LECTURE 8

Date of Lecture: September 5, 2019

As usual, K is a complete non-archimedean field whose absolute value is non-trivial and $k = \widetilde{K}$, i.e., $k = \mathscr{O}_K/\mathfrak{m}_K$.

T_n^* is the group of units in T_n.

For any ring A, $\mathrm{Max}(A)$ is the collection of maximal ideal of A.

As before $\mathbf{N} = \{0, 1, 2, \dots, m, \dots\}$. Rings mean commutative rings with 1.

The symbol ☡ is for flagging a cautionary comment or a tricky argument. It occurs in the margins and is Knuth's version of Bourbaki's "dangerous bend symbol".

1. Some basic results

1.1. These results could have been stated and proved earlier.

Lemma 1.1.1. *T_n^* is open in T_n.*

Proof. Let $f \in T_n^*$ and let $r = \|f\|$. Note that $r > 0$. By Theorem 1.1.2 of Lecture 5 we know that $|f(0)| = r$ and $\|f - f(0)\| < r$. It follows (again by [Lecture 5, Theorem 1.1.2]) that if $g \in T_n$ is such that $\|g\| < r$, then $f + g \in T_n^*$. We have shown that $f + B_r \subset T_n^*$ where $B_r = \{x \in T_n \mid \|x\| < r\}$. $\square$

Lemma 1.1.2. *Let $f \in T_n$. If $\|f\| = 1$ then there exists $c \in K$, $|c| = 1$ such that $f + c$ is not a unit.*

Proof. If $|f(0)| = 1$, take $c = -f(0)$. If $|f(0)| < 1$, then take $c = 1$. In either case f is a non unit by Theorem 1.1.2 in Lecture 5. $\square$

Lemma 1.1.3. *The Jacobson radical of T_n is zero. In other words*

$$\bigcap_{\mathrm{Max}(T_n)} \mathfrak{m} = 0.$$

Proof. Suppose we have a non-zero element f in every maximal ideal of T_n. Without loss of generality, we may assume $\|f\| = 1$. Then we have $c \in K$, $|c| = 1$ such that $f + c$ is a non-unit, and hence $f + c \in \mathfrak{m}$ for some $\mathfrak{m} \in \mathrm{Max}(T_n)$. Since $f \in \mathfrak{m}$ this forces c to lie in $\mathfrak{m}$ contradicting the fact that c is a unit in K and hence in T_n. $\square$

2. Other results

2.1. $T_n/\mathfrak{a}$ is Jacobson. For a ring A, let $j(A)$ denote its *Jacobson radical*. In other words,

$$j(A) = \bigcap_{\mathrm{Max}(A)} \mathfrak{m}.$$

A ring A is said to be *Jacobson* if every prime ideal is the intersection of the maximal ideals containing it. Equivalently, A is Jacobson if the radical $\sqrt{\mathfrak{a}}$ of any ideal $\mathfrak{a}$ is the intersection of maximal ideals. Clearly A is Jacobson if and only if $j(A/\mathfrak{p}) = \sqrt{(0)}$ for every prime ideal $\mathfrak{p}$ of A.

Theorem 2.1.1. *Let $\mathfrak{a}$ be an ideal in T_n. Then $T_n/\mathfrak{a}$ is Jacobson.*

Proof. Let $\mathfrak{p}$ be prime ideal in $A = T_n/\mathfrak{a}$. We have to show that $j(A/\mathfrak{p}) = 0$. We may as well replace A by $A/\mathfrak{p}$ and show that $j(A) = (0)$. Then $A = T_n/\mathfrak{q}$ where $\mathfrak{q}$ is a prime ideal of T_n. By Noether normalisation we have a finite injective ring homomorphism $T_d \hookrightarrow T_n/\mathfrak{q}$. Let $\mathfrak{n} \in \mathrm{Max}(T_d)$ and $\kappa = T_d/\mathfrak{n}$. Since every fibre of $\mathrm{Spec}\, A \to \mathrm{Spec}\, T_d$ is non-empty (the map is finite and dominant, and hence surjective in the set-theoretic sense), therefore $\mathfrak{n}A \neq A$, i.e., $A/(\mathfrak{n}A) \neq 0$. Now $A/(\mathfrak{n}A) = A \otimes_{T_d} \kappa$ is finite over κ, and since κ is a field, this means $A/(\mathfrak{n}A)$ is zero-dimensional. It follows that every prime ideal of $A/(\mathfrak{n}A)$ is maximal. Each of these corresponds to a maximal ideal of A containing $\mathfrak{n}A$, whence every one of them contracts to $\mathfrak{n}$. Thus there exists at least one $\mathfrak{m} \in \mathrm{Max}(A)$ such that $\mathfrak{m} \cap T_d = \mathfrak{n}$. From this it is easy to see that $j(A) \cap T_d = j(T_d)$, whence from Lemma 1.1.3, we get

$$(*) \qquad j(A) \cap T_d = (0).$$

We have to show that $j(A) = (0)$. Suppose it is not. Say $0 \neq x \in j(A)$. Let r be the minimum degree of any integral relation for x. In other words r is the minimum positive integer such that we have an integral relation

$$x^r + b_1 x^{r-1} + \cdots + b_{r-1}x + b_r = 0$$

with $b_1, \ldots, b_r \in T_d$. Then $b_r \in j(A) \cap T_d$. Using $(*)$, this means $b_r = 0$ violating the minimality of r. □

2.2. All ideals in T_n are closed. Recall that every maximal ideal $\mathfrak{m}$ in T_n is closed (see [Lecture 7, Remark 1.3.2]). We will show that all (proper) ideals in T_n are closed. Let $\mathfrak{a} \subset T_n$ be an ideal. Note that by convention in commutative algebra this means that $\mathfrak{a} \neq T_n$. Since T_n^* is open by Lemma 1.1.1, the closure $\mathfrak{a}'$ of $\mathfrak{a}$ in T_n is also an ideal, i.e. $\mathfrak{a}' \neq T_n$. Since T_n is noetherian, we can find a finite number of elements $f_1, \ldots, f_m \in \mathfrak{a}'$ such that $\mathfrak{a}' = \langle f_1, \ldots, f_m \rangle$. We have a surjective map

$$T_n^{\oplus m} \xrightarrow{\pi} \mathfrak{a}'$$

given by $(g_1, \ldots, g_m) \longmapsto \sum_{i=1}^m f_i g_i$. Now $T_n^{\oplus m}$ is a K-Banach space with $\|\boldsymbol{g}\| = \max_{1\le i\le m} \|g_i\|$ where $\boldsymbol{g} = (g_1, \cdots, g_m)$. Let $M := \max_i \|f_i\|$. Then

$$\|\pi(\boldsymbol{g})\| = \left\| \sum_i g_i f_i \right\| \le \max_{1\le i\le m} \|g_i\| \|f_i\| \le M\|\boldsymbol{g}\|.$$

It follows that π is continuous. Let B_1 be the unit ball in T_n centred at 0. Then B_1^m is open in $\mathbf{T}_n^{\oplus m}$. Since π is continuous and surjective, the Open Mapping Theorem tells us that

$$U := \pi(B_1^{\oplus m}) = \sum_{i=1}^m f_i B_1$$

is open in $\mathfrak{a}'$. Since $\mathfrak{a}$ is dense in $\mathfrak{a}'$, if $f \in \mathfrak{a}'$ there exists $g \in \mathfrak{a}$ such that $f \in g + U$. Hence

$$\mathfrak{a}' = \mathfrak{a} + U = \mathfrak{a} + \sum_{i=1}^m f_i B_1.$$

We therefore have elements $h_1, \ldots, h_m \in \mathfrak{a}$, and $\phi_{ij} \in B_1$, $1 \le, i, j \le m$ such that

$$f_i = h_i + \sum_{j=1}^m \phi_{ij} f_j, \quad 1 \le i \le m.$$

If $\boldsymbol{f}$ and $\boldsymbol{h}$ are the $m \times 1$ column vectors determined by f_i and h_i, then

$$\boldsymbol{f} = \boldsymbol{h} + \Phi \boldsymbol{f}$$

where $\Phi = (\phi_{ij})$. This means that

$$(1 - \Phi)\boldsymbol{f} = \boldsymbol{h}.$$

Now $\phi_{ij} \in B_1$, i.e. $\|\phi_{ij}\| < 1$ for every (i, j). Thus $\det(1 - \Phi) = 1 + u$ where $\|u\| < 1$, i.e. $\det(1 - \Phi)$ is a unit in T_n. This means $1 - \Phi$ is invertible and

$$\boldsymbol{f} = (1 - \Phi)^{-1}\boldsymbol{h}$$

whence $f_1, \dots, f_m \in \mathfrak{a}$. Thus $\mathfrak{a} = \mathfrak{a}'$. We have thus proven:

Theorem 2.2.1. *All ideals in T_n are closed.*

3. General remarks about the norm on T_n and norms on $T_n/\mathfrak{a}$

3.1. Orbits of $\mathrm{Aut}(\overline{K}/K)$. Recall that given a point $\boldsymbol{x} \in \mathbb{B}^n(\overline{K})$, say $\boldsymbol{x} = (x_1, \dots, x_n)$, the evaluation map $\varphi_{\boldsymbol{x}} \colon T_n \to \overline{K}$, $f \mapsto f(x)$ takes $K(x_1, \dots, x_n)$, by the completeness of $K(x_1, \dots, x_n)$, and in fact (since $x_i = \varphi_{\boldsymbol{x}}(\zeta_i)$), the image of $\varphi_{\boldsymbol{x}}$ is $K(x_1, \dots, x_n)$. Then $\mathfrak{m}_{\boldsymbol{x}} := \ker \varphi_{\boldsymbol{x}}$ is a maximal ideal of $\mathbf{T}_n$. We showed in Theorem 1.3.1 of Lecture 7 that the map $\mathbb{B}^n(\overline{K}) \to \mathrm{Max}(T_n)$ given by $f \mapsto \mathfrak{m}_{\boldsymbol{x}}$, is surjective. The proof showed that if $\mathfrak{m} \in \mathrm{Max}(T_n)$, then the points $\boldsymbol{x}$ in $\mathbb{B}^n(T_n)$ such that $\mathfrak{m} = \mathfrak{m}_{\boldsymbol{x}}$ are all obtained in the following way. Choose an embedding $\eta \colon T_n/\mathfrak{m} \hookrightarrow \overline{K}$. Such an embedding exists since $T_n/\mathfrak{m}$ is a finite extension of K by Noether normalisation. Let $\varphi^\eta \colon T_n \to \overline{K}$ be the composite $T_n \twoheadrightarrow T_n/\mathfrak{m} \xrightarrow{\eta} \overline{K}$. Set $x_i^\eta = \varphi^\eta(\zeta_i)$, $i = 1, \dots, n$, and $\boldsymbol{x}^\eta = (x_1^\eta, \dots, x_n^\eta)$. Then $\mathfrak{m} = \mathfrak{m}_{\boldsymbol{x}^\eta}$. Each embedding η therefore gives us a point in the fibre of $\mathfrak{m}$ under the map $\mathbb{B}^n(\overline{K}) \to \mathrm{Max}(T_n)$ and every point in the fibre is so obtained. Since the number of embeddings $\eta \colon T_n/\mathfrak{m} \hookrightarrow \overline{K}$ is fnite, the fibres are finite. There is an obvious re-interpretation, namely:

The Galois group $\mathrm{Aut}(\overline{K}/K)$ acts on $\mathbb{B}^n(\overline{K})$ in an obvious way, since it acts on $\overline{K}$ in a norm preserving way, and the fibres of $\mathbb{B}^n(\overline{K}) \to \mathrm{Max}(T_n)$ are precisely the orbits of this action.

One consequence of this observation is this: If $\mathfrak{m}_{\boldsymbol{x}} = \mathfrak{m}_{\boldsymbol{y}}$ for two points $\boldsymbol{x}$ and $\boldsymbol{y}$ in $\mathbb{B}^n(\overline{K})$, then $|f(\boldsymbol{x})| = |f(\boldsymbol{y})|$ for every $f \in T_n$. Indeed, $f(\boldsymbol{x})$ and $f(\boldsymbol{y})$ are conjugates in $\overline{K}$ and hence their absolute values are the same.

3.2. Intrinsic definition of $\|f\|$. We now change notations, and try to remove references to $\mathbb{B}^n(\overline{K})$ in our definition of $\| \; \|$ on T_n. We denote elements of $\mathrm{Max}(T_n)$ by symbols of the form $\boldsymbol{x}$ and regard it as a point in a space of interest (namely $\mathrm{Max}(T_n)$). If we wish to remember such an element's role as a maximal ideal in T_n we write $\mathfrak{m}_{\boldsymbol{x}}$ for it. We write $K(\boldsymbol{x})$ for the residue field $T_n/\mathfrak{m}_{\boldsymbol{x}}$. As has been observed a number of times now, $K(\boldsymbol{x})$ is a finite extension of K and hence has a unique absolute value on it extending the one on K. For $f \in T_n$, we write $f(\boldsymbol{x})$ for its image in $K(\boldsymbol{x})$. From the maximum modulus principle (see [Lecture 4, Theorem 3.2.1]) and the observations in Subsection 3.1 above, we see that

$$\|f\| = \max_{\boldsymbol{x} \in \mathrm{Max}(T_n)} |f(\boldsymbol{x})| \qquad (f \in T_n). \tag{3.2.1}$$

The formula (3.2.1) is intrinsic and does not need $\zeta_1, \dots, \zeta_n$ and hence is invariant under K-algebra automorphisms of T_n. Thus all K-algebra automorphisms of $\mathbf{T}_n$

are continuous and norm preserving. In particular, it gives another explanation of why the map $\sigma_{\boldsymbol{\alpha}}$ of Subsection 2.2 of Lecture 5 is a K-Banach algebra norm preserving automophism (see Lemma 2.2.1 of Lecture 5).

Moreover, since every ideal $\mathfrak{a}$ of T_n is closed, the K-algebra $A = T_n/\mathfrak{a}$ has a residue norm (See problem 4 of HW 3) on it which makes it a K-Banach space. This norm is in fact a K-algebra norm but need not be multiplicative (i.e. $\|fg\| \leq \|f\|\|g\|$ and the two sides are not necessarily equal for $f, g \in A$). Some things to keep in mind.

- If $\mathfrak{a}$ is not a radical ideal then we have non-zero nilpotents in $A = T_n/\mathfrak{a}$. Let $\| \ \|$ be the residue norm on A. If $f \neq 0$ is a nilpotent in A, say $f^j = 0$ for some $j > 0$, then $\|f\|^j > 0 = \|f^j\|$. Similarly, if A has zero divisors, say $fg = 0$ with neither f nor g equal to zero, then $\|f\|\|g\| > 0 = \|fg\|$. Hence in this case we cannot expect relations of the form $\|fg\| = \|f\|\|g\|$ to hold.
- If A is isomorphic to $T_n/\mathfrak{a}$ and to $T_m/\mathfrak{b}$, then residue norm on A from these two quotients is not necessarily the same. However, the two residue norms on A are equivalent.
- If one defines $\| \ \|$ on $A = T_n/\mathfrak{a}$ by the formula in (3.2.1), replacing T_n in the formula with A, then this may not define a K-norm. Indeed if A is non-reduced, then every non-zero nilpotent f in A must have $\|f\| = 0$. In fact the $\| \ \|$ so defined is *power multiplicative*, i.e., $\|f^n\| = \|f\|^n$ for $n \in \mathbf{N}$, which means $\|f\| = 0$ for any $f \in \sqrt{(0)}$. However if A is reduced than the formula in (3.2.1) (with T_n replaced by A) does provide a norm on A. In this case this intrinsic norm is equivalent to the various residue norms on A from different presentations.