LECTURE 7

Date of Lecture: September 3, 2019

As usual, K is a complete non-archimedean field and $k = \widetilde{K}$, i.e., $k = \mathscr{O}_K/\mathfrak{m}_K$.

T_n^* is the group of units in T_n.

For any ring A, $\mathrm{Max}(A)$ is the collection of maximal ideal of A.

As before $\mathbf{N} = \{0, 1, 2, \dots, m, \dots\}$. Rings mean commutative rings with 1.

The symbol ☡ is for flagging a cautionary comment or a tricky argument. It occurs in the margins and is Knuth's version of Bourbaki's "dangerous bend symbol".

A *Weierstrass polynomial* in T_n is an element $w \in T_{n-1}[\zeta_n]$ which is monic and $\|w\| = 1$. A Weierstrass polynomial w is clearly ζ_n-distinguished of order equal to $\deg(w)$. The Weierstrass Preparation Theorem says that any ζ_n-distinguished element $g \in T_n$ is of the form $g = ew$ where e is a unit in T_n, and w is Weierstrass polynomial. Moreover, the representation $g = ew$ is unique.

1. Consequences of the Weierstrass theorems

1.1. A basic lemma.

We begin with the following important lemma.

Lemma 1.1.1. *Let f be a non-zero element in T_n. Then there is a finite monomorphism*

$$T_{n-1} \hookrightarrow T_n/(f).$$

Proof. By Lemma 2.2.2 of Lecture 5, we have a norm preserving K-automorphism $\sigma\colon T_n \to T_n$ such that $g = \sigma(f)$ is ζ_n distinguished, say of order s. Then

$$T_n/(f) \xrightarrow{\sim} T_n/(g).$$

By the Weierstrass Division Theorem, if $h \in T_n$, there is a unique polynomial $r_h \in T_{n-1}[\zeta_n]$, $\deg(r_h) < s$ such that $h \equiv r_h \pmod{g}$. It follows that the natural composition of maps of rings $T_{n-1} \hookrightarrow T_n \twoheadrightarrow T_n/(g)$ is a finite map and as a T_{n-1}-module $T_n/(g)$ is free of rank s. For clarity, if $r_h = \sum_{i=0}^{s-1} g_i\zeta_n^i$, with $g_i \in T_{n-1}$, then the T_{n-1}-module isomorphism $T_n/(g) \xrightarrow{\sim} T_{n-1}^{\oplus s}$ is $h + (g) \mapsto (g_0, g_1, \dots, g_{s-1})$. Since $T_n/(g) \xrightarrow{\sim} T_n/(f)$, we are done. $\square$

Remark 1.1.2. The monomorphism $T_{n-1} \hookrightarrow T_n/(f)$ is not (necessarily) the composite of the standard inclusion $T_{n-1} \subset T_n$ followed by the canonical map $T_n \twoheadrightarrow T_n/(f)$ but is instead the composite $T_{n-1} \subset T_n \xrightarrow{\sim} T_n \twoheadrightarrow T_n/(f)$ where the inclusion on the left is the standard one, the surjective map on the right the canonical one, and the automorphism on T_n the inverse of an automorphism which sends f to a ζ_n-distinguished element via Lemma 2.2.2 of Lecture 5. If f is already ζ_n-distinguished then one can take this automorphism to be the identity map.

1.2. First properties of T_n.

In this sub-section we show that T_n is a noetherian UFD of Krull dimension n. In particular it is normal. We also show that there is a natural surjective $\mathbb{B}^n(\overline{K}) \to \mathrm{Max}(T_n)$.

Theorem 1.2.1. *T_n is noetherian with finite Krull dimension. Its Krull dimension is n.*

Proof. We will first prove that T_n is noetherian. Now, $T_0 = K$ is noetherian. Assume $n > 0$ and T_{n-1} is noetherian. Let I be a anon-zero ideal in T_n. Pick $0 \neq f \in I$ and let $\overline{I}$ be its image in $T_n/(f)$. By Lemma 1.1.1, we have a finite monomorphism of rings $T_{n-1} \hookrightarrow T_n/(f)$. Since T_{n-1} is noetherian by our induction hypothesis, $T_n/(f)$ is a noetherian T_{n-1} module, and therefore so is $\overline{I}$. If $g_1 + (f), \dots, g_m + (f)$ are generators of $\overline{I}$ as a T_{n-1}-module, then they are generators of $\overline{I}$ as a $T_n/(f)$ ideal. It follows that I is generated by $g_1, \dots, g_m$ and f. Thus T_n is noetherian.

For a ring A, let $\dim A$ denote its Krull dimension, which could well be ∞. Note that $\dim T_0 = 0$. Let $n > 1$ and assume that $\dim T_{n-1} = n - 1$. Suppose

$$(0) \subsetneq \mathfrak{p}_1 \subsetneq \dots \subsetneq \mathfrak{p}_m$$

is an increasing chain of prime ideals in T_n with $m \in \mathbf{N}$. We claim that $m \leq n$. Indeed let $0 \neq f \in \mathfrak{p}_1$ and for $i = 0, \dots, m-1$ set $\mathfrak{q}_i = \mathfrak{p}_{i+1}$. Then

$$\mathfrak{q}_0 \subsetneq \mathfrak{q}_1 \subsetneq \dots \subsetneq \mathfrak{q}_{m-1}$$

is an increasing chain of prime ideals in $T_n/(f)$ of length $m-1$. Since we have a finite monomorphism $T_{n-1} \hookrightarrow T_n/(f)$, and $\dim T_{n-1} = n-1$, the Krull dimension of $T_n/(f)$ is also $n-1$. It follows that $m - 1 \leq n - 1$, i.e. $m \leq n$. Thus $\dim T_n \leq n$. The existence of the chain of primes ideals $(0) \subsetneq (\zeta_1) \subsetneq \dots \subsetneq (\zeta_n)$ shows that $\dim T_n = n$. □

Theorem 1.2.2. *T_n is a UFD and hence is normal.*

Remark: If A is a UFD then it must be normal. Indeed, let $Q = Q(A)$ be its field of fractions and suppose $x \in Q$ is integral over A, say

$$x^r + a_{r-1}x^{r-1} + a_{r-2}x^{r-2} + \dots + a_1 x + a_0 = 0$$

with $r > 0$. Write $x = \alpha/\beta$ with $\alpha, \beta \in A$, $\beta \neq 0$, and $(\alpha, \beta) = 1$. Clearing denominators in the above integral relation, we get

$$\alpha^r = -(a_{r-1}\alpha^{r-1}\beta + a_{r-2}\alpha^{r-2}\beta^2 + \dots + a_1\alpha\beta^{r-1} + a_0\beta^r).$$

In other words $\alpha^r = \beta a$ for some $a \in A$. Since $(\alpha, \beta) = 1$ this forces β to be a unit in A. Thus $x = \alpha/\beta \in A$.

Proof of Theorem. By way of induction, we assume T_{n-1} is a UFD. By Gauss's lemma, so is $T_{n-1}[\zeta_n]$

Let $f \in T_n$. We have to factor f into irreducible factors. By Lemma 2.2.2 of Lecture 5 we may assume f is ζ_n-distinguished, in fact a Weierstrass poynomial. Thus $f \in T_{n-1}[\zeta_n]$, is monic, and $\|f\| = 1$. Using the fact that $T_{n-1}[\zeta_n]$ is a UFD, we can write $f = \pi_1 \dots \pi_m$ where $\pi_i \in T_{n-1}[\zeta_n]$ are prime in $T_{n-1}[\zeta_n]$. Since f is monic, we can re-arrange matters (by multiplying each π_i by a unit from T_{n-1}) so that each π_i is monic. It follows that $\|\pi_i\| \geq 1$ for each i. Since $\|f\| = 1$, we have $\|\pi_i\| = 1$ for each i. Thus each π_i is a Weierstrass polynomial, whence ζ_n-distingushed. It remains to show that each π_i is prime in T_n, in other words to show that $T_n/(\pi_i)$ is an integral domain. To see this note that the canonical map

$$T_{n-1}[\zeta_n]/(\pi_i) \longrightarrow T_n/(\pi_i),$$

induced by the inclusion $T_{n-1}[\zeta_n] \subset T_n$, is an isomorphism. Indeed, if $s = \deg_{\zeta_n}(\pi_i)$, then both sides are free T_{n-1}-modules of rank s freely generated by the residue classes of $\zeta_n, \ldots, \zeta_n^{s-1}$, the left side because of Euclid Division and the right because of Weierstrass Division. The ring $T_{n-1}[\zeta_n]/(\pi_i)$ is an integral domain since π_i is prime in $T_{n-1}[\zeta_n]$. □

Remark 1.2.3. In fact T_n is a regular ring. We will see this later in the course (hopefully).

Theorem 1.2.4. (Noether normalisation) *Let $\mathfrak{a}$ be an ideal of T_n and $d = \dim T_n/\mathfrak{a}$. Then there exists a K-algebra finite monomorphism $T_d \hookrightarrow T_n/\mathfrak{a}$.*

Proof. The theorem is clearly true for $n = 0$. Let $n > 0$ and suppose the theorem is true for T_{n-1}. If $\mathfrak{a} = 0$ then the statement is clearly true with $d = n$. So suppose we have a non-zero element f in $\mathfrak{a}$. By Lemma 1.1.1 we have a finite injective ring homomorphsim $T_{n-1} \hookrightarrow T_n/(f)$. Let $\mathfrak{a}_1$ be the kernel of the composite $T_{n-1} \hookrightarrow T_n/(f) \twoheadrightarrow T_n/\mathfrak{a}$. The following commutative diagram might help:

$$\begin{tikzcd}
 & T_n/(f) \arrow[dr, two heads] & \\
T_{n-1} \arrow[ur, hook] \arrow[dr, two heads] & & T_n/\mathfrak{a} \\
 & T_{n-1}/\mathfrak{a}_1 \arrow[ur, hook] &
\end{tikzcd}$$

The twoheaded arrows of the form $\twoheadrightarrow$ denote (as always) surjective maps. These are the two southeast pointing arrows in the above diagram. A word of caution: the northeast pointing monomorphism on the left is *not* (necessarily) the composite $T_{n-1} \subset T_n \twoheadrightarrow T_n/(f)$ where the inclusion on the left is the standard one. See Remark 1.1.2 for clarification.

Since composite of the northeast pointing arrow from T_{n-1} followed by the southeast arrow from $T_n/(f)$ is finite, so is the composite of the southeast followed by the northeast pointing arrow in the lower half of the diagram. The inclusion $T_{n-1}/\mathfrak{a}_1 \hookrightarrow T_n/\mathfrak{a}$ is therefore a finite monomorphism of K-algebras.[1] By our induction hypothesis, the theorem is true for T_{n-1}, whence we have a finite monomorphsim $T_d \hookrightarrow T_{n-1}/\mathfrak{a}_1$. Composing, we get a finite monomorphism

$$T_d \hookrightarrow T_n/\mathfrak{a}.$$

It is clear (from the various going up and going down theorems) that $\dim T_n/\mathfrak{a} = d$. □

We then have the following important corollary

Corollary 1.2.5. *Let $\mathfrak{m}$ be a maximal ideal of T_n. Then the field extension*

$$K \to T_n/\mathfrak{m}$$

(given by the composite $K \subset T_n \twoheadrightarrow T_n/\mathfrak{m}$) is finite.

Proof. Since the Krull dimension of $T_n/\mathfrak{m}$ is zero, and since $T_0 = K$, we are done by Noether normalisation. □

[1]Indeed any generating set for $T_n/\mathfrak{a}$ as a T_{n-1}-module is also a generating set for $T_n/\mathfrak{a}$ as a $T_{n-1}/\mathfrak{a}_1$-module since the map $T_{n-1} \twoheadrightarrow T_{n-1}/\mathfrak{a}_1$ is surjective.

1.3. **The maximal spectrum of T_n.** Let $\boldsymbol{x} = (x_1, \dots, x_n) \in \mathbb{B}^n(\overline{K})$ and let

$$\mathfrak{m}_{\boldsymbol{x}} = \{f \in \mathbb{B}^n(\overline{K}) \mid f(\boldsymbol{x}) = 0\}.$$

If $f = \sum_{\boldsymbol{\nu}} c_{\boldsymbol{\nu}} \boldsymbol{\zeta}^{\boldsymbol{\nu}} \in T_n$, then $f(\boldsymbol{x}) = \sum_{\boldsymbol{\nu}} c_{\boldsymbol{\nu}} \boldsymbol{x}^{\boldsymbol{\nu}}$ lies in $K(x_1, \dots, x_n)$ since $K(x_1, \dots, x_n)$ is complete. This gives us the *evaluation map at* $\boldsymbol{x}$

$$\varphi_{\boldsymbol{x}} \colon T_n \longrightarrow K(x_1, \dots, x_n), \qquad f \longmapsto f(\boldsymbol{x}).$$

Clearly

$$\mathfrak{m}_{\boldsymbol{x}} = \ker \varphi_{\boldsymbol{x}}.$$

Clearly $\mathfrak{m}_{\boldsymbol{x}} \in \operatorname{Max}(T_n)$ since $T_n/\mathfrak{m}_{\boldsymbol{x}} \cong K(x_1, \dots, x_n)$.

Theorem 1.3.1. *The map*

$$\mathbb{B}^n(\overline{K}) \longrightarrow \operatorname{Max}(T_n), \qquad \boldsymbol{x} \longmapsto \mathfrak{m}_{\boldsymbol{x}}$$

is surjective.

Proof. Let $\mathfrak{m} \in \operatorname{Max}(T_n)$. Let $L = T_n/\mathfrak{m}$. By Corollary 1.2.5, L is a finite extension of K and hence there is an embedding $L \hookrightarrow \overline{K}$. Let

$$\varphi \colon T_n \to \overline{K}$$

be the induced map, i.e., φ is the composite $T_n \twoheadrightarrow L \hookrightarrow \overline{K}$. We claim that

$$(*) \qquad |\varphi(f)| \leq \|f\| \qquad (f \in T_n).$$

This will prove that φ is continuous. Moreover, setting $x_i = \varphi(\zeta_i)$, $i = 1, \dots, n$, the inequality $(*)$ shows that $|x_i| \leq \|\zeta_i\| = 1$, and hence that $\boldsymbol{x} = (x_1, \dots, x_n) \in \mathbb{B}^n(\overline{K})$. Since φ is continuous (assuming $(*)$), $\varphi = \varphi_{\boldsymbol{x}}$, and so $\ker\varphi = \ker\varphi_{\boldsymbol{x}}$, i.e. $\mathfrak{m} = \mathfrak{m}_{\boldsymbol{x}}$.

It remains to prove $(*)$. Suppose $(*)$ is not true. There is an $f \in T_n$ such that $|\varphi(f)| > \|f\|$. This means $f \neq 0$, and we may assume $\|f\| = 1$. Let $\alpha = \varphi(f)$. Note that

$$|\alpha| > 1.$$

Let

$$\theta = X^r + c_1 X^{r-1} + \cdots + c_{r-1} X + c_r \in K[X]$$

be the minimal polynomial of α over K, and $\alpha_1, \dots, \alpha_r$ the roots of θ in $\overline{K}$. Since K is not necessarily separable, the roots are not necessarily distinct. Since the fields $K(\alpha_i)$ are canonically isomorphic to each other, and since the extensions of the norm from K to $K[X]/(\theta)$ is unique, we have

$$|\alpha_i| = |\alpha| \qquad (i = 1, \dots, r).$$

Now,

$$\theta = \prod_{i=1}^{r} (X - \alpha_i).$$

Thus $c_r = (-1)^r \alpha_1 \dots \alpha_r$. It follows that

$$|c_r| = |\alpha|^r > 1.$$

Since $c_i = (-1)^i \sigma_i(\alpha_1, \dots, \alpha_r)$, $i = 1, \dots, r$, where σ_i is the i^{th} elementary symmetric polynomial in r variables, we have

$$(**) \qquad |c_i| \leq |\alpha|^i < |\alpha|^r = |c_r| \quad (1 \leq i < r).$$

We have used the fact that $|\alpha| > 1$ in establishing $(**)$. Thus we have

$$\theta(f) := c_r + c_{r-1}f + \cdots + c_1 f^{r-1} + f^r = c_r\Big(1 + \sum_{i=1}^{r} b_{r-i} f^r\Big)$$

where $b_0 = c_r^{-1}$ and $b_i = c_i c_r^{-1}$ for $i = 1, \ldots, r-1$. Setting

$$g = -\sum_{i=1}^{r} b_{r-i} f^i,$$

we see via $(**)$ that $\|g\| < 1$. Thus $1 - g$ is a unit in T_n with inverse $\sum_{m \in \mathbf{N}} g^m$. This means $\theta(f) = c_r(1-g)$ is a unit in T_n, and hence $\varphi(\theta(f))$ is a unit in $\overline{K}$. However,

$$\varphi(\theta(f)) = \theta(\varphi(f)) = \theta(\alpha) = 0,$$

giving a contradiction. Thus $|\varphi(f)| \leq \|f\|$ for every $f \in T_n$. $\square$

Remark 1.3.2. Since $\varphi_{\boldsymbol{x}}$ is continuous therefore $\mathfrak{m}_{\boldsymbol{x}}$ is closed in T_n. And by the theorem, this means every maximal ideal of T_n is closed. In fact every ideal in T_n is closed as we will see in a future (next?) lecture.