LECTURE 22

Date of Lecture: November 5, 2019

Let $\mathcal{A}b$ denote the category of abelian groups. Fix a Grothendieck topology $(\mathscr{C}, \mathscr{C}ov)$. $\mathcal{P}sh$ and $\mathcal{S}h$ denote the category of presheaves and sheaves (of abelian groups) on $(\mathscr{C}, \mathscr{C}ov)$. The functor

$$i\colon \mathcal{S}h \longrightarrow \mathcal{P}sh$$

is the forgetful functor and

$$(\ )^{\#}\colon \mathcal{P}sh \longrightarrow \mathcal{S}h$$

is the sheafification functor. (See §§2.1 of Lectures 18 and 19.)

The symbol ☡ is for flagging a cautionary comment or a tricky argument. It occurs in the margins and is Knuth's version of Bourbaki's "dangerous bend symbol".

1. The Čech to derived functor map

1.1. Let $\mathscr{F} \in \mathcal{S}h$ and let $\mathscr{F} \to \mathscr{E}^\bullet$ be an injective resolution of $\mathscr{F}$ in $\mathcal{S}h$. Let $U \in \mathscr{C}$ and $\mathfrak{U} = \{U_\alpha\} \in \mathscr{C}ov(U)$. We have an obvious double complex

$$C^{\bullet\bullet} := C^\bullet(\mathfrak{U},\, i(\mathscr{E}^\bullet)). \tag{1.1.1}$$

By [Lectures 18-19, Proposition 2.3.2], $i(\mathscr{E}^p)$ is an injective presheaf for every $p \in \mathbf{N}$. By [*ibid*, Proposition 1.2.6], the q^{th}-row of $C^{\bullet\bullet}$ is a resolution of $\Gamma(U, \mathscr{E}^q)$. Moreveover if we apply H^0 to the p^{th}-column of $C^{\bullet\bullet}$ we get $C^p(\mathfrak{U}, i(\mathscr{F}))$, i.e. $\mathrm{H}^0(C^{p\bullet}) = C^p(\mathfrak{U}, i(\mathscr{F}))$. The upshot is we have a diagram of complexes

$$\begin{array}{ccc} \Gamma(U,\, \mathscr{E}^\bullet) & \xrightarrow[\psi^\bullet]{\text{quasi-isomorphism}} & \operatorname{Tot} C^{\bullet\bullet} \\ & & \uparrow{\scriptstyle \varphi^\bullet} \\ & & C^\bullet(\mathfrak{U},\, \mathscr{F}) \end{array} \tag{1.1.2}$$

We have therefore have the so called *Čech to derived functor maps with respect to* $\mathfrak{U}$, one for each $n \in \mathbf{N}$, given by

$$\rho_n\colon H^n(\mathfrak{U},\, \mathscr{F}) \longrightarrow H^n(U,\, \mathscr{F}) \qquad (n \in \mathbf{N}) \tag{1.1.3}$$

where $\rho_n = \mathrm{H}^n(\psi^\bullet)^{-1} \circ \mathrm{H}^n(\varphi^\bullet)$. On the other hand, in [Lecture 21, (1.2.2.2)] we defined another Čech to derived functor map

$$\check{\rho}_n\colon \check{H}^n(U,\, \mathscr{F}) \longrightarrow H^n(U,\, \mathscr{F}) \qquad (n \in \mathbf{N}). \tag{1.1.4}$$

We leave it to the reader to check that the following diagram commutes

$$
\begin{array}{ccc}
H^n(\mathfrak{U},\,\mathscr{F}) & & \\
 & \overset{\rho_n}{\searrow} & \\
\text{natural}\downarrow & & H^n(U,\,\mathscr{F}) \\
 & \underset{\check{\rho}_n}{\nearrow} & \\
\check{H}^n(U,\,\mathscr{F}) & &
\end{array}
\tag{1.1.5}
$$

The following theorem of Serre (re-phrased in terms of Grothendieck topologies and derived functors) is very useful.

Theorem 1.1.6. *Let $\mathscr{F} \in \mathscr{Sh}$. Suppose J is a collection of objects in $\mathscr{C}$ with the following properties:*

(i) *If U, V, and W are objects in J and $U \to W$, $V \to W$ are morphisms in $\mathscr{C}$, then $U \times_W V$ is in J.*

(ii) *For every U in J, and every $\mathfrak{U} \in \mathscr{C}ov(U)$, there exists $\mathfrak{V} \in \mathscr{C}ov(U)$ such that $\mathfrak{V}$ is a refinement of $\mathfrak{U}$ and the members of $\mathfrak{V}$ lie in J.*

(iii) *$\check{H}^n(U,\mathscr{F}) = 0$ for every U in J.*

Then $H^n(U,\,\mathscr{F}) = 0$ for U in J and $n \geq 1$.

Proof. Let $\mathscr{F} \to \mathscr{E}^\bullet$ be an injective resolution of sheaves and $\mathscr{I}^{\bullet\bullet}$ be a Cartan-Eilenberg resolution of $i(\mathscr{E}^\bullet)$ in $\mathscr{Psh}$.

According to [Lecture 21, Proposition 1.2.3] and property (iii) of objects in J, $H^1(U,\mathscr{F}) = 0$ for every U in J. Let $n > 1$ and assume

$$
H^q(V,\mathscr{F}) = 0 \qquad (V \text{ in } J,\ 1 \leq q \leq n-1). \tag{IH}
$$

We have to prove that $H^n(V,\,\mathscr{F}) = 0$ for all V in J.

Let U be an object in J. Let $D^{\bullet\bullet}(= D_U^{\bullet\bullet}) = \check{H}^0(U,\,\mathscr{I}^{\bullet\bullet})$. According to the discussion in [Lecture 21, 1.2.2], we have a diagram

$$
\begin{array}{ccc}
\Gamma(U,\,\mathscr{E}^\bullet) & \xrightarrow[\check{\psi}^\bullet]{\text{quasi-isomorphism}} & \operatorname{Tot} D^{\bullet\bullet} \\
 & & \uparrow{\scriptstyle \check{\varphi}^\bullet} \\
 & & \check{H}^0(U,\,\mathscr{E}^\bullet)
\end{array}
$$

and $\check{\rho}_n = \mathrm{H}^n(\check{\psi}^\bullet)^{-1} \circ \mathrm{H}^n(\check{\varphi}^\bullet)$. As in [Lecture 21, (1.2.2.1)], we write E_2^{pq} for the abelian group $H_{II}H_I^{pq}$ associated with $D^{\bullet\bullet}$, to bring it more in line with standard notations. Once again by the discussion in [Lecture 21, 1.2.2] we see that

$$
E_2^{pq} \xrightarrow{\sim} \check{H}^p(U,\,\mathrm{R}^q i(\mathscr{F})).
$$

By Problem (5) of HW 6, it is enough to show that $E_2^{pq} = 0$ for $p+q = n$, $p, q \in \mathbf{N}$. According to [Lecture 21, Proposition 1.2.1] and property (iii) of objects in J we have $E_2^{0n} = E_2^{n0} = 0$. It remains to show that $E_2^{pq} = 0$ for $(p,q) \in S$ where

$$
S = \{(p,q) \in \mathbf{N} \times \mathbf{N} \mid p+q = n,\ 1 \leq q \leq n-1\}.
$$

Let $\mathscr{C}ov_J(U)$ denote the sub-collection of $\mathscr{C}ov(U)$ consisting of covers made up of members of J. According to property (ii) of objects in J, $E_2^{pq} = \check{H}^p(U, \mathrm{R}^q i(\mathscr{F}))$ is the direct limit of $H^p(\mathfrak{U},\, \mathrm{R}^q i(\mathscr{F}))$ as $\mathfrak{U}$ varies over $\mathscr{C}ov_J(U)$. It is therefore enough for us to show that

$$(*) \qquad H^p(\mathfrak{U},\, \mathrm{R}^q i(\mathscr{F})) = 0 \qquad ((p,q) \in S,\, \mathfrak{U} \in \mathscr{C}ov_J(U)).$$

So suppose $1 \leq q \leq n-1$. Let $\mathfrak{U} = \{U_i\}_{i \in I} \in \mathscr{C}ov_J(U)$. By property (i) for objects in J, the product $U_{i_o \dots i_p} := U_{i_0} \times_U \cdots \times_U U_{i_p}$ lies in J for every $(i_0, \dots, i_p) \in I^{p+1}$. Therefore for $p \in \mathbf{N}$ we have,

$$\begin{aligned} C^p(\mathfrak{U},\, \mathrm{R}^q i(\mathscr{F})) &= \prod_{(i_0,\dots,i_p)} \Gamma(U_{i_0 \dots i_p},\, \mathrm{R}^q i(\mathscr{F})) \\ &= \prod_{(i_0,\dots,i_p)} H^q(U_{i_0 \dots i_p},\, \mathscr{F}) \\ &= 0 \qquad\qquad \text{(by (IH), since } U_{i_0 \dots i_p} \text{ is in } J). \end{aligned}$$

The assertion in $(*)$ follows immediately. □

Theorem 1.1.7. *Let $\mathscr{F} \in \mathcal{S}h$, $U \in \mathscr{C}$ and $\mathfrak{U} \in \mathscr{C}ov(U)$. Suppose for every finite sequence $U_0, \dots, U_p$ of members of $\mathfrak{U}$ we have $H^q(U_{i_0 \dots i_p},\, \mathscr{F}) = 0$ for $q \geq 1$, where, as usual, $U_{i_0 \dots i_p} := U_{i_0} \times_U \cdots \times_U U_{i_p}$. Then*

$$H^n(\mathfrak{U},\, \mathscr{F}) \xrightarrow[\text{(1.1.3)}]{\sim} H^n(U,\, \mathscr{F}), \qquad (n \in \mathbf{N}).$$

Proof. Let $\mathscr{F} \to \mathscr{E}^\bullet$ be an injective resolution in $\mathcal{S}h$. Let $C^{\bullet\bullet} = C^\bullet(\mathfrak{U},\, i(\mathscr{E}^\bullet))$ as in (1.1.1). The p^{th}-th column of $C^{\bullet\bullet}$ is the product of the complexes $\Gamma(U_{i_0 \dots i_p}, \mathscr{E}^\bullet)$ and hence by definition of derived functor cohomology, the q^{th}-th cohomology of the p^{th}-th column is the product of $H^q(U_{i_0 \dots i_p},\, \mathscr{F})$. This is zero by hypothesis for $q \geq 1$ and equals $C^p(\mathfrak{U},\, \mathscr{F})$ for $q = 0$. Therefore, the map $\varphi^\bullet$ in diagram (1.1.2) is a quasi-isomorphism. The assertion follows by definition of the map in (1.1.3) (i.e. of the map ρ_n). □

As an immediate corollary of Theorems 1.1.6 and 1.1.7 we have:

Corollary 1.1.8. *Let J be a collection of objects in $\mathscr{C}$ satisfying* (i), (ii)*, and* (iii) *in the hypotheses of Theorem* 1.1.6*. Let $U \in \mathscr{C}$.*

(a) *Let $\mathfrak{U} \in \mathscr{C}ov(U)$ and suppose that all the members of $\mathfrak{U}$ are in J. Then*

$$H^n(\mathfrak{U},\, \mathscr{F}) \xrightarrow[\text{(1.1.3)}]{\sim} H^n(U,\, \mathscr{F}) \qquad (n \in \mathbf{N}).$$

(b) *If every member of $\mathscr{C}ov(U)$ has a refinement consisting of objects in J then*

$$\check{H}^n(U,\, \mathscr{F}) \xrightarrow[\text{(1.1.4)}]{\sim} H^n(U,\, \mathscr{F}) \qquad (n \in \mathbf{N}).$$

Proof. Part (a) is obvious from Theorems 1.1.6 and 1.1.7, and part (b) follows from (a) by using (1.1.5), hypothesis (ii) on J, and taking direct limits.

Remark 1.1.9. In classical topology, part (b) of the Corollary is due to Serre and Cartan and was essential for computing cohomology of projective space as Serre did in FAC. In somewhat greater detail, Cartan's Theorem (B) is the assertion that for any pseudoconvex holomophic domain U in $\mathbf{C}^n$ and any coherent sheaf $\mathscr{F}$ on U, $\check{H}^n(U,\, \mathscr{F}) = 0$. Such domains are also called Stein domains (and generalise to Stein manifolds). Serre proved the analogue of Cartan's Theorem (B) for affine varieties (the history is tangled because the two of them spoke to each other regularly, and Cartan mentions Serre in his papers and attributes some results to him). All this was before derived functors came about. In any case, from today's point of view (and this is perhaps a little revisionist), for Cartan, J was the collection of Stein open domains in a complex manifold and $\mathscr{F}$ was a coherent analytic sheaf, and for Serre, J was the collection of affine open subvarieties of a quasi-projective variety, and $\mathscr{F}$ was a coherent algebraic sheaf, and both focussed on the isomorphism (b) in the Corollary.