbf{u} = (u_1,\, \dots,\, u_e)$. Suppose $(\mathbf{t},\, \mathbf{u})$ is a quasi-regular sequence in R (so that $\bar{\mathbf{u}}$ is quasi-regular in A). The map

$$\mathbf{e}_{\mathbf{t}}\colon M^\bullet \otimes_R K^\bullet(\mathbf{t}) \longrightarrow M^\bullet \tag{C.6.3}$$

corresponding to $\mathbf{ev}_I$ under the isomorphism $M^\bullet \otimes K^\bullet(\mathbf{t}) \xrightarrow{\ \sim\ } \mathbf{R}\mathrm{Hom}^\bullet_R(A,\, M^\bullet)$ (cf. (C.2.3))) is the map which in degree n is

$$(M^\bullet \otimes_R K^\bullet(\mathbf{t}))^n = \bigoplus_{p+q=n} M^p \otimes_R K^q(\mathbf{t}) \xrightarrow{\text{projection}} M^n \otimes_R K^0(\mathbf{t}) = M^n. \tag{C.6.4}$$

(Note that the map $\mathbf{e_t}$ is defined even if $\mathbf{t}$ is not quasi-regular, so that in particular $\mathbf{e_u}$ makes sense.) The following diagram clearly commutes

$$\begin{tikzcd}
M^\bullet \otimes_R K^\bullet(\mathbf{t}) \otimes_R K^\bullet(\mathbf{u}) \arrow[r, equal] \arrow[d, "\mathbf{e_u}"'] & M^\bullet \otimes_R K^\bullet(\mathbf{t}, \mathbf{u}) \arrow[d, "\mathbf{e}_{(\mathbf{t},\mathbf{u})}"] \\
M^\bullet \otimes_R K^\bullet(\mathbf{t}) \arrow[r, "\mathbf{e_u}"'] & M^\bullet
\end{tikzcd}$$

An obvious re-interpretation of this, in our case, is that the following diagram commutes (we are implicitly using the fact that if N is an A-module, then $\mathbf{e_u} = \mathbf{e_{\bar{u}}}$ on $N \otimes_R K^\bullet(\mathbf{u}) = N \otimes_A K^\bullet(\bar{\mathbf{u}})$):

$$\begin{tikzcd}
M^\bullet \otimes_R K^\bullet(\mathbf{t}, \mathbf{u}) \arrow[r, "\sim", "(C.2.3)"'] \arrow[d, equal] & \mathbf{R}\mathrm{Hom}^\bullet_R(B,\, M^\bullet) \\
M^\bullet \otimes_R K^\bullet(\mathbf{t}) \otimes_R K^\bullet(\mathbf{u}) \arrow[d, "(C.2.3)"', "\wr"] & \\
\mathbf{R}\mathrm{Hom}^\bullet_R(A,\, M^\bullet) \otimes_R K^\bullet(\mathbf{u}) \arrow[d, equal] & \\
\mathbf{R}\mathrm{Hom}^\bullet_R(A,\, M^\bullet) \otimes_A K^\bullet(\bar{\mathbf{u}}) \arrow[r, "\sim", "(C.2.3)"'] & \mathbf{R}\mathrm{Hom}^\bullet_A(B,\, \mathbf{R}\mathrm{Hom}^\bullet_R(A,\, M^\bullet)) \arrow[uuu, "\wr"', "(C.6.1)"]
\end{tikzcd} \tag{C.6.5}$$

Setting $n = d + e$ we have an isomorphism of rank one free A-modules

$$\alpha\colon (\wedge^d_R I/I^2)^* \otimes_R (\wedge^e_A \bar{L}/\bar{L}^2)^* \xrightarrow{\ \sim\ } (\wedge^n_A L/L^2)^*$$

given by $1/\mathbf{t} \otimes 1/\bar{\mathbf{u}} \mapsto 1/(\mathbf{t}, \mathbf{u})$.

The role of the hyptheses on the $\mathrm{Tor}(-, \bullet)$ functors in the statement of Proposition C.6.6 below is the following: Suppose S is a ring, P, Q S-modules such that $\mathrm{Tor}^S_i(P, Q) = 0$ for $i \neq 0$. Then $P \overset{\mathbf{L}}{\otimes}_S Q$ is canonically isomorphic to $P \otimes_S Q$ and we treat this as an identity, i.e., in this case we write $P \overset{\mathbf{L}}{\otimes}_S Q = P \otimes_S Q$. In particular if J is an S-ideal generated by a quasi-regular sequence, and $\mathrm{Tor}^S_i(P, S/J) = 0$, then we have $P \overset{\mathbf{L}}{\otimes}_S (\wedge^m_{S/J} J/J^2)^* = P \otimes_S (\wedge^m_{S/J} J/J^2)^* = (P \otimes_S S/J) \otimes_{S/J} (\wedge^m_{S/J} J/J^2)^*$. In other words, if $\mathscr{G} = \widetilde{P}$, the quasi-coherent sheaf on $W = \operatorname{Spec} S$ corresponding to P, and $u\colon Z = \operatorname{Spec} S/J \hookrightarrow W$ the natural closed immersion, we have

$$\begin{aligned}
u^\blacktriangle \mathscr{G}[m] &= \mathbf{L}u^*\mathscr{G}[m] \otimes_{\mathscr{O}_Z} (\mathscr{N}^m[-m]) \\
&= \mathbf{L}u^*\mathscr{G} \otimes_{\mathscr{O}_Z} (\mathscr{N}^m[-m])[m] \\
&= (\mathbf{L}u^*\mathscr{G} \otimes_{\mathscr{O}_Z} \mathscr{N}^m)[0] \\
&= (u^*\mathscr{G} \otimes_{\mathscr{O}_Z} \mathscr{N}^m_u)[0].
\end{aligned}$$

Proposition C.6.6. *Let R, A, B, I, L, $\bar{L}$, $\mathbf{t}$, $\mathbf{u}$, i, j, be as above with $(\mathbf{t}, \mathbf{u})$ being a quasi-regular sequence in R. Let M be an R-module, $\mathscr{F} = \widetilde{M}$, the quasi-coherent $\mathscr{O}_{\operatorname{Spec} R}$-module corresponding to M. Suppose we have $\mathrm{Tor}^R_i(M, A) =$*

$\mathrm{Tor}_i^R(M,B) = \mathrm{Tor}_i^A(M/IM,B) = 0$ *for* $i \neq 0$. *Then the following diagram commutes*

$$\begin{tikzcd}
(j^*i^*(\mathscr{F}) \otimes \mathscr{N}_{ij}^n)[0] \arrow[r, equal] & (ij)^{\blacktriangle}(\mathscr{F}[n]) \arrow[r, "\sim", "\eta'_{ij}"'] & (ij)^!(\mathscr{F}[n]) \\
(j^*i^*(\mathscr{F}) \otimes j^*\mathscr{N}_i^d \otimes \mathscr{N}_j^e)[0] \arrow[u, "\text{via }\alpha", "\wr"'] \arrow[r, equal] & j^{\blacktriangle}(i^{\blacktriangle}(\mathscr{F}[d])[e]) \arrow[d, "\text{via }\eta'_j\text{ and }\eta'_i"', "\wr"] & \\
 & j^!(i^!(\mathscr{F}[d])[e]) \arrow[r, "\sim"] & j^!i^!(\mathscr{F}[n]) \arrow[uu, "\wr"']
\end{tikzcd}$$

Proof. For any noetherian ring S and S-ideal J generated by a quasi-regular sequence $\mathbf{v} = (v_1, \dots, v_m)$, and every S-module P, we have, with $\overline{S} = S/J$, a map of complexes

$$w_{S,\mathbf{v}} = w_{S,\mathbf{v},P} \colon P[m] \otimes_S K^\bullet(\mathbf{v}) \to P \otimes_S \wedge_{\overline{S}}^m (J/J^2)^*[0]$$

defined on 0-cochains by

$$x \mapsto (-1)^m x \otimes 1/\mathbf{v}, \qquad x \in P = (P[m] \otimes_S K^\bullet(\mathbf{v}))^0.$$

Since $P[m] \otimes_S K^\bullet(\mathbf{v})$ is a complex which is zero in positive degrees and the complex $P \otimes_S \wedge_{\overline{S}}^m (J/J^2)^*[0]$ is concentrated in degree 0, the above recipe defines $w_{S,\mathbf{v}}$. Moreover, if $\mathrm{Tor}_S^j(P, \overline{S}) = 0$ for $j \neq 0$, then in $\mathbf{D}(\mathrm{Mod}_S)$ the image of the map $w_{S,\mathbf{v}}$ under the localization functor is the composite

$$\begin{aligned}
P[m] \otimes_S K^\bullet(\mathbf{v}) \;\xrightarrow{\;\sim\;}\; P[m] \overset{\mathbf{L}}{\otimes}_S \overline{S}[-m] \;&\xrightarrow{\;\sim\;}\; P[m] \overset{\mathbf{L}}{\otimes}_S (\wedge_{\overline{S}}^m (J/J^2)^*[-m]) \\
&= \;\; P[m] \otimes_S (\wedge_{\overline{S}}^m (J/J^2)^*[-m]) \\
&= \;\; P \otimes_S (\wedge_{\overline{S}}^m (J/J^2)^*)[0]
\end{aligned}$$

where the first arrow is (C.2.1) and the second arrow $1 \otimes \lambda_{\mathbf{v}}[-m]$. In other words the following diagram in $\mathbf{D}(\mathrm{Mod}_S)$, consisting of isomorphisms, commutes :

$$\begin{tikzcd}
 & P[m] \otimes_S K^\bullet(\mathbf{v}) \arrow[dl, "w_{S,\mathbf{v}}"', "\sim"] \arrow[dr, "(\mathrm{C.2.3})", "\sim"'] & \\
P \otimes_S (\wedge_{\overline{S}}^m (J/J^2)^*)[0] \arrow[rr, "\sim", "\eta_{S,\overline{S}}"'] & & \mathbf{R}\mathrm{Hom}_S(\overline{S},\, P[m])
\end{tikzcd}$$

(see (C.2.5) for the definition of $\eta_{S,\overline{S}}$). In view of these observations, as well the commutativity of (C.6.2) and (C.6.5), we are done if we show that the following diagram commutes where for convenience we use N to denote $\wedge^d_A(I/I^2)^*$.

$$\begin{tikzcd}
M[n]\otimes_R K^\bullet(\mathbf{t},\mathbf{u}) \arrow[r, "w_{R,(\mathbf{t},\mathbf{u})}"] \arrow[d, equal] & M\otimes_R(\wedge^n_B L/L^2)^*[0] \\
(M[d]\otimes_R K^\bullet(\mathbf{t}))[e]\otimes_R K^\bullet(\mathbf{u}) \arrow[d, "w_{R,\mathbf{t}}"', "\wr"] & \\
(M\otimes_R N)[e]\otimes_A K^\bullet(\bar{\mathbf{u}}) \arrow[r, "w_{A,\bar{\mathbf{u}}}"'] & M\otimes_R N\otimes_A(\wedge^e_B \bar{L}/\bar{L}^2)^*[0] \arrow[uu, "(1\otimes\alpha)[0]"', "\wr"]
\end{tikzcd}$$

Indeed, we only have to check on 0-cochains as we argued earlier. Let $m\in M$ be an element. Regard it as a 0-cochain of the complex on the northwest corner. Its image in $M\otimes_R(\wedge^d_A I/I^2)^*\otimes_A(\wedge^e_B\bar{L}/\bar{L}^2)^*$ in the southeast corner under the composite $w_{A,\,\bar{\mathbf{u}}}\circ w_{R,\,\mathbf{t}}$ is $(-1)^n m\otimes 1/\mathbf{t}\otimes 1/\bar{\mathbf{u}}$, and its image in $M\otimes_R(\wedge^n_B L/L^2)^*$ in the northeast corner (via $w_{R,(\mathbf{t},\mathbf{u})}$) is $(-1)^n m\otimes 1/(\mathbf{t},\mathbf{u})$. This proves our assertion. $\square$

C.7. Another look at the composition of closed immersions

This is a slightly different but related exploration of duality for compositions of closed immersions. So, as before, suppose R is a noetherian ring, I, J ideals in R, $I\subset J$, I (resp. J) generated by a regular sequence $\{t_1,\ldots,t_r\}$ (resp. $\{g_1,\ldots,g_r\}$). Note that the number of t's equals the number of g's.

In this set-up, let $t_i=\sum_j u_{ij}g_j$, $A=R/I$, $B=R/J=A/\overline{J}$, where $\overline{J}=JA$. Let $i\colon \operatorname{Spec}A\hookrightarrow\operatorname{Spec}R$, $j\colon\operatorname{Spec}B\hookrightarrow\operatorname{Spec}R$, and $h\colon\operatorname{Spec}B\hookrightarrow\operatorname{Spec}A$ be the closed immersions corresponding respectively to the surjections $R\twoheadrightarrow A$, $R\twoheadrightarrow B$, and $A\twoheadrightarrow B$. Then $i\circ h=j$. We have a composite

$$\phi^!_h\colon h_*j^!\ \xrightarrow{\ \sim\ }\ h_*h^!i^!\xrightarrow{\mathrm{Tr}_h} i^!.$$

Using (C.6.2) and (C.2.13) (the latter for i and j), this corresponds to a map

$$\phi^\blacktriangle_h\colon h_*j^\blacktriangle\to i^\blacktriangle.$$

In particular, for an R-module M, $H^n(\phi^\blacktriangle_h(M))$ gives us a map

$$\phi_h\colon M\otimes_R(\wedge^r_B J/J^2)^*\longrightarrow M\otimes_R(\wedge^r_A I/I^2)^*. \tag{C.7.1}$$

From the definition of $U^\bullet(M)$ in (C.1.1), it is straightforward that for an R-module M, the following diagram commutes.

$$\begin{tikzcd}
\mathbf{R}\mathrm{Hom}^\bullet_R(A,\,M) \arrow[d, "(C.2.3)"'] & \mathbf{R}\mathrm{Hom}^\bullet_R(B,\,M) \arrow[l] \arrow[d, "(C.2.3)"] \\
M\otimes_R K^\bullet(\mathbf{t}) & M\otimes_R K^\bullet(\mathbf{g}) \arrow[l, "U^\bullet"]
\end{tikzcd}$$

commutes. The unlabelled arrow is the one arising from the map $A\twoheadrightarrow B$. Unwinding all the definitions, we see that $\phi_h(m\otimes\mathbf{1/g})=\det(u_{ij})m\otimes\mathbf{1/t}$. Moreover if $\mathrm{H}^r_J(M)\to\mathrm{H}^r_I(M)$ is the natural map arising from the inclusion $\Gamma_J\hookrightarrow\Gamma_I$, the element $\left[{m\atop g_1,\ldots,g_r}\right]$ maps to $\left[{\det(u_{ij})m\atop t_1,\ldots,t_r}\right]$.

These are well-known results (for example, see [**HK1**, §3, pp.71–72] or [**L2**, Chap. III, §7, pp.59–60]). We record them for completeness. It should be pointed out that (ii) and (iii) in Theorem C.7.2 do not need $\mathbf{g}$ or $\mathbf{t}$ to be regular sequences.

THEOREM C.7.2. *Let R, I, J, $\mathbf{t}$, $\mathbf{g}$ and u_{ij} be as above.*

(i) *Let $\phi_h \colon M \otimes (\wedge_B^r J/J^2)^* \to M \otimes (\wedge_A^r I/I^2)^*$ be as in (C.7.1). Then*

$$\phi_h\left(m \otimes \frac{\mathbf{1}}{\mathbf{g}}\right) = \det(u_{ij})\, m \otimes \frac{\mathbf{1}}{\mathbf{t}}.$$

(ii) *If $\psi \colon \mathrm{H}_J^r(M) \to \mathrm{H}_I^r(M)$ is the natural map arising from the inclusion $\Gamma_J \hookrightarrow \Gamma_I$, then*

$$\psi\left(\begin{bmatrix} m \\ g_1, \dots, g_r \end{bmatrix}\right) = \begin{bmatrix} \det(u_{ij})m \\ t_1, \dots, t_r \end{bmatrix}$$

(iii) *If $\sqrt{I} = \sqrt{J}$, so that $\mathrm{H}_I^r(M) = \mathrm{H}_J^r(M)$, then*

$$\begin{bmatrix} m \\ g_1, \dots, g_r \end{bmatrix} = \begin{bmatrix} \det(u_{ij})m \\ t_1, \dots, t_r \end{bmatrix}$$

PROOF. Part (i) has been established. We elaborate a bit on (ii) and (ii). Let $Z = \operatorname{Spec} A$ and $W = \operatorname{Spec} B$. The natural maps $i_*i^\flat \to \mathbf{R}\Gamma_Z$ and $j_*j^\flat \to \mathbf{R}\Gamma_W$ fit into a commutative diagram, as the reader can readily verify:

$$\begin{tikzcd}
j_*j^\flat \arrow[r, "\sim"] \arrow[dd] & i_*h_*h^\flat i^\flat \arrow[dr, "\text{via } \mathrm{Tr}_h^\flat"] & \\
& & i_*i^\flat \arrow[dl, "\mathrm{Tr}_i^\flat"] \\
\mathbf{R}\Gamma_W \arrow[r, hook] & \mathbf{R}\Gamma_Z &
\end{tikzcd}$$

Here $\mathrm{Tr}_i^\flat$ is the map in (B.1.1). Parts (ii) and (iii) then follow from (i). $\square$

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