LECTURE 4

Date of Lecture: August 22, 2019

A reminder: The set $\mathbf{N}$ includes 0.

1. An important Banach Algebra

The aim of this section is to prove that the ring of strictly convergent power series in n-variables over a complete non-archimedean field is a Banach algebra.

1.1. **Basic Lemma.** If $(M, \|\cdot\|)$ is a normed linear space over a normed field K, then a series $\sum_{n=0}^{\infty} a_n$ in M is a said to be *absolutely convergent* if $\sum_{n=0}^{\infty} \|a_n\| < \infty$. It is *convergent* if there exists $a \in M$ such that $\lim_{n\to\infty} \sum_{l=0}^{n} a_l = a$. The series $\sum_{n=1}^{\infty} a_n$ is said to be *Cauchy* if its sequence of partial sums is Cauchy.

Lemma 1.1.1. *Let $(K, |\cdot|)$ be a complete normed field (archimedean or non-archimedean) and $(M, \|\cdot\|)$ a normed vector space over K. M is a K-Banach space if and only if every absolutely convergent series in M is convergent.*

Proof. First note that for any series $\sum_{n=1}^{\infty} a_n$ in M, we must have $\|\sum_{l=m}^{n} a_l\| \le \sum_{l=m}^{n} \|a_l\|$, for $m \le n$, whence an absolutely convergent series in M is necessarily Cauchy. It follows that if M is Banach, every absolutely convergent series in M converges.

Conversely suppose every absolutely convergent series in M converges. Let $\{s_n\}$ be a Cauchy sequence in M. For each $k \in \mathbf{N}$ there exists n_k such that

$$\|s_n - s_m\| \le 2^{-k} \qquad (n, m \ge n_k).$$

We choose our $n_{k+1} > n_k$ for all $k \ge 0$. Clearly this can always be arranged. Let

$$\begin{aligned} a_0 &= s_{n_0} \\ a_k &= s_{n_k} - s_{n_{k-1}}, \qquad k \ge 1. \end{aligned}$$

Now for $1 \le m \le n$ we have

$$\sum_{k=0}^{\infty} \|a_k\| \le \|a_0\| + \sum_{k=1}^{\infty} \|s_{n_k} - s_{n_{k-1}}\| \le \|a_0\| + \sum_{k=1}^{\infty} 2^{-k+1} < \infty.$$

Thus $\sum_k a_k$ converges absolutely. By our hypothesis it therefore converges to a limit a. Now $\sum_{l=0}^{k} a_l = s_{n_k}$ for all $k \ge 0$ and hence

$$\lim_{k\to\infty} s_{n_k} = a.$$

Thus the Cauchy sequence $\{s_n\}$ has a convergent subsequence $\{s_{n_k}\}$. It follows that $\lim_{n\to\infty} s_n = a$. In somewhat greater detail, given $\epsilon > 0$, there exists $N \in \mathbf{N}$ such that $\|s_n - s_m\| < \epsilon$ for $m, n \ge N$. Pick $k \in \mathbf{N}$ such that $n_k \ge N$ and $\|s_{n_k} - a\| < \epsilon$. Then

$$\|s_n - a\| \le \|s_n - s_{n_k}\| + \|s_{n_k} - a\| < 2\epsilon \quad (n \ge N).$$

$\square$

1.2. **The Tate algebra is a Banach algebra.** Let $(K,|\cdot|)$ be a complete non-archimedean field with non-trivial absolute value and as before let

$$T_n = K < \zeta_1,\dots,\zeta_n >= \{\sum_{\boldsymbol{\nu}\in\mathbf{N}^n} c_{\boldsymbol{\nu}}\boldsymbol{\zeta}^{\boldsymbol{\nu}} \mid \lim_{|\boldsymbol{\nu}|\to\infty} |c_{\boldsymbol{\nu}}| = 0\}.$$

Recall T_n has a norm $\|\cdot\|\colon T_n \to [0,\infty)$ on it, namely the *Gauss norm*,

$$\Big\| \sum_{\boldsymbol{\nu}\in\mathbf{N}^n} c_{\boldsymbol{\nu}}\boldsymbol{\zeta}^{\boldsymbol{\nu}} \Big\| = \max_{\boldsymbol{\nu}} |c_{\boldsymbol{\nu}}|.$$

It is easy to see (as we did on pp. 2–3 of Lecture 3) that T_n is a normed K-algebra, i.e., it is a normed vector space over K satisfying $\|fg\| = \|f\|\|g\|$ for f and g in T_n.

Theorem 1.2.1. *T_n is a Banach K-algebra.*

Proof. Let $\sum_{j\in\mathbf{N}} f_j$ be an absolutely convergent series in T_n. According to Lemma 1.1.1, we have to show that $\sum_j f_j$ is then it is convergent. Let $c_{j,\boldsymbol{\nu}} \in K$, for $j \in \mathbf{N}$ and $\boldsymbol{\nu} \in \mathbf{N}^n$, be defined by

$$f_j = \sum_{\boldsymbol{\nu}} c_{j,\boldsymbol{\nu}}\boldsymbol{\zeta}^{\boldsymbol{\nu}}.$$

For a fixed $\boldsymbol{\nu}$ we have

$$\sum_j |c_{j,\boldsymbol{\nu}}| \le \sum_j \|f_j\| < \infty$$

whence $\sum_j c_{j,\boldsymbol{\nu}}$ is convergent in K. Set

$$c_{\boldsymbol{\nu}} = \sum_j c_{j,\boldsymbol{\nu}}.$$

Let $\epsilon > 0$ be given. Since $\sum_j \|f_j\| < \infty$, there exists $N \in \mathbf{N}$ such that $\|f_j\| < \epsilon$ for $j \ge N$, whence

$$|c_{j,\boldsymbol{\nu}}| < \epsilon \qquad (j \ge N, \boldsymbol{\nu} \in \mathbf{N}^n).$$

For each $j \in \{0,\dots,N-1\}$, since $f_j \in T_n$, for all but a finite number of $\boldsymbol{\nu}$ we have $|c_{j,\boldsymbol{\nu}}| < \epsilon$. Varying j over $\{0,\dots,N-1\}$, we see that for all but a finite number of $(j,\boldsymbol{\nu}) \in \mathbf{N}\times\mathbf{N}^n$, $|c_{j,\boldsymbol{\nu}}| < \epsilon$. It follows that $|c_{\boldsymbol{\nu}}| < \epsilon$ for all but a finite number of $\boldsymbol{\nu} \in \mathbf{N}^n$. In other words $\lim_{|\boldsymbol{\nu}|\to\infty} |c_{\boldsymbol{\nu}}| = 0$, whence $\sum_{\boldsymbol{\nu}} c_{\boldsymbol{\nu}}\boldsymbol{\zeta}^{\boldsymbol{\nu}} \in T_n$. Let

$$f = \sum_{\boldsymbol{\nu}} c_{\boldsymbol{\nu}}\boldsymbol{\zeta}^{\nu}.$$

It is easy to see that $\sum_j f_j$ converges to f. □

2. The valuation ring associated with $(K,|\cdot|)$

Throughout this section K is a non-archimedean field with a non-trivial absolute value.

2.1. **The ring $\mathscr{O}_K$.** Let

$$\begin{aligned} \mathscr{O}_K &= \{x \in K \mid |x| \le 1\}, \\ \mathfrak{m}_K &= \{x \in K \mid |x| < 1\}, \\ \widetilde{K} &= \mathscr{O}_K/\mathfrak{m}_K. \end{aligned} \tag{2.1.1}$$

We therefore have a canonical surjective map

$$\mathscr{O}_K \longrightarrow \widetilde{K}. \tag{2.1.2}$$

The image of any element $x \in \mathscr{O}_K$ in $\widetilde{K}$ is denoted $\widetilde{x}$.

If $0 \ne x \in K$, then either x or x^{-1} lies in $\mathscr{O}_K$, whence $\mathscr{O}_K$ is a valuation ring in K with $\mathfrak{m}_K$ its unique maximal ideal. Since the absolute value on K is assumed to be non-trivial, $\mathfrak{m}_K \ne 0$. The field $\widetilde{K}$ is called the *residue field* of K as well as the residue field of $\mathscr{O}_K$.

Proposition 2.1.3. *If K is algebraically closed then $\mathscr{O}_K$ is non-noetherian.*

Proof. Let $x \in \mathfrak{m} \subset K$. Since K is algebraically closed, there exists $y \in K$ such that $y^2 = x$. It follows that $\mathfrak{m}_K^2 = \mathfrak{m}_K$. Moreover, since the absolute value on K is non-trivial, $\mathfrak{m}_K \ne 0$. By Nakayama's lemma, $\mathscr{O}_K$ is non-noetherian. □

2.2. **Behaviour with respect to extensions.** If L is a field extension of K, and L has a norm which extends the one on K, we often write $(L, |\cdot|_L)$ is an extension of $(K, |\cdot|_K)$, or simply $(L, |\cdot|)$ is an extension of $(K, |\cdot|)$. We also sometimes describe this by saying $(K, |\cdot|) \to (L, |\cdot|)$ is an extension of normed fields. It is an algebraic or finite extension if the underlying field extension is algebraic of finite. Suppose

$$(K, |\cdot|) \longrightarrow (L, |\cdot|)$$

is an extension of normed fields. Clearly $\mathscr{O}_K \subset \mathscr{O}_L$ and $\mathfrak{m}_L \cap \mathscr{O}_K = \mathfrak{m}_K$. Hence we have an extension of fields

$$\widetilde{K} \to \widetilde{L}.$$

Theorem 2.2.1. *Let $(L, |\cdot|)$ be an algebraic extension of $(K, |\cdot|)$. Then $\widetilde{L}$ is an algebraic extension of $\widetilde{K}$.*

Proof. Let $\widetilde{\theta} \in \widetilde{L}$ be an element, and $\theta \in \mathscr{O}_L$ a pre-image of θ. Let

$$f(X) = \sum_{i=0}^{d} a_i X^i$$

be a polynomial over K such that $f(\theta) = 0$ and $a_d = 1$. If all the a_i lie in $\mathscr{O}_K$, then $\widetilde{f}(\widetilde{\theta}) = 0$, where $\widetilde{f} = \sum_{i=0}^{d} \widetilde{a_i} X^i$, and since $\widetilde{a_d} = 1$, this shows $\widetilde{\theta}$ is algebraic. Otherwise, let $l \in \{0, \dots, d\}$ be an index such that $|a_i| \le |a_l|$ for all $i \in \{0, \dots, d\}$. Then $b_i = a_i/a_l$, $i = 0, \dots, d$ lie in $\mathscr{O}_K$. We have $g(X) = \sum_{i=0}^{d} b_i X^i \in \mathscr{O}_K[X]$, and since $b_l = 1$, this is a non-zero polynomial. Clearly $\widetilde{g}(\widetilde{\theta}) = 0$, where $\widetilde{g}(X) \in \widetilde{K}[X]$ has the obvious meaning. Thus $\widetilde{\theta}$ is algebraic. □

Theorem 2.2.2. *If K is algebraically closed, then $\widetilde{K}$ is algebraically closed.*

Proof. Let $\tilde{c} \in \overline{\widetilde{K}}$ and say its minimal polynomial over $\widetilde{K}$ is $\tilde{g} \in \widetilde{K}[X]$. Lift $\tilde{g}$ to a monic polynomial $g \in \mathscr{O}_K[X]$. Since K is algebraically closed, $g = \prod_i (X - c_i)$ for some $c_i \in K$. Since $\mathscr{O}_K$ is a valuation ring of K, the c_i lie in $\mathscr{O}_K$. Hence $\tilde{g} = \prod_i (X - \tilde{c}_i)$. Note that $\tilde{c}_i \in \widetilde{K}$, and $\tilde{c}$ is one of the $\tilde{c}_i$. Thus $\tilde{c} \in \widetilde{K}$. □

Theorem 2.2.3. *Let $\overline{K}$ be an algebraic closure of K and fix a norm on $\overline{K}$ such that $(\overline{K}, |\cdot|)$ is an extension of $(K, |\cdot|)$. Then $\widetilde{\overline{K}}$ is an algebraic closure of $\widetilde{K}$.*

Proof. This is immediate from Theorem 2.2.1 and Theorem 2.2.2 $\square$

Theorem 2.2.3 is often written succinctly as

$$\widetilde{\overline{K}} = \overline{\widetilde{K}}.$$

From now onwards we will do so, the implicit assumption being that algebraic closures have somehow been fixed. Recall that two algebraic closures of a field are isomorphic, but not uniquely isomorphic. In fact even the separable closures are not uniquely isomorphic, though they are isomorphic.

3. The ring T_n°

In this section K is a *complete* non-trivial and non-archimedean field and we set

$$k = \widetilde{K}.$$

3.1. Notations and definitions.

Let $n \in \mathbf{N}$. Set

$$T_n^\circ = \{f \in T_n \mid \|f\| \le 1\}. \tag{3.1.1}$$

Sometimes T_n° is also written as $\mathscr{O}_K\langle\zeta_1, \dots, \zeta_n\rangle$ and we might have occasion to write it in this manner. The reason for the alternative notation is clear; if $f = \sum_{\boldsymbol{\nu}} c_{\boldsymbol{\nu}}\boldsymbol{\zeta}^{\boldsymbol{\nu}}$, then $c_{\boldsymbol{\nu}} \in \mathscr{O}_K$ for all $\boldsymbol{\nu} \in \mathbf{N}^n$.

If $f = \sum_{\boldsymbol{\nu}} c_{\boldsymbol{\nu}}\boldsymbol{\zeta}^{\boldsymbol{\nu}}$, then for all but a finite number of $c_{\boldsymbol{\nu}}$, we have $|c_{\boldsymbol{\nu}}| < 1$, for $\lim_{|\boldsymbol{\nu}|\to\infty} |c_{\boldsymbol{\nu}}| = 0$ and $|c_{\boldsymbol{\nu}}| \le 1$ for all $\boldsymbol{\nu} \in \mathbf{N}^n$. In other words, $c_{\boldsymbol{\nu}} \in \mathscr{O}_K$ and for all but a finite number of $\boldsymbol{\nu}$, $c_{\boldsymbol{\nu}} \in \mathfrak{m}_K$. We therefore have a natural ring homomorphism:

$$\pi_n \colon T_n^\circ \longrightarrow k[\zeta_1, \dots, \zeta_n].$$

The preferred notation in the subject is

$$\widetilde{f} = \pi_n(f) \qquad (f \in T_n^\circ). \tag{3.1.2}$$

3.2. The maximum modulus principle.

Consider the "unit disc" in $\overline{K}^n$,

$$\mathbb{B}^n(\overline{K}) = \left\{(x_1, \dots, x_n) \in \overline{K}^n \,\middle|\, |x_i| \le 1,\, 1 \le i \le n\right\}.$$

Theorem 3.2.1. (The Maximum Modulus Principle) *Let $f \in T_n$. Then $|f(x_1, \dots, x_n)|$ attains a maximum in $\mathbb{B}^n(\overline{K})$ and*

$$\|f\| = \max\left\{|f(\mathbf{x})| \,\middle|\, \mathbf{x} \in \mathbb{B}^n(\overline{K})\right\}.$$

Proof. First, from Problem 6 of HW 1, we know that $f(\mathbf{x})$ makes sense for $\mathbf{x} \in \mathbb{B}^n(\overline{K})$. Without loss of generality we may assume $\|f\| = 1$. Suppose $f = \sum_{\boldsymbol{\nu}} c_{\boldsymbol{\nu}}\boldsymbol{\zeta}^{\boldsymbol{\nu}}$. Since $|f| = 1$, each $|c_{\boldsymbol{\nu}}| \le 1$, whence for every $(x_1, \dots, x_n) \in \mathbb{B}^n(\overline{K})$,

$$|f(x_1, \dots, x_n)| \le \left|\sum_{\boldsymbol{\nu}} c_{\boldsymbol{\nu}} x_1^{\nu_1} \dots x_n^{\nu_n}\right| \le 1$$

since each $|c_{\boldsymbol{\nu}} x_1^{\nu_1} \dots x_n^{\nu_n}| \le 1$. Thus

$$\sup\left\{|f(\mathbf{x})| \,\middle|\, \mathbf{x} \in \mathbb{B}^n(\overline{K})\right\} \le 1.$$

Since $\|f\| = 1$, there is a $\boldsymbol{\nu}_0 \in \mathbf{N}^n$ such that $|c_{\boldsymbol{\nu}_0}| = 1$, whence $\widetilde{f}$ is a non-zero polynomial, for $\widetilde{c}_{\boldsymbol{\nu}_0}$ is a non-zero coefficient in the expansion of $\widetilde{f} \in k[\boldsymbol{\zeta}]$. By

Theorem 2.2.3, k is algebraically closed and hence by the Hilbert Nullstellensatz (for example), there exists $\widetilde{\mathbf{x}} = (\widetilde{x_1}, \ldots, \widetilde{x_n}) \in k^n$ such that $\widetilde{f}(\widetilde{\mathbf{x}}) \neq 0$. Pick any pre-image $\mathbf{x} \in \mathbb{B}^n(\overline{K})$ of $\widetilde{\mathbf{x}}$. Then $\widetilde{f(\mathbf{x})} = \widetilde{f}(\widetilde{\mathbf{x}}) \neq 0$. Hence $|f(\mathbf{x})| = 1$. This proves the supremum displayed above is attained at $\mathbf{x} \in \mathbb{B}^n(\overline{K})$, and hence is the maximum asserted. $\square$