Aug 20, 2019.

Lecture 3

Let K be complete and non-archimedean.

A K-algebra B is called a normed algebra if the underlying K vector space has a norm $\|\cdot\| : B \longrightarrow [0,\infty)$ satisfying $\|ab\| = \|a\|\cdot\|b\|$ $\forall a,b \in B$. A normed algebra is a Banach algebra if the underlying normed space is a Banach space.

The Tate algebra: Let T_n be the K-subalgebra of $K[[\zeta_1,\dots,\zeta_n]]$ consisting of elements

$$f = \sum_{\nu\in\mathbb{N}^n} c_\nu \zeta^\nu = \sum_{\nu\in\mathbb{N}^n} c_{\nu_1\dots\nu_n}\zeta_1^{\nu_1}\cdots\zeta_n^{\nu_n} \in K[[\zeta_1,\dots,\zeta_n]]$$

satisfying the condition

$$\lim_{|\nu|\to\infty} |c_\nu| = 0.$$

Here $|\nu| = \nu_1+\dots+\nu_n$.

It is easy to verify that T_n is a K-algebra, in fact a K-subalgebra of $K[[\zeta_1,\dots,\zeta_n]]$. From your HW, there is another description of T_n, namely the collection of formal power series $f = \sum_{\nu\in\mathbb{N}^n} c_\nu\zeta^\nu$ such that f converges on every point of

$$B^n(\bar{K}) = \{(x_1,\dots,x_n)\in\bar{K}^n \mid |x_i|\le 1,\ i=1,\dots,n\}.$$

For this reason elements of T_n are often called strictly convergent formal power series, or restricted power series.

T_n is called the n^{th} Tate algebra and is often denoted $K\langle\zeta_1,\dots,\zeta_n\rangle$.

T_n has a norm on it, the so called Gauss norm which makes it into a Banach algebra. The Gauss norm is defined as follows:

For $\displaystyle f = \sum_{\nu \in \mathbb{N}^n} c_\nu \zeta^\nu = \sum_{\nu \in \mathbb{N}^n} c_{\nu_1 \ldots \nu_n} \zeta_1^{\nu_1} \ldots \zeta_n^{\nu_n} \in T_n$

$$\|f\| = \max_{\nu \in \mathbb{N}^n} |c_\nu|.$$

Since $\lim_{|\nu| \to \infty} |c_\nu| = 0$, the maximum on the right side makes sense.

It is not hard to see that $\|\cdot\| : T_n \longrightarrow k$ is a norm. To check T_n is a Banach algebra with this norm we have to check that

$$\|fg\| = \|f\| \cdot \|g\| \qquad \text{for } f, g \in T_n.$$

To that end, first suppose $\|f\| = \|g\| = 1$. Note that $\|fg\| \leq 1$. Suppose $f = \sum c_\nu \zeta^\nu$ and $g = \sum d_\nu \zeta^\nu$. Suppose $\sigma, \tau \in \mathbb{N}^n$ are such that $|c_\sigma| = |d_\tau| = 1$. Consider the coefficient θ of $\zeta_1^{\sigma_1+\tau_1} \zeta_2^{\sigma_2+\tau_2} \ldots \zeta_n^{\sigma_n+\tau_n}$ in fg. Then $\theta = \sum_{\nu+\mu=\sigma+\tau} c_\nu d_\mu$. Now $|c_\nu d_\mu| \leq 1$, and $|c_\sigma d_\tau| = 1$, and $c_\sigma d_\tau$ is one of the summands in the sum $\sum_{\nu+\mu=\sigma+\tau} c_\mu d_\nu$. It follows that $|\theta| = |c_\sigma d_\tau| = 1$. Hence $\|fg\| = 1$. Now suppose f and g are general. We may assume both are non-zero. Hence $f = c\Phi$, $g = d \cdot \Psi$ where $c, d \in k$, $\Phi, \Psi \in T_n$, $|c| = \|f\|$, $|d| = \|g\|$, $\|\Phi\| = 1$, $\|\Psi\| = 1$. Since $\|\Phi\Psi\| = 1$ from the argument we gave above, we have $\|fg\| = |cd| \cdot \|\Phi\Psi\| = |cd| = |c| \cdot |d| = \|f\| \cdot \|g\|$.

We will prove T_n is a Banach algebra next time.

Some homological algebra

All the action below is taking place in an abelian category $\mathcal{A}$.

Mapping Cones: This is one of the fundamental tools in homological alg.

Recall that a map of complexes (i.e., a chain map) of A-modules

$$\varphi^\bullet : M^\bullet \longrightarrow N^\bullet$$

is a quasi-isomorphism (i.e., $H^j(\varphi^\bullet)$ is an isomorphism for all $j \in \mathbb{Z}$) if and only if the mapping cone $C^\bullet_\varphi$ is exact where

$$C^n_\varphi = M^{n+1} \oplus N^n$$

and

$$\partial^n_{C_\varphi} = \begin{pmatrix} -\partial^{n+1}_M & 0 \\ \varphi^{n+1} & \partial^n_N \end{pmatrix}$$

Check:

$$\begin{pmatrix} -\partial^{n+2}_M & 0 \\ \varphi^{n+2} & \partial^{n+1}_N \end{pmatrix}\begin{pmatrix} -\partial^{n+1}_M & 0 \\ \varphi^{n+1} & \partial^n_N \end{pmatrix} = \begin{pmatrix} -\partial^{n+2}_M\partial^{n+1}_M & -\partial^{n+2}_M \cdot 0 + 0 \cdot \partial^n_N \\ \underbrace{-\varphi^{n+2}\partial^{n+1}_M + \partial^{n+1}_N\varphi^{n+1}} & \partial^{n+1}_N\partial^n_N \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

zero because $\varphi^\bullet$ is map of cplxes

This follows from the exact sequence of complexes

$$0 \longrightarrow N^\bullet \longrightarrow C^\bullet_\varphi \longrightarrow M^\bullet[1] \longrightarrow 0$$

and the fact that in the resulting long exact sequence of homologies, the connecting maps are $\pm H^n(\varphi)$.

Double Complexes: These are bigraded objects $\{D^{p,q}\}$ with horizontal and vertical differentials:

$$\partial^{p,q}_h : D^{p,q} \longrightarrow D^{p+1,q}, \quad \partial^{p,q}_v : D^{p,q} \longrightarrow D^{p,q+1}, \quad p,q \in \mathbb{Z}$$

satisfying the following two conditions

1. $(D^{\bullet,q}, \partial_h)$ and $(D^{p,\bullet}, \partial_v)$ are complexes for each $p, q \in \mathbb{Z}$.

2. The diagram

$$\begin{tikzcd} D^{p,q+1} \arrow[r, "\partial_h"] & D^{p+1,q+1} \\ D^{p,q} \arrow[u, "\partial_v"] \arrow[r, "\partial_h"'] & D^{p+1,q} \arrow[u, "\partial_v"'] \end{tikzcd}$$

commutes for every $p, q \in \mathbb{Z}$.

Assuming the existence of countable direct sums, or alternately that for fixed $n \in \mathbb{N}$, all but a finite number of $D^{p,q}$ with $p+q = n$ are zero, we can form a complex $(\mathrm{Tot}^\bullet(D), \partial)$ called the total complex of $D^{\bullet\bullet}$ given by

$$\mathrm{Tot}^n(D) = \bigoplus_{p+q=n} D^{p,q}$$

with

$$\partial^n = \sum_{p+q=n} \left\{ \partial_h^{p,q} + (-1)^p \partial_v^{p,q} \right\}$$

It is not hard to see that $(\mathrm{Tot}^\bullet(D), \partial)$ is a complex.

Remark: The above is called the total complex associated to a commuting double complex. If the rectangles in 2 anti-commute for every p, q, then $\partial^n = \sum_{p+q=n} \{ \partial_h^{p,q} + \partial_v^{p,q} \}$ would give us a complex on $\mathrm{Tot}^\bullet(D)$, where again $\mathrm{Tot}^n(D) = \bigoplus_{p+q=n} D^{p,q}$. The two theories are completely equivalent. In fact if we have a commuting double complex, then one way of getting an anti-commuting double complex (the way we have implicitly followed) is to change the differentials on every odd vertical column to its negative. And we can do the same thing to move from an anti-commuting double complex to a commuting double complex. The total complexes after these transforms will be identical.

The n^{th} translate: For a complex $C^\bullet$ and an integer n, $C^\bullet[n]$ is the complex

$$\left(C^\bullet[n]\right)^p = C^{n+p}$$

$$\partial^p_{C^\bullet[n]} = (-1)^n \, \partial^{n+p}_C$$

An sub-complex of $\mathrm{Tot}^\bullet(D)$:

If $D^{\bullet\bullet}_{\geq n}$ is the double complex whose $(p,q)^{th}$ term is $D^{p,q}$ if $p \geq n$ and equal to zero otherwise, and if $T^\bullet = \mathrm{Tot}^\bullet(D)$, $T^\bullet_{\geq n} = \mathrm{Tot}^\bullet(D_{\geq n})$ then it is easy to see that $T^\bullet_{\geq n}$ is a subcomplex of $T^\bullet$. Similarly $D^{\bullet\bullet}_{\leq n}$ is the double complex whose $(p,q)^{th}$ term is $D^{p,q}$ if $p \leq n$ and zero otherwise, and $T^\bullet_{\leq n} = \mathrm{Tot}^\bullet(D_{\leq n})$. For each n we have an exact sequence of complexes:

$$0 \longrightarrow T^\bullet_{\geq n} \longrightarrow T^\bullet \longrightarrow T^\bullet_{\leq n-1} \longrightarrow 0.$$

Note that $T^\bullet_{\leq n}$ is in general not a subcomplex of $T^\bullet$, but is always a quotient complex of $T^\cdot$.

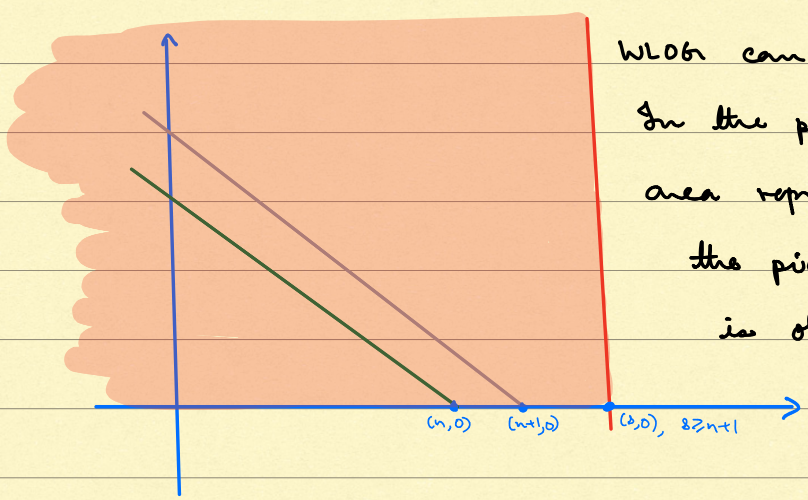
We write $D^{\cdot\cdot,\leq n}$ (resp. $D^{\cdot\cdot,\geq n}$) for the double complex whose $(p,q)^{th}$ term is $D^{p,q}$ for $q \leq n$ (resp. $q \geq n$) and is zero otherwise, and set $T^{\cdot\leq n} = \mathrm{Tot}^\cdot(D^{\leq n})$, $T^{\cdot\geq n} = \mathrm{Tot}^\cdot(D^{\geq n})$. (The notation is decidedly awkward and I have to think of a better one.)

$D^{\cdot\cdot}$ is **bounded on the left** if $\exists\, m$ such that $D^{\cdot\cdot} = D^{\cdot\cdot}_{\geq m}$, in which case we say it is bounded on the left by m. I will leave it to you to define bounded on the right, bounded below, bounded above etc along the same lines.

Lemma: Let $D^{\cdot\cdot}$ be bounded below by m. Then $\forall n \in \mathbb{N}$ we have:

$$H^n(T^\cdot) = H^n(T^\cdot_{\leq s}) \qquad \forall\, s \geq n-m+1.$$

Proof:



WLOG can assume $m = 0$. In the picture, the shaded area represents $D^{\cdot\cdot}_{\leq s}$. From the picture the assertion is obvious.

Proposition: Let $D^{\bullet\bullet}$ be bounded below and to the left. If every column of $D^{\bullet\bullet}$ is exact, then $T^\bullet = \mathrm{Tot}^\bullet(D)$ is exact.

Proof:

Let us first prove the statement for $D^{\bullet\bullet}$ bounded to the right (in addition to the other boundedness hypotheses it satisfies). WLOG we may $D^{\bullet\bullet}$ is bounded on the left by 0 and below by 0. Let m be such that $D^{\bullet\bullet}_{\leq m} = D^{\bullet\bullet}$. We proceed by induction on m. Let $C^\bullet_p = D^{p,\bullet}$, the p^{th} column of $D^{\bullet\bullet}$. Then $T^\bullet_{\geq m} = C^\bullet_m[m]$ and we have an exact sequence of complexes

$$0 \longrightarrow C^\bullet_m[-m] \longrightarrow T^\bullet \longrightarrow T^\bullet_{\leq m-1} \longrightarrow 0.$$

By hypothesis $C^\bullet_m[-m]$ is exact, and by induction hypothesis $T^\bullet_{\leq n-1}$ is exact (the base case of $m=0$ is obvious). The associated long exact sequence in cohomology shows that $T^\bullet$ is exact.

This proves the result when $D^{\bullet\bullet}$ is assumed to be bounded on the right. The general case follows from the lemma above, since $H^n(T^\bullet) = H^n(T^\bullet_{\leq n+1})$ and $D^{\bullet\bullet}_{\leq n+1}$ is bounded on the right, whence $H^n(T^\bullet_{\leq n+1}) = 0$. q.e.d.

Remarks: ① Using the fact that cohomology commutes with direct limits, it follows that the Proposition is true even when $D^{\bullet\bullet}$ is not bounded on the left. However, the statement about direct limits in an arbitrary abelian category is not standard, though it is proven in courses for categories of modules over a ring.

② Clearly we have an analogous result for complexes whose rows are exact. In other words if $D^{\bullet\bullet}$ is bdd below and to the left and every row is exact, then $T^{\bullet} = \mathrm{Tot}^{\bullet}(D)$ is exact.

Let $D^{\bullet\bullet}$ be bounded below by (say) m and bounded on the left, and for each n, let $C_n^{\bullet}$ be the n^{th} column of $D^{\bullet\bullet}$, i.e., $C_n^{\bullet} = (D^{n,\bullet}, \partial_v)$. Suppose

$$H^j(C_n^{\bullet}) = 0 \qquad \forall\, j > m.$$

Set

$$K^{n+m} = H^m(C_n^{\bullet}).$$

The map $\partial_h^{n,m} : D^{n,m} \longrightarrow D^{n+1,m}$ induces a map (by "restriction")

$$\partial_K^{n+m} : K^{n+m} \longrightarrow K^{n+m+1}$$

making $(K^{\bullet}, \partial_K)$ into a complex.

Consider the graded map

$$\phi^{\bullet} : K^{\bullet} \longrightarrow T^{\bullet}$$

given at level p by the composite

$$K^p \hookrightarrow D^{p-m,m} \hookrightarrow T^p.$$

It is clear that ϕ is a map of complexes since ∂_K is the restriction of ∂_h to $K^{\bullet}$, and $\partial_v(K^{\bullet}) = 0$.

Proposition: In the above situation, the map $\varphi^\bullet : K^\bullet \longrightarrow T^\bullet$ is a quasi-isomorphism.

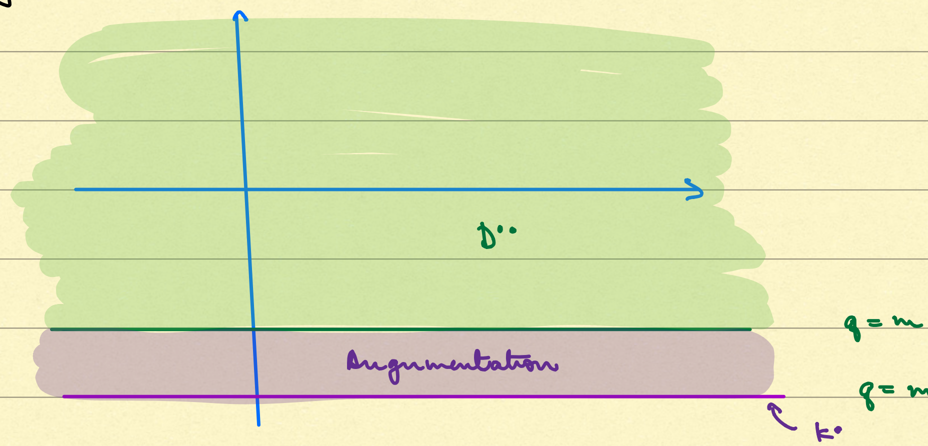
Proof:

Let $\tilde{D}^{\bullet\bullet}$ be the double complex

$$\tilde{D}^{p,q} = \begin{cases} D^{p,q} & \text{if } q \neq m-1 \\ K^{p+m} & \text{if } q = m-1 \end{cases}$$

with vertical differential and horizontal differential the same as that of $D^{\bullet\bullet}$ for $q \geq m$, and for $q = m-1$ the vertical differential is the natural inclusion $K^{p+m} \hookrightarrow D^{p,m}$. The horizonal diff'l on $\tilde{D}^{p,m-1}$ is ∂_K^{p+m}.

In other words $\tilde{D}^{\bullet\bullet}$ is built from $D^{\bullet\bullet}$ by augumenting it with an extra row at level $q = m-1$, namely the complex $K^\bullet$.



Let $\tilde{T}^\bullet = \mathrm{Tot}^\bullet(\tilde{D})$. From the previous proposition $\tilde{T}^\bullet$ is exact since the columns of $\tilde{D}^{\bullet\bullet}$ are exact. One checks that $\tilde{T}$ is isomorphic to the mapping cone $C^\bullet_\varphi$ of $\varphi^\bullet : K^\bullet \longrightarrow T^\bullet$. Thus $\varphi^\bullet$ is a quasi-isomorphism as required. q.e.d.

Remark: Clearly the analogous statement for rows is also true.

Čech to derived functor cohomology

Let X be a topological space and $\mathcal{U} = (U_i)$ an open cover. Then for every p and every sheaf of abelian groups $\mathcal{F}$ on X there is a natural map

$$\check{H}^p(\mathcal{U}, \mathcal{F}) \longrightarrow H^p(X, \mathcal{F}) \qquad \text{——— } (*)$$

where the left side is the p^{th} Čech cohomology w.r.t. $\mathcal{U}$ and the right side is p^{th} derived functor cohomology.

Recall how $(*)$ is defined in (for example) Hartshorne's Algebraic Geometry. First, if $\mathscr{C}^\bullet(\mathcal{U}, \mathcal{F})$ is the sheaf Čech complex, where $\Gamma(V, \mathscr{C}^p(\mathcal{U}, \mathcal{F})) := C^p(\mathcal{U} \cap V, \mathcal{F}|_V)$ for every V open in X, then we have a resolution

$$\psi_{\mathcal{F}} : \mathcal{F} \longrightarrow \mathscr{C}^\bullet(\mathcal{U}, \mathcal{F}).$$

Now let

$$\phi : \mathcal{F} \longrightarrow \mathcal{I}^\bullet$$

be an injective resolution of $\mathcal{F}$. Then, as $\mathcal{I}^\bullet$ is an injective resolution of $\mathcal{F}$, we have a homotopy unique map

$$f : \mathscr{C}^\bullet(\mathcal{U}, \mathcal{F}) \longrightarrow \mathcal{I}^\bullet$$

which lifts the identity map on $\mathcal{F}$. Taking global sections we get a map of complexes of abelian groups

$$\Gamma(X, f) : C^\bullet(\mathcal{U}, \mathcal{F}) \longrightarrow \Gamma(X, \mathcal{I}^\bullet)$$

Taking cohomology we get (*)

It is well-known and easy to see that if $\mathcal{E}$ is an injective sheaf, i.e. an injective object in the category Sh_X of sheaves of abelian groups on X, then $\mathcal{C}^p(\mathcal{U},\mathcal{E})$ is injective for each p, whence

$$\psi_{\mathcal{E}} : \mathcal{E} \longrightarrow \mathcal{C}^\bullet(\mathcal{U},\mathcal{E})$$

is an injective resolution of $\mathcal{E}$.

Returning to our injective resolution

$$\phi : \mathcal{F} \longrightarrow \mathcal{J}^\bullet,$$

we have a double complex $\mathcal{D}^{\bullet\bullet}$ of sheaves given by

$$\mathcal{D}^{p,q} = \mathcal{C}^p(\mathcal{U},\mathcal{J}^q)$$

The q^{th} row is an injective resolution of $\mathcal{J}^q$ and the p^{th} column is an injective resolution of $\mathcal{C}^p(\mathcal{U},\mathcal{F})$.

$$\begin{tikzcd}
{} & {} & & {} & \\
\mathcal{J}^q \arrow[u] \arrow[r, "\text{via } \psi"] & \mathcal{C}^0(\mathcal{U},\mathcal{J}^q) \arrow[u] \arrow[r] & \cdots \arrow[r] & \mathcal{C}^p(\mathcal{U},\mathcal{J}^q) \arrow[u] \arrow[r] & \cdots \\
\vdots \arrow[u] & \vdots \arrow[u] & \mathcal{D}^{\bullet\bullet} & \vdots \arrow[u] & \\
\mathcal{J}^0 \arrow[u] \arrow[r] & \mathcal{C}^0(\mathcal{U},\mathcal{J}^0) \arrow[u] \arrow[r] & \cdots \arrow[r] & \mathcal{C}^p(\mathcal{U},\mathcal{J}^0) \arrow[u] \arrow[r] & \cdots \\
\mathcal{F} \arrow[u, dashed] \arrow[ur, dashed] \arrow[r, dashed] & \mathcal{C}^0(\mathcal{U},\mathcal{F}) \arrow[u] \arrow[r] & \cdots \arrow[r] & \mathcal{C}^p(\mathcal{U},\mathcal{F}) \arrow[u] \arrow[r] & \cdots \arrow[u, "\text{via } \phi"']
\end{tikzcd}$$

We therefore have a diagram of quasi-isomorphisms

$$\begin{array}{ccc} \mathcal{I}^\bullet & \longrightarrow & \mathcal{J}^\bullet \\ \uparrow & & \uparrow \\ \mathcal{F} & \longrightarrow & \mathcal{C}^\bullet(\mathcal{U},\mathcal{F}) \end{array}$$

where $\mathcal{J}^\bullet = \mathrm{Tot}^\bullet(\mathcal{D})$.

Since

$$\begin{array}{ccc} \mathcal{F} & \longrightarrow & \mathcal{I}^\bullet \\ \| & & \downarrow \\ \mathcal{F} & \longrightarrow & \mathcal{J}^\bullet \end{array}$$

commutes and $\mathcal{I}^\bullet$ and $\mathcal{J}^\bullet$ are injective resolutions of $\mathcal{F}$, we have a homotopy unique homotopy inverse $\mathcal{J}^\bullet \xrightarrow{\theta} \mathcal{I}^\bullet$ which is a quasi-isomorphism. (More generally, if $\mathcal{I}^\bullet \longrightarrow \mathcal{E}^\bullet$ is a quasi-isomorphism between bdd below injective complexes in an abelian category with enough injectives, then it has a homotopy unique homotopy inverse which is a quasi-isomorphism.) It follows that the composite

$$\mathcal{C}^\bullet(\mathcal{U},\mathcal{F}) \longrightarrow \mathcal{J}^\bullet \xrightarrow{\theta} \mathcal{I}^\bullet$$

is, up to homotopy equivalence, the same as $f: \mathcal{C}^\bullet(\mathcal{U},\mathcal{F}) \longrightarrow \mathcal{I}^\bullet$.

Theorem: In the above situation, set $U_{\alpha_0 \ldots \alpha_p} = U_{\alpha_0} \cap \ldots \cap U_{\alpha_p}$. Suppose $\mathcal{F}$ is such that $H^j(U_{\alpha_0 \ldots \alpha_p}, \mathcal{F}) = 0$ $\forall j \geq 1$ and all $(\alpha_0, \ldots, \alpha_p)$. Then the map (*)

$$H^p(\mathcal{U},\mathcal{F}) \longrightarrow H^p(X,\mathcal{F})$$

is an isomorphism $\forall p \geq 0$.

Proof:

Let $D^{\bullet\bullet} = \Gamma(X, \mathcal{D}^{\bullet\bullet})$. Recall $D^{\bullet\bullet}$ is:

$$
\begin{tikzcd}
\uparrow & \uparrow & & \uparrow & \\
\mathcal{J}^q \arrow[r, "\text{via } \psi"] & \mathcal{C}^0(\mathcal{U},\mathcal{J}^q) \arrow[r] \arrow[u] & \cdots \arrow[r] & \mathcal{C}^p(\mathcal{U},\mathcal{J}^q) \arrow[r] \arrow[u] & \cdots \\
\vdots \arrow[u] & \vdots \arrow[u] & \mathcal{D}^{\bullet\bullet} & \vdots \arrow[u] & \\
\mathcal{J}^0 \arrow[r] \arrow[u] & \mathcal{C}^0(\mathcal{U},\mathcal{J}^0) \arrow[r] \arrow[u] & \cdots \arrow[r] & \mathcal{C}^p(\mathcal{U},\mathcal{J}^0) \arrow[r] \arrow[u] & \cdots \\
\mathcal{F} \arrow[r, dashed] \arrow[u, dashed] \arrow[ur, dashed] & \mathcal{C}^0(\mathcal{U},\mathcal{F}) \arrow[r] \arrow[u] & \cdots \arrow[r] & \mathcal{C}^p(\mathcal{U},\mathcal{F}) \arrow[r] \arrow[u] & \cdots \arrow[u, "\text{via } \phi"']
\end{tikzcd}
$$

The global sections of the above gives $D^{\bullet\bullet}$. The qth row of $D^{\bullet\bullet}$ is

$$C^0(\mathcal{U},\mathcal{J}^q) \longrightarrow C^1(\mathcal{U},\mathcal{J}^q) \longrightarrow \cdots \longrightarrow C^p(\mathcal{U},\mathcal{J}^q) \longrightarrow \cdots$$

Since $\mathcal{J}^q \longrightarrow \mathcal{C}^\bullet(\mathcal{U},\mathcal{J}^q)$ is an injective resolution, and since $\mathcal{J}^q$ is Γ-acyclic, being injective, we have

$$0 = H^p(X,\mathcal{J}^q) \underset{\uparrow}{=} H^p(\Gamma(X, \mathcal{C}^\bullet(\mathcal{U},\mathcal{J}^q))) = H^p(C^\bullet(\mathcal{U},\mathcal{J}^q)) \quad \forall\, p \geq 1.$$

Since $\mathcal{C}^\bullet(\mathcal{U},\mathcal{J}^q)$ is an injective resln of $\mathcal{J}^q$.

and $$H^0(C^\bullet(\mathcal{U},\mathcal{J}^q)) = \Gamma(X,\mathcal{J}^q).$$

Let $T^\bullet = \mathrm{Tot}^\bullet(D)$.

From what we have proved,

$$\Gamma(X, \mathcal{J}^\bullet) \longrightarrow T^\bullet$$

is a quasi-isomorphism. This was the row-wise analysis. Now let us examine the columns of $D^{\bullet\bullet}$. The pth column is

$$\prod_{(\alpha_0,\dots,\alpha_p)} \Gamma(U_{\alpha_0\dots\alpha_p}, \mathcal{J}^0) \longrightarrow \prod_{(\alpha_0\dots\alpha_p)} \Gamma(U_{\alpha_0\dots\alpha_p}, \mathcal{J}^1) \longrightarrow \cdots \longrightarrow \prod_{(\alpha_0\dots\alpha_p)} \Gamma(U_{\alpha_0\dots\alpha_p}, \mathcal{J}^q) \to \cdots$$

where the differentials are induced on each factor by $\mathcal{J}^q \longrightarrow \mathcal{J}^{q+1}$. Since $\mathcal{J}^\bullet|_{U_{\alpha_0 \dots \alpha_p}}$ is an injective resolution of $\mathcal{F}|_{U_{\alpha_0 \dots \alpha_p}}$, we then have that the q^{th} cohomology of the p^{th} column of $D^{\bullet\bullet}$ is

$$\prod H^q(U_{\alpha_0 \dots \alpha_p}, \mathcal{F}) = \begin{cases} 0 & \text{if } q \geq 1 \quad \text{(by hypothesis on } \mathcal{F}\text{)} \\ C^p(\mathcal{U}, \mathcal{F}) & \text{if } q = 0 \end{cases}$$

It then follows that

$$C^\bullet(\mathcal{U}, \mathcal{F}) \longrightarrow T^\bullet$$

is also a quasi-isomorphism.

Since $T^\bullet = \Gamma(X, \mathcal{J}^\bullet)$, the map $\Gamma(X, \theta) : T^\bullet \longrightarrow \Gamma(X, \mathcal{J}^\bullet)$ is a homotopy inverse of $\Gamma(X, \mathcal{J}^\bullet) \longrightarrow T^\bullet$.

$$\begin{tikzcd}
\Gamma(X, \mathcal{J}^\bullet) \arrow[rr] & & T^\bullet \arrow[ll, bend left, "\Gamma(X,\theta)"] \\
& & C^\bullet(\mathcal{U}, \mathcal{F}) \arrow[u] \arrow[ull, "\Gamma(X,f)"]
\end{tikzcd}$$

commutative up to homotopy.

Since $C^\bullet(\mathcal{U}, \mathcal{F}) \longrightarrow T^\bullet$ and $\Gamma(X, \theta)$ are quasi-isomorphisms, so is $\Gamma(X, f)$. This proves the theorem. q.e.d.