LECTURE 21

Date of Lecture: October 31, 2019

Let $\mathcal{A}b$ denote the category of abelian groups. Fix a Grothendieck topology $(\mathscr{C}, \mathscr{C}ov)$. $\mathscr{P}sh$ and $\mathscr{S}h$ denote the category of presheaves and sheaves (of abelian groups) on $(\mathscr{C}, \mathscr{C}ov)$.

The symbol ☡ is for flagging a cautionary comment or a tricky argument. It occurs in the margins and is Knuth's version of Bourbaki's "dangerous bend symbol".

1. Sheafications

1.1. As in §§2.1 of Lectures 18 and 19, let $i\colon \mathscr{S}h \to \mathscr{P}sh$ be the forgetful functor and $(\ )^{\#}\colon \mathscr{P}sh \to \mathscr{S}h$ the sheafifcation functor. Recall that the notion of kernels and cokernels in $\mathscr{P}sh$ are such that for a complex $\mathscr{P}^\bullet$ in $\mathscr{P}sh$,

$$(\mathrm{H}^q(\mathscr{P}^\bullet))(U) = \mathrm{H}^q(\mathscr{P}^\bullet(U)) \qquad (U \in \mathscr{C}). \tag{1.1.1}$$

Lemma 1.1.2. *Let $\mathscr{F} \in \mathscr{S}h$ and $q \in \mathbf{N}$. Then*

$$\mathrm{R}^q i(\mathscr{F}) = \{U \mapsto H^q(U, \mathscr{F})\}.$$

Proof. Let $\mathscr{F} \to \mathscr{E}^\bullet$ be an injective resolution of $\mathscr{F}$ in $\mathscr{S}h$. Using (1.1.1) we see that

$$\mathrm{R}^q i(\mathscr{F})(U) = (\mathrm{H}^q(i(\mathscr{E}^\bullet)))(U) = \mathrm{H}^q(\mathscr{E}^\bullet(U)) = H^q(U, \mathscr{F})$$

for every $U \in \mathscr{C}$. □

Proposition 1.1.3. *Let $\mathscr{F}$ be a sheaf. Then $(\mathrm{R}^q i(\mathscr{F}))^{\#} = 0$ for $q \geq 1$.*

Proof. We know that $(\ )^{\#} \circ i = \mathbf{1}_{\mathscr{S}h}$ (see (2.2.5) of the notes on Lectures 18 and 19). Thus $\mathrm{R}^q((\ )^{\#} \circ i) = 0$ for $q \geq 1$. Since $(\ )^{\#}$ is exact, $\mathrm{R}^q((\ )^{\#} \circ i) = (\ )^{\#} \circ \mathrm{R}^q i = (\mathrm{R}^q i)^{\#}$. This proves the proposition. □

1.2. Čech to derived functor (first steps). We begin with a result which has important consequences.

Proposition 1.2.1. *Let $\mathscr{F}$ be a sheaf. Then $\check{H}^0(U, \mathrm{R}^q i(\mathscr{F})) = 0$ for every $q \geq 1$ and every $U \in \mathscr{C}$.*

Proof. First note that since $\mathscr{P}^+$ is separated for $\mathscr{P} \in \mathscr{P}sh$, the natural map from $\mathscr{P}^+$ to $\mathscr{P}^{++}$ is an inclusion, $\mathscr{P}^+ \hookrightarrow \mathscr{P}^{++}$.

Let $U \in \mathscr{C}$ and $q \geq 1$. Proposition 1.1.3 gives:

$$\check{H}^0(U, \mathrm{R}^q i(\mathscr{F})) = (\mathrm{R}^q i(\mathscr{F}))^+(U) \hookrightarrow (\mathrm{R}^q i(\mathscr{F}))^{++}(U) = (\mathrm{R}^q i(\mathscr{F}))^{\#}(U) = 0.$$

□

1.2.2. For $U \in \mathscr{C}$ we have $\check{H}^0(U, -) \circ i = \Gamma(U, -)$. Fix $\mathscr{F} \in \mathscr{S}h$, as well as an injective resolution $\mathscr{F} \to E^\bullet$ in the category of sheaves. Let $I^{\bullet\bullet}$ be a Cartan-Eilenberg resolution of $i(E^{\bullet\bullet})$ in $\mathscr{P}sh$, and $D^{\bullet\bullet} = \check{H}^0(U, I^{\bullet\bullet})$.

We now switch to more standard notation in place of $H_{II}H_I^{pq}$ associated to $D^{\bullet\bullet}$. Let

$$E_2^{pq} := H_{II}H_I^{pq} \tag{1.2.2.1}$$

where the right side is the iterated cohomology group associated with $D^{\bullet\bullet}$, as in HW 4, HW 6. E_2^{pq} is technically the "E_2-term of the spectral sequence associated with $D^{\bullet\bullet}$".

The discussion in [Lecture 20, §§1.2], especially Corollary 1.2.4 and §§§1.2.5 of *loc.cit.* applies to our situation if we set $F = i$ and $G = \check{H}^0(U, -)$. Recall that $\check{H}^p(U, -)$, $p \in \mathbf{N}$, are the right derived functors of $\check{H}^0 \colon \mathscr{P}\!sh \to \mathcal{A}b$ (see Theorem 1.2.8 of notes on Lectures 18 and 19). Thus we have natural isomorphisms

$$E_2^{pq} \xrightarrow{\ \sim\ } \check{H}^p(U, \mathrm{R}^q i(\mathscr{F}))$$

and

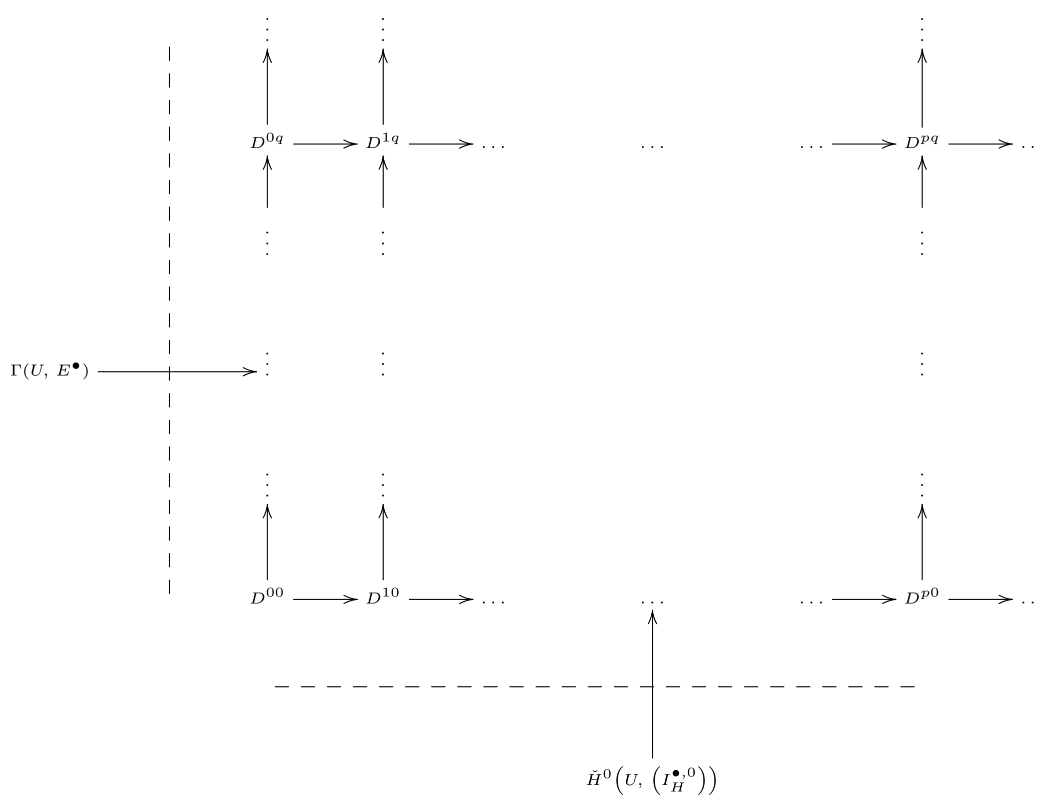
$$H^n(U, \mathscr{F}) \xrightarrow{\ \sim\ } \mathrm{H}^n(\mathrm{Tot}\, D^{\bullet\bullet})$$

via [Lecture 20, Corollary 1.2.4 and (1.2.5.2)]. Finally, by [Lecture 20, (1.2.6.1)] we have maps, one for each $n \in \mathbf{N}$,

$$\check{H}^n(U, \mathscr{F}) \longrightarrow H^n(U, \mathscr{F}) \qquad (n \in \mathbf{N}). \tag{1.2.2.2}$$

These are the *Čech to derived functor cohomology maps.*

The important picture to keep in mind is the following with the understanding that the horizontal arrow passing through the vertical broken line is a quasi-isomorphism.



Here the complex $I_H^{\bullet 0}$ is the injective resolution in $\mathcal{Psh}$ of $\mathrm{H}^0(i(E^\bullet)) = i(\mathscr{F})$ coming from the Cartan-Eilenberg resolution $I^{\bullet\bullet}$ of $i(E^\bullet)$. In other words, $I_H^{p0} = \mathrm{H}^0(I^{p\bullet}) = \ker(I^{p0} \to I^{p1})$, $p \in \mathbf{N}$.

The following result is well known in classical topology.

Proposition 1.2.3. *Let $\mathscr{F}$ be a sheaf and $U \in \mathscr{C}$. Then* (1.2.2.2) *gives an isomorphism*

$$\check{H}^1(U, \mathscr{F}) \xrightarrow{\sim} H^1(U, \mathscr{F}).$$

Proof. According to Proposition 1.2.1, $E_2^{01} = 0$. The assertion follows from Remark 1.2.6 of Lecture 20, or, what amounts to the same thing, from Problem (6) of HW 6. □