LECTURES 9 AND 10

Dates of Lectures: September 10 and 12, 2019

For any ring A, $\mathrm{Max}(A)$ is the collection of maximal ideals of A. A^* is the multiplicative group of units of A.

As before $\mathbf{N} = \{0, 1, 2, \ldots, m, \ldots\}$. Rings mean commutative rings with 1.

The symbol ☡ is for flagging a cautionary comment or a tricky argument. It occurs in the margins and is Knuth's version of Bourbaki's "dangerous bend symbol".

1. B-rings

For this section K is a non-archimedean field, *not necessarily complete* and $k = \widetilde{K}$, i.e., $k = \mathscr{O}_K/\mathfrak{m}_K$. Moreover (again, only for this section) we do allow the absolute value on K to be trivial.

We fix a subring A of K.

1.1. We set
$$\partial A = \Big\{ a \in A \,\Big|\, |a| = 1 \Big\}.$$
Note that $\partial \mathscr{O}_K = \mathscr{O}_K^*$. However it is not always true that $\partial A = A^*$.

Definitions 1.1.1. Let A^*, ∂A etc., be as above.

1. A is said to be *bounded* if $A \subset \mathscr{O}_K$. Equivalently, A is bounded if $|x| \le 1$ for every $x \in A$.
2. A is called a *B-ring* if A is bounded and $A^* = \partial A$.
3. A B-subring of a B-ring is a subring which is also a B-ring.

1.1.2. The intersections of B-subrings of a B-ring is also a B-ring. It therefore makes sense to talk about the smallest B-subring containing a subset of K, provided there is at least one B-subring containing it.

Proposition 1.1.3. *If $A \subset \mathscr{O}_K$ then the localisation $A_{\partial A}$ is the smallest B-ring containing A. In this case A and ∂A have the same value semi-group, i.e. $|A| = |A_{\partial A}|$.*

Proof. This is obvious. ∎

1.1.4. For our ring A, set
$$\check{A} = A \cap \mathfrak{m}_K.$$
In other words $\check{A} = \{x \in A \mid |x| < 1\}$.

If A is a B-ring we prefer the notation $\mathfrak{m}_A$ for $\check{A}$. Thus
$$\mathfrak{m}_A = A \cap \mathfrak{m}_K$$
when A is a B-ring.

Suppose A is a B-ring. Then it is clear that $\mathfrak{m}_A$ is a maximal ideal and in fact the unique maximal ideal of A. Thus $(A, \mathfrak{m}_A)$ is a local ring. We set
$$\widetilde{A} = A/\mathfrak{m}_A$$

and for $x \in A$, $\tilde{x} \in \widetilde{A}$ is defined to be the image of x under the canonical surjection $A \twoheadrightarrow \widetilde{A}$. The notation $\check{A}$ is also used for $\mathfrak{m}_A$, but we will not use it now. It is a useful notation when A is not a B-ring, for in that case, elements of absolute value less than 1 do not form a unique maximal ideal.

Proposition 1.1.5. *If A is a B-ring, then its completion $\widehat{A}$ is a B-ring in $\widehat{K}$.*

Cautionary Remark: The ring $\widehat{A}$ above is the completion of A as a normed space, and not necessarily as a local ring. In other words $\widehat{A}$ need not equal $\varprojlim_n A/\mathfrak{m}_A^n$. If A is noetherian then $A = \varprojlim_n A/\mathfrak{m}_A^n$. There is a natural surjective map

$$\widehat{A} \longrightarrow\!\!\!\!\rightarrow \varprojlim_n A/\mathfrak{m}_A^n$$

whose kernel is $\cap_n(\mathfrak{m}_A^n \widehat{A})$. If A is noetherian then $\cap_n \mathfrak{m}_A^n = (0)$. An example to remember is this: If $A = \mathscr{O}_K$, and K is algebraically closed, then $\mathfrak{m}_A = \mathfrak{m}_A^2$, since every element of K has a square root in K, and if the element is in $A = \mathscr{O}_K$, then its square root is also in $\mathscr{O}_K$. It is then straightforward to see that $\varprojlim_n A/\mathfrak{m}_A^n = \mathscr{O}_K/\mathfrak{m}_K = \widetilde{K}$, which is not $\widehat{A}$, unless $|\cdot|$ is trivial.

Proof. Let $x \in \partial\widehat{A}$. Since A is dense in $\widehat{A}$, we can find $a \in A$ such that $|a - x| < 1$. Then $|a| = 1$ and hence is a unit in A, for A is a B-ring. Set $z = a^{-1}(a - x)$. Since $|a^{-1}| = 1$, this means $|z| < 1$, whence $1 - z$ is a unit in $\widehat{A}$, with inverse $\sum_n z^n$. Now $x = a(1 - z)$, and hence x is a unit in $\widehat{A}$. □

Lemma 1.1.6. *Let A be a B-ring and y an element of $\mathscr{O}_K$. Then there exists a polynomial $g \in A[\zeta]$ with $g(y) \in \mathfrak{m}_K$ such that the following is true: Each $f \in A[\zeta]$ with $f(y) \in \mathfrak{m}_K$ admits a $A[\zeta]$-decomposition*

$$f = qg + r$$

where all the coefficients of r are in $\mathfrak{m}_A$.

Proof. The residue field $\widetilde{K}$ is an extension of $\widetilde{A}$.

If $\tilde{y}$ is transcendental over $\widetilde{A}$, then any $f \in A[\zeta]$ satisfying $f(y) \in \mathfrak{m}_K$ must have all its coefficients in $\mathfrak{m}_A$, for

$$\tilde{f}(\tilde{y}) = \widetilde{f(y)} = 0$$

forcing all the coefficients of $\tilde{f}$ to be zero since $\tilde{y}$ is transcendental over $\widetilde{A}$. So in this case we may take $g = q = 0$ and $r = f$.

If on the other hand $\tilde{y}$ is algebraic over $\widetilde{A}$, then let $\tilde{g} \in \widetilde{A}[\zeta]$ be the minimal polynomial of $\tilde{y}$ over $\widetilde{A}$, and let $g \in A[\zeta]$ be a monic lift of $\tilde{g}$. Then clearly $g(y) \in \mathfrak{m}_K$. Moreover if $f \in A[\zeta]$ is such that $f(y) \in \mathfrak{m}_K$, then by Euclidean division we have $g, r \in A[\zeta]$ such that

$$f = qg + r$$

with $\deg r < \deg g$. Clearly $\tilde{r}(\tilde{y}) = 0$, and since $\tilde{g}$ is the minimal polynomial of $\tilde{y}$ and since $\deg \tilde{r} < \deg \tilde{g}$, we must have $\tilde{r} = 0$. It follows that the coefficients of r are in $\mathfrak{m}_A$. □

2. Bald rings

Once again, we don't assume K is complete in what follows. To avoid annoying trivialities we assume that $|\cdot|$ is non-trivial.

2.1. Bald rings are a way to handle non-noetherian rings which occur in the subject, for example $\mathscr{O}_K$ when K is algebraically closed.

Definition 2.1.1. A is called *bald* if

$$\sup\left\{|x| \;\middle|\; x \in A,\ |x| < 1\right\} < 1.$$

Remark: We are not assuming A is bounded, i.e. we are not assuming $A \subset \mathscr{O}_K$.

Proposition 2.1.2. *Let $A \subset \mathscr{O}_K$ and suppose A is bald. Let $M \subset \check{A}$ be such that*

$$\sup_{y \in M} |y| < 1.$$

Then $S = A[M]$ is bald. More precisely, if $\widehat{S}$ is the (norm) completion of S then

$$\sup_{\widehat{S} \cap \mathfrak{m}_{\widehat{K}}} |z| \leq \max\left\{\sup_{a \in \check{A}} |a|, \sup_{y \in M} |y|\right\}.$$

Proof. Let $z \in \widehat{S} \cap \mathfrak{m}_{\widehat{K}}$, say

$$z = a_0 + z'$$

with $a_0 \in A$ and

$$z' = \sum_{\nu_1 + \cdots + \nu_n > 0} a_{\nu_1 \ldots \nu_n} y_1^{\nu_1} \ldots y_n^{\nu_n}, \qquad y_i \in M.$$

We have (using the fact that in the expansion of z', the case $\nu_1 = \nu_2 = \cdots = \nu_n = 0$ is excluded),

$$\begin{aligned} |z'| &\leq \max |a_{\nu_1 \ldots \nu_n}| |y_1^{\nu_1} \ldots y_n^{\nu_n}| \\ &\leq \max |y_1^{\nu_1} \ldots y_n^{\nu_n}| \quad (\text{since } a_{\boldsymbol{\nu}} \in \mathscr{O}_K) \\ &\leq \sup_{y \in M} |y| \\ &< 1. \end{aligned}$$

It follows that $|a_0| < 1$. Hence

$$|z| \leq \max\left\{\sup_{a \in \check{A}} |a|, \sup_{y \in M} |y|\right\}.$$

□

Proposition 2.1.3. *If $A \subset \mathscr{O}_K$ is bald and $y \in \mathscr{O}_K$, then $A[y]$ is bald.*

Proof. Since $A_{\partial A}$ is a B-ring and the value semi-groups of A and $A_{\partial A}$ are the same (see Proposition 1.1.3), therefore we may assume A is a B-ring. Let

$$\varepsilon := \sup_{x \in \mathfrak{m}_A} |x|.$$

Let $g \in A[\zeta]$ be as in Lemma 1.1.6 for the element y. Let

$$\varepsilon' = \max\{\varepsilon, |g(y)|\}.$$

Then $\varepsilon' < 1$. It is enough to show that if $z \in A[y]$ and $|z| \neq 1$, then $|z| < \varepsilon'$. Pick such an element z. Note that $z \in A[y] \cap \mathfrak{m}_K$. Write $z = f(y)$, where $f \in A[\zeta]$. By our choice of g, we have $f = qg + r$ with q and r in $A[\zeta]$, and such that the coefficients of r are in $\mathfrak{m}_A$. By definition of ε, all coefficients of r have value $\leq \varepsilon$. We therefore have

$$\begin{aligned}|f(y)| &\leq \max\{|q(y)||g(y)|, |r(y)|\}\\ &\leq \max\{|g(y)|, |r(y)|\}\\ &\leq \max\{|g(y)|, \varepsilon\}\\ &= \varepsilon'\end{aligned}$$

□

Theorem 2.1.4. *Let K be complete and $M \subset \mathscr{O}_K$ a subset such that $M \cap \partial\mathscr{O}_K$ is finite and such that*

$$\sup_{y \in M \cap \mathfrak{m}_K} |y| < 1.$$

Then the smallest complete B-ring R in K such that $M \subset R \subset \mathscr{O}_K$ is bald.

Proof. Let S be the smallest ring containing 1. Then $S = \mathbf{Z}$ or $S = \mathbf{Z}/p\mathbf{Z}$ for some prime number p. In the latter case the valuation on S is trivial and S is bald. In case $S = \mathbf{Z}$ then either the valuation on S is trivial or else it is equivalent to the p-adic valuation. In either case it is bald. So in every case S is bald. Hence $S[M]$ is bald from by Proposition 2.1.2 and Proposition 2.1.3. By localising if necessary, we have a bald B-ring containing M. By Proposition 2.1.2, baldness is not destroyed by completing. Thus the collection of bald complete B-subrings of $\mathscr{O}_{\widehat{K}}$ containing M is non-empty. We are done by intersecting over this non-empty collection. □

We immediately have:

Corollary 2.1.5. *Let $\{y_\nu\}$ be a sequence in $\mathscr{O}_K$ such that $\lim_{\nu\to\infty} y_\nu = 0$. Then the smallest complete B-subring of K containing all the y_ν, $\nu \in \mathbf{N}$, is bald.*

3. Miscellaneous results for T_n

3.1. Algebraic closure and completion. Here is the basic theorem which shows that $\mathbf{C}_p$, the completion of $\overline{\mathbf{Q}}_p$, is algebraically closed.

Theorem 3.1.1. *If K is algebraically closed then its completion $\widehat{K}$ is also algebraically closed.*

Proof. It is typographically more convenient in this proof to use K' for $\widehat{K}$ and we will do so. Let $\alpha \in \overline{K'}$. Let $L = K'[\alpha]$. Let

$$g = \zeta^r + c_1\zeta^{r-1} + \cdots + c_r \in K'[\zeta]$$

be the minimal polynomial of α over K'. For $n \in \mathbf{N}$ pick a monic polynomial $g_n \in K[\zeta]$ of degree r such that $\|g_n - g\| \leq 2^{-n}$. This is always possible since K is dense in K' and by the definition of $\|\cdot\|$. Fix $n \in \mathbf{N}$. Since K is algebraically closed, we have

$$g_n = \prod_{\alpha' \text{ root of } g_n} (X - \alpha'),$$

where each root $\alpha' \in K$ of g_n occurs as often as its multiplicity. This gives

$$\prod_{\alpha' \text{ root of } g_n} |\alpha - \alpha'| = |g_n(\alpha)| = |g_n(\alpha) - g(\alpha)| \leq 2^{-n}.$$

It follows that at least one root of g_n, call it α_n, is such that

$$|\alpha_n - \alpha| \le 2^{-n/r}.$$

Clearly $\alpha_n \to \alpha$ as $n \to \infty$. Each $\alpha_n \in K$, and since K' is the completion of K, therefore $\alpha \in K'$. Thus K' is algebraically closed. □

3.2. T_n is a regular ring. We return to a familiar situation, namely the case where K is complete, and to avoid annoying trivialities, the valuation on K is non-trivial.

Proposition 3.2.1. *Let K be complete. Let $\mathfrak{m} \in \mathrm{Max}(T_n)$ and set $\mathfrak{n} = \mathfrak{m} \cap K[\zeta_1, \dots, \zeta_n]$. Then*

(a) $\mathfrak{m} = \mathfrak{n}T_n$*;*

(b) *The natural map*

$$K[\boldsymbol{\zeta}]/\mathfrak{n} \longrightarrow T_n/\mathfrak{m}$$

induced by the inclusion $K[\boldsymbol{\zeta}] \subset T_n$ is an isomorphim.

Proof. Recall that given $\boldsymbol{a} = (a_1, \dots, a_n) \in \mathbb{B}^n(\overline{K})$ we have the "evaluation map"

$$\varphi_{\boldsymbol{a}} \colon T_n \longrightarrow K(a_1, \dots, a_n)$$

given by $f \mapsto f(\boldsymbol{a})$. By [Lecture 7, Theorem 1.3.1], we know there exists $\boldsymbol{a} \in \mathbb{B}^n(\overline{K})$ such that

$$\mathfrak{m} = \ker \varphi_{\boldsymbol{a}}.$$

It follows that

$$\mathfrak{n} = \Big\{ g \in K[\boldsymbol{\zeta}] \;\Big|\; g(\boldsymbol{a}) = 0 \Big\}.$$

Part (b) follows from the following commutative diagram, since the downward arrows are isomorphisms.

$$\begin{array}{ccc} K[\boldsymbol{\zeta}]/\mathfrak{n} & \longrightarrow & T_n/\mathfrak{m} \\ {\scriptstyle \text{via } \varphi_{\boldsymbol{a}}} \big\downarrow \wr & & \wr \big\downarrow {\scriptstyle \text{via } \varphi_{\boldsymbol{a}}} \\ K(\boldsymbol{a}) & = & K(\boldsymbol{a}) \end{array}$$

For (a) consider the commutative diagram with all the arrows the obvious ones.

$$\begin{array}{ccc} K[\boldsymbol{\zeta}] & \overset{\text{dense}}{\hookrightarrow} & T_n \\ \big\downarrow & & \big\downarrow \\ K[\boldsymbol{\zeta}]/\mathfrak{n} & \overset{i}{\longrightarrow} & T_n/(\mathfrak{n}T_n) \end{array}$$

The ideal $\mathfrak{n}$ is closed in $K[\boldsymbol{\zeta}]$ since $\mathfrak{n} = \mathfrak{m} \cap K[\boldsymbol{\zeta}]$ and $\mathfrak{m}$ is closed in T_n. The surjective downward arrow on the left is continuous with the residue norm on $K[\boldsymbol{\zeta}]/\mathfrak{n}$ which is the same as the residue norm on $T_n/\mathfrak{m}$ from T_n via the identification of $K[\boldsymbol{\zeta}]/\mathfrak{n}$ with $T_n/\mathfrak{m}$. The composite $K[\boldsymbol{\zeta}] \subset T_n \twoheadrightarrow T_n/(\mathfrak{n}T_n)$ is also continuous. Since $K[\boldsymbol{\zeta}]/\mathfrak{n}$ is finite dimensional over K, the map labelled i is continuous from a well-known result (see [Lecture 11, Corollary 1.1.8]). Since $K[\boldsymbol{\zeta}]/\mathfrak{n}$ is a field, i is an inclusion. Now $K[\boldsymbol{\zeta}]$ is dense in T_n, and hence $i(K[\boldsymbol{\zeta}]/\mathfrak{n})$ is dense in $T_n/(\mathfrak{n}T_n)$. However $i(K[\boldsymbol{\zeta}]/\mathfrak{n})$ is complete under every K-norm (see [Lecture 11, Theorem 1.1.7]). Hence $i(K[\boldsymbol{\zeta}]/\mathfrak{n}) = T_n/(\mathfrak{n}T_n)$. Thus i is an isomorphism, whence $T_n/(\mathfrak{n}T_n)$ is a field, i.e. $\mathfrak{n}T_n$ is a maximal ideal. Since $\mathfrak{m} \supset \mathfrak{n}T_n$, it follows that $\mathfrak{m} = \mathfrak{n}T_n$. □

Corollary 3.2.2. *There exist n polynomials, $p_i \in K[\zeta_1, \ldots, \zeta_i] \subset K[\zeta_1, \ldots, \zeta_n]$, with p_i monic in x_i such that*

(a) $\mathfrak{m} = (p_1, \ldots, p_n)T_n$ *and* $\mathfrak{n} = (p_1, \ldots, p_n)K[\boldsymbol{\zeta}]$.
(b) *If* $\boldsymbol{a} \in \mathbb{B}^n(\overline{K})$ *is an element such that* $\mathfrak{m} = \ker \varphi_{\boldsymbol{a}}$*, then*

$$K[\zeta_1, \ldots, \zeta_i]/((p_1, \ldots, p_i)K[\zeta_1, \ldots, \zeta_i]) \cong K(a_1, \ldots, a_i) \qquad (i = 1, \ldots, n).$$

Proof. It is well known and easy to show that $p_i \in K[\zeta_1, \ldots, \zeta_i]$ exist such that $\mathfrak{n} = (p_1, \ldots, p_n)K[\boldsymbol{\zeta}]$ and (b) is satisfied. Use Proposition 3.2.1 to reduce to this case. □

Theorem 3.2.3. *Let K be complete. Then $(T_n)_{\mathfrak{m}}$ is a regular local ring of Krull dimension n for every $\mathfrak{m} \in \mathrm{Max}(T_n)$. In particular T_n is a regular ring, i.e. $(T_n)_{\mathfrak{p}}$ is a regular local ring for every prime ideal $\mathfrak{p}$ of T_n.*

Proof. Let $\mathfrak{m} \in \mathrm{Max}(T_n)$ and $\mathfrak{n} = K[\boldsymbol{\zeta}] \cap \mathfrak{m}$. Pick $p_1, \ldots, p_n$ as in Corollary 3.2.2. Let $\mathfrak{p}_i = (p_1, \ldots, p_i)K[\boldsymbol{\zeta}]$ and $\mathfrak{q}_i = (p_1, \ldots, p_i)T_n$. One checks that

$$T_n/\mathfrak{q}_i \xrightarrow{\sim} K(a_1, \ldots, a_i)\langle \zeta_{i+1}, \ldots, \zeta_n \rangle.$$

Thus $\mathfrak{q}_i$ is a prime ideal for each i. This means $\mathrm{ht}(\mathfrak{m}) \geq n$. On the other hand $\dim T_n = n$ and so $\mathrm{ht}(\mathfrak{m}) \leq n$. Thus $\mathrm{ht}(\mathfrak{m}) = n$, i.e. $\dim (T_n)_{\mathfrak{m}} = n$. Since $\mathfrak{m}(T_n)_{\mathfrak{m}}$ is generated by n elements, $(T_n)_{\mathfrak{m}}$ is regular. □

Remark 3.2.4. In the event $|\cdot|$ is trivial, then $T_n = K[\boldsymbol{\zeta}]$, and Proposition 3.2.1, Corollary 3.2.2 and Theorem 3.2.3 are obviously true.