LECTURE 5

Date of Lecture: August 27, 2019

In this lecture, we fix a complete non-archimedean field K and assume that the absolute value is non-trivial.

A quick recap of some notations and definitions may be in order. As before, T_n is the n^{th} Tate algebra $K\langle\zeta_1,\dots,\zeta_n\rangle$ of restricted power series in $\zeta_1,\dots,\zeta_n$ over K and T_n° is the subring of T_n consisting of elements $f\in T_n$ such that $\|f\|\le 1$. In other words $T_n^\circ=\mathscr{O}_K\langle\zeta_1,\dots,\zeta_n\rangle$, the ring of restricted power series with coefficients in $\mathscr{O}_K$. Note that $\mathscr{O}_K=T_0^\circ$.

The residue field $\widetilde{K}$ is denoted k in this lecture. In other words

$$k=\mathscr{O}_K/\mathfrak{m}_K.$$

For $f\in T_n$, $\tilde{f}\in k[\zeta_1,\dots,\zeta_n]$ is the image of f under the natural map of rings

$$T_n^\circ=\mathscr{O}_K\langle\zeta_1,\dots,\zeta_n\rangle\longrightarrow k[\zeta_1,\dots,\zeta_n].$$

Note that all but a finite number of coefficients of an restricted power series f in $\mathscr{O}_K\langle\zeta_1,\dots,\zeta_n\rangle$ lie in $\mathfrak{m}_K$, explaining the above ring homomorphism.

A reminder: The set $\mathbf{N}$ includes 0.

1. Units in T_n° and T_n

1.1. The following two theorems (equivalent to each other) characterise units in T_n° and T_n.

Theorem 1.1.1. *Let $f\in T_n^\circ$. The following are equivalent:*

(1) *f is a unit in T_n°.*
(2) *$\tilde{f}$ is a unit in $k[\zeta_1,\dots,\zeta_n]$, i.e., $\tilde{f}\in k^*$.*
(3) *$|f(0)|=1$ and $\|f-f(0)\|<1$.*

Proof. It is clear that (1) implies (2) and that (2) $\Longleftrightarrow$ (3). Suppose f satisfies (3). Then $\|f\|=1$. Without loss of generality, we may assume $f(0)=1$. Set

$$g=1-f.$$

Then $\|g\|<1$ since $g=-(f-f(0))$. It follows that the series $\sum_{\nu=0}^{\infty}g^\nu$ is absolutely convergent. Since T_n is a Banach K-algebra (see Theorem 1.2.1] of Lecture 4), this means $\sum_{\nu=0}^{\infty}g^\nu$ converges in T_n by Lemma 1.1.1 of Lecture 2. Let

$$h=\sum_\nu g^\nu.$$

Now $\|h\|=1$ since $\|g^0\|=1$, and $\|g^n\|<1$ for $n\ge 1$. Thus $h\in T_n^\circ$. Clearly $(1-g)h=1$, i.e. $fh=1$. $\square$

Theorem 1.1.1 is equivalent to:

Theorem 1.1.2. *Let $f\in T_n$. The following are equivalent:*

(1) *f is a unit in T_n.*
(2) *$f\neq 0$ and $f/\|f\|$ is a unit in T_n°.*

(3) $\|f - f(0)\| < |f(0)|$.

Proof. Statements (1) and (2) are clearly equivalent, for an element of norm 1 is a unit in T_n if and only if it is a unit in T_n°. By Theorem 1.1.1, (2) is equivalent to saying $|f(0)/\|f\|| = 1$ and $\|f - f(0)\| < \|f\|$. But thesecond statement is clearly equivalent to saying $\|f\| = |f(0)|$ and $\|f - f(0)\| < |f(0)|$. However, if $\|f - f(0)\| < |f(0)|$, then $\|f\| = |f(0)|$ showing that (2) is equivalent to (3). ☐

2. Distinguished elements

In this section we make the first moves towards stating and proving the Weierstrass Division Theorem and the Weierstrass Preparation Theorem. The consequences of the two theorems are many. They show that T_n is a noetherian UFD with finite Krull dimension equal to n.

2.1. ζ_n-distinguished elements. If $f \in T_n = K\langle\zeta_1, \dots, \zeta_n\rangle$ then clearly f has a unique decomposition into a series

$$(*) \qquad f = \sum_{\nu \in \mathbf{N}} g_\nu \zeta_n^\nu \qquad (g_\nu \in T_{n-1},\ \nu \in \mathbf{N}).$$

Definition 2.1.1. An element $f \in T_n$ is said to be ζ_n*-distinguished of order* s if in the decomposition $f = \sum_{\nu \in \mathbf{N}} g_\nu \zeta_n^\nu$ in (*) above, the following hold:

(i) g_s is a unit in T_{n-1}.
(ii) $\|g_s\| = \|g\|$ and $\|g_s\| > \|g_\nu\|$ for $\nu > s$.

Remark 2.1.2. Suppose $\|g\| = 1$. Then g is distinguished of order s if and only if

$$\tilde{g} = \tilde{g}_s\zeta_n^s + \tilde{g}_{s-1}\zeta^{s-1} + \cdots + \tilde{g}_1\zeta_n + \tilde{g}_s$$

with $\tilde{g_s} \in k^*$. In particular, g is distinguished of order 0 if and only if $\tilde{g} \in k^*$, i.e., if and only if g is a unit. It follows that an arbitrary $g \in T_n$ (not necessarily with $\|g\| = 1$) is ζ_n-distinguished if and only if it is a unit.

2.2. An automorphism of T_n. Let $\alpha_1, \dots, \alpha_{n-1}$ be positive integers. We have a map

$$\sigma_{\boldsymbol{\alpha}} \colon K[|\boldsymbol{\zeta}|] \longrightarrow K[|\boldsymbol{\zeta}|]$$

defined by

$$\sigma_{\boldsymbol{\alpha}}(\zeta_i) = \begin{cases} \zeta_i + \zeta_n^{\alpha_i}, \text{for } i < n \\ \zeta_n, \text{for } i = n. \end{cases}$$

According to Problem (4) of HW 2, $\sigma_{\boldsymbol{\alpha}}(f)$ makes sense for any $f \in K[|\boldsymbol{\zeta}|]$. It has an inverse given by (once again using Problem 4 of HW 2)

$$\zeta_i \longmapsto \begin{cases} \zeta_i - \zeta_n^{\alpha_i}, \text{for } i < n \\ \zeta_n, \text{for } i = n. \end{cases}$$

There are two observations worth making

(i) $\|\zeta_i + \zeta_n^{\alpha_i}\| = 1 = \|\zeta_i\|$.
(ii) For $\boldsymbol{\nu} \in \mathbf{N}^n$ let $g_{\boldsymbol{\nu}} = \prod_{i=1}^{n-1}(\zeta_i + \zeta_n^{\alpha_i})^{\nu_i} \cdot \zeta_n^{\nu_n}$. Then $\|g_{\boldsymbol{\nu}}\| = 1$.

Let $f = \sum_{\boldsymbol{\nu}\in\mathbf{N}^n} c_{\boldsymbol{\nu}}\boldsymbol{\zeta}^{\boldsymbol{\nu}}$ be in T_n. We have

$$\sigma_{\boldsymbol{\alpha}}(f) = \sum_{\boldsymbol{\nu}\in\mathbf{N}^n} c_{\boldsymbol{\nu}} g_{\boldsymbol{\nu}}.$$

Let $\epsilon > 0$ be given. There exists N such that for all $\boldsymbol{\nu}$ with $|\boldsymbol{\nu}| \geq N$, $|c_{\boldsymbol{\nu}}| < \epsilon$. Then for any finite subset I of the set of indices $\boldsymbol{\nu}$ with $|\boldsymbol{\nu}| \geq N$, we have

$$\|\sum_I c_{\boldsymbol{\nu}} g_{\boldsymbol{\mu}}\| \leq \max_I \|c_{\boldsymbol{\nu}} g_{\boldsymbol{\nu}}\| = \max_I |c_{\boldsymbol{\nu}}| < \epsilon.$$

It follows that $\sum_{\boldsymbol{\nu}} c_{\boldsymbol{\nu}} g_{\boldsymbol{\nu}}$ converges in T_n, since T_n is complete, and clearly $\sum_{\boldsymbol{\nu}} c_{\boldsymbol{\nu}} g_{\boldsymbol{\nu}} = \sigma_{\boldsymbol{\alpha}}(f)$. Moreover $\|\sigma_{\boldsymbol{\alpha}}(f)\| \leq \sup_{\boldsymbol{\nu}} \|c_{\boldsymbol{\nu}} g_{\boldsymbol{\nu}}\| = \sup_{\boldsymbol{\nu}} |c_{\boldsymbol{\nu}}| = \|f\|$. Similar considerations for $\sigma_{\boldsymbol{\alpha}}^{-1}$ show that in fact

$$\|\sigma_{\boldsymbol{\alpha}}(f)\| = \|f\| \quad (f \in T_n)$$

whence $\sigma_{\boldsymbol{\alpha}}$ is an isometric automorphism of Banach K-algebras, and in particular is continuous. We record that as follows

Lemma 2.2.1. *The map*

$$\sigma_{\boldsymbol{\alpha}} \colon T_n \longrightarrow T_n$$

is a Banach K-algebra automorphism which preserves norms.

Lemma 2.2.2. *Let $f_1, \ldots, f_r$ be finitely many non-zero elements of T_n. Then there exists $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_{n-1})$ such that $\sigma_{\boldsymbol{\alpha}}(f_1), \ldots, \sigma_{\boldsymbol{\alpha}}(f_r)$ are ζ_n-distinguished.*

Proof. Let $r = 1$ and $f = f_1$. Let $f = \sum_{\boldsymbol{\nu}\in\boldsymbol{\nu}^n} c_{\boldsymbol{\nu}}\boldsymbol{\zeta}^{\boldsymbol{\nu}}$. Without loss of generality we assume $\|f\| = 1$. Let

$$S = \left\{\boldsymbol{\nu} \in \mathbf{N}^n \,\middle|\, \tilde{c}_{\boldsymbol{\nu}} \neq 0\right\}.$$

Then S is finite and

$$\tilde{f} = \sum_{\boldsymbol{\nu}\in S} \tilde{c}_{\boldsymbol{\nu}}\boldsymbol{\zeta}^{\boldsymbol{\nu}}.$$

For each $\boldsymbol{\nu} \in S$, consider

$$\phi_{\boldsymbol{\nu}} = \nu_n + \nu_{n-1}X + \cdots + \nu_1 X^{n-1} \in \mathbf{Z}[X].$$

Since there are only a finite number of plynomials in play there exists a real number $r_0 > 0$ such that

$$\phi_{\boldsymbol{\nu}}(x) - \phi_{\boldsymbol{\mu}}(x) \neq 0 \qquad (x \geq r_0,\ \boldsymbol{\mu}, \boldsymbol{\nu} \in S \text{ and } \boldsymbol{\nu} \neq \boldsymbol{\mu}).$$

Pick $t \in \mathbf{N}$ with $r > r_o$. Let $\alpha_i = t^{n-i}$, $i = 1, \ldots, n-1$, and let $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_{n-1})$. Let $\sigma = \sigma_{\boldsymbol{\alpha}}$. Then

$$\begin{aligned}
\widetilde{\sigma(f)} &= \sum_{\boldsymbol{\nu}\in S} \tilde{c}_{\boldsymbol{\nu}} \prod (\zeta_i + \zeta_n^{\alpha_i})^{\nu_i} \cdot \zeta_n^{\nu_n} \\
&= \sum_{\boldsymbol{\nu}\in S} \tilde{c}_{\boldsymbol{\nu}} \zeta_n^{\alpha_1\nu_1 + \cdots + \alpha_{n-1}\nu_{n-1} + \nu_n} + \tilde{g} \\
&= \sum_{\boldsymbol{\nu}\in S} \tilde{c}_{\boldsymbol{\nu}} \zeta_n^{\phi_{\boldsymbol{\nu}}(t)} + \tilde{g}
\end{aligned}$$

where $\tilde{g} \in k[\boldsymbol{\zeta}]$ is a polynomial in ζ_n whose degree in ζ_n is strictly less than the maximum of all exponents $\phi_{\boldsymbol{\nu}}(t)$ with $\boldsymbol{\nu}$ varying over S.

Since $t > r_0$, these exponents are pairwise distinct, and hence there is a maximum exponent s which is assumed at a unique $\boldsymbol{\nu}^* \in S$. Then

$$\widetilde{\sigma(f)} = \tilde{c}_{\boldsymbol{\nu}^*} \zeta_n^s + h$$

where h is a polynomial of degree $< s$ in ζ_n. Since $\tilde{c}_{\boldsymbol{\nu}^*} \neq 0$ (for $\boldsymbol{\nu}^* \in S$), $\sigma(f)$ is ζ_n-distinguished of order s.

The general case, i.e. when $r \geq 1$, is dealt with in the same way. We simply have to pick a t which is large enough that it works for $f_1, \ldots, f_r$ simultaneously. $\square$