LECTURE 7 SUPPLEMENT

Date of Lecture: September 3, 2019

This is a supplement to the lecture on Sep 3, in view of certain discussions regarding dimensions and finite maps in the class.

Recall that a morphism of schemes $f\colon X \to Y$ is called *dominant* if $f(X)$ is dense in Y. If a dominant map is also closed (e.g. a proper map) then clearly f is surjective (as a set theoretic map). A map of rings $A \to B$ is said to be dominant if the corresponding map of schemes $\operatorname{Spec} B \to \operatorname{Spec} A$ is dominant. Here is a well known criterion for dominance of maps of affine schemes.

Theorem 1. *A map of rings $\varphi\colon A \to B$ is dominant if and only if $\ker \varphi \subset \sqrt{(0)}$. In particular if φ is a monomorphism, it is dominant.*

Proof. Let J be an ideal of B. Let $V(J)$ be the usual closed subset in $\operatorname{Spec} B$ associated to J, namely the collection of prime ideals $\mathfrak{q}$ in B such that $\mathfrak{q} \supset J$. Let $X = \operatorname{Spec} B$ and $Y = \operatorname{Spec} A$, and let $f\colon X \to Y$ be the map corresponding to φ. It is easy to see

$$\overline{f(V(J))} = V(\varphi^{-1}(J)).$$

Setting $J = 0$, so that $V(J) = X$, we see that

$$\overline{f(X)} = V(\ker \varphi).$$

The right side equals Y if and only if $\ker \varphi \subset \sqrt{(0)}$. $\square$

Now suppose $\varphi\colon A \to B$ is a monomorphism which is finite. Using the notations of the proof of the theorem, since $f\colon X \to Y$ is finite, it is proper. In particular f is closed. Since φ is a monomorphism, f is dominant. It follows that $f\colon X \to Y$ is surjective. This means $\dim X = \dim Y$ (see for example Lemma 28.42.9 of the Stacks Project). We record this as follows:

Theorem 2. *Let $\varphi\colon A \to B$ be a finite monomorphism of rings. Then the Krull dimension of A is equal to the Krull dimension of B.*