LECTURE 20

Date of Lecture: October 29, 2019

Let $\mathcal{A}b$ denote the category of abelian groups.

The symbol ☡ is for flagging a cautionary comment or a tricky argument. It occurs in the margins and is Knuth's version of Bourbaki's "dangerous bend symbol".

1. Cartan-Eilenberg resolutions

Fix an abelian category $\mathscr{A}$ with enough injectives.

1.1. The Horseshoe Lemma.

The following result is fundamental, though easy to prove.

Lemma 1.1.1. *Let*

$$0 \to A \longrightarrow B \longrightarrow C \to 0$$

be an exact sequence in $\mathscr{A}$, and $A \to E_A^\bullet$ and $C \to E_C^\bullet$ injective resolutions in $\mathscr{A}$. Then there exists an injective resolution $B \to E_B^\bullet$ and an exact sequence of complexes

$$0 \to E_A^\bullet \longrightarrow E_B^\bullet \longrightarrow E_C^\bullet \to 0. \tag{†}$$

Proof. Write ∂_A^p and ∂_C^p for the p^{th}-coboundary maps in $E_A^\bullet$ and $E_C^\bullet$. Since $E_A^\bullet$ and $E_C^\bullet$ are injective complexes, if $E_B^\bullet$ exists as in the assertion, then necessarily E_B^p is the direct sum $E_A^p \oplus E_C^p$. Therefore set

$$E_B^p = E_A^p \oplus E_C^p \qquad (p \in \mathbf{N}).$$

We have to find maps $\partial_B^p \colon E_B^p \to E_B^{p+1}$ such that the resulting complex $E_B^\bullet$ resolves B and fits into the sequence (†) making it exact. At each level $p \in \mathbf{N}$ we have a split exact sequence

$$0 \to E_A^p \xrightarrow{\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)} E_B^p \xrightarrow{(0\ 1)} E_C^p \to 0.$$

Since E_A^0 is an injective object and A is a subobject of B, the map $A \to E_A^0$ extends (in perhaps many ways) to B giving us a map $\varphi \colon B \to E_A^0$. Let $\psi \colon B \to E_C^0$ be the composite $B \to C \to E_C^0$. It is clear that the following diagram with exact rows commutes:

$$\begin{tikzcd}
0 \arrow[r] & E_A^0 \arrow[r, "\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)"] & E_A^0 \oplus E_C^0 \arrow[r, "(0\ 1)"] & E_C^0 \arrow[r] & 0 \\
0 \arrow[r] & A \arrow[r] \arrow[u] & B \arrow[r] \arrow[u, "\left(\begin{smallmatrix}\varphi\\\psi\end{smallmatrix}\right)"] & C \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

It is easy to check that the middle vertical arrow is injective. Thus we have an exact sequence $0 \to B \to E_B^0$. Let $A^0 = \operatorname{coker} A \to E_A^0$, $B^0 = \operatorname{coker} B \to E_B^0$, and

$C^0 = \operatorname{coker} C \to E_C^0$. Then we have a short exact sequence (use the snake lemma on the above commutative diagram with exact rows)

$$0 \to A^0 \longrightarrow B^0 \longrightarrow C^0 \to 0.$$

Repeating the argument we gave earlier, since E_A^1 is an injective object, we have a map $\varphi^0 \colon B^0 \to E_A^1$ extending the natural map $A^0 \to E_A^1$ and a map $\psi^0 \colon B^0 \to E_C^1$ which is the composite $B^0 \to C^0 \to E_C^1$. Repeating earlier arguments one notes that

$$B^0 \xrightarrow{\left(\begin{smallmatrix}\varphi^0\\ \psi^0\end{smallmatrix}\right)} E_B^1$$

is injective and that the diagram below, whose rows are exact, commutes.

$$\begin{tikzcd}
0 \arrow[r] & E_A^1 \arrow[r, "\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)"] & E_A^1 \oplus E_C^1 \arrow[r, "(0\ 1)"] & E_C^0 \arrow[r] & 0 \\
0 \arrow[r] & A^0 \arrow[r] \arrow[u] & B^0 \arrow[r] \arrow[u, "\left(\begin{smallmatrix}\varphi^0\\ \psi^0\end{smallmatrix}\right)"] & C^0 \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

Set $\partial_B^0 \colon E_B^0 \to E_B^1$ to be the composite

$$E_C^0 \twoheadrightarrow B^0 \xrightarrow{\left(\begin{smallmatrix}\varphi^0\\ \psi^0\end{smallmatrix}\right)} E_B^1.$$

Then $0 \to B \to E_B^0 \to E_B^1$ is exact. The process can be repeated ad infinitum. For example, set A^1, B^1, C^1 to be the cokernels of $A^0 \to E_A^1$, $B^0 \to E_B^1$, and $C^0 \to E_C^1$ respectively. Then $A^1 \hookrightarrow E_A^2$, $C^1 \hookrightarrow E_C^2$ and we can find appropriate $\varphi^1 \colon B^1 \to E_A^2$ and $\psi^1 \colon B^1 \to E_C^2$ and set ∂_B^1 to be the composite of $E_B^1 \twoheadrightarrow B^1$ followed by $\begin{pmatrix}\varphi^1\\ \psi^1\end{pmatrix} \colon B^1 \to E_B^2$. One checks that $H^1(0 \to E_B^0 \to E_B^1 \to E_B^2 \to 0) = 0$.

A standard induction argument then gives the result. $\square$

1.2. **Cartan-Eilenberg resolutions.** The following is a cut and paste from another set of notes (on spectral sequences) that I wrote. So the notations may not be consistent with what we had in the lecture. Note that our injective resolutions in class were "vertical". Here they are horizontal, and indeed, given other conventions, having them horizontal is better.

Suppose $\mathscr{A}$ is an abelian category with enough injectives, and $C^\bullet$ a bounded below complex in $\mathscr{A}$, say $C^q = 0$ if $q < q_0$. One can find a double-complex $I^{\bullet,\bullet}$ (whose total complex is written $I^\bullet$) of injectives in $\mathscr{A}$ and maps $\varepsilon^q \colon C^q \to I^{0,q}$ fitting into the diagram below satisfying the following:

1. $I^{p,q} = 0$ if either $p < 0$ or $q < q_0$.
2. The horizontal rows are exact, i.e., for each $q \geq q_0$, $C^q \to I^{\bullet,q}$ is an injective resolution.
3. Let $I_Z^{p,q}$ be the kernel of the "vertical differential" $I^{p,q} \to I^{p,q+1}$. Then $I_Z^{\bullet,q}$ is an injective resolution of $Z^q = Z^q(C^\bullet)$, where $Z^q \to I_Z^{\bullet,q}$ is the natural map induced by $C^\bullet \to I^\bullet$.
4. Let $I_B^{p,q}$ be the image of the vertical differential $I^{p,q} \to I^{p,q+1}$. Then $I_B^{\bullet,q}$ is an injective resolution of $B^q = B^q(C^\bullet)$. The map $B^q \to I_B^{0,q}$ is (again) the natural map arising from $C^\bullet \to I^\bullet$.
5. Let $I_H^{p,q}$ be the q-th cohomology of the complex $I^{p,\bullet}$. Then $I_H^{\bullet,q}$ is an injective resolution of $H^q(C^\bullet)$ (again via $C^\bullet \to I^\bullet$).

$$\begin{tikzcd}
 & \vdots & \vdots & \vdots & \\
0 \arrow[r] & C^{q+1} \arrow[u] \arrow[r, "\varepsilon^{q+1}"] & I^{0,q+1} \arrow[u] \arrow[r] & I^{1,q+1} \arrow[u] \arrow[r] & \cdots \\
0 \arrow[r] & C^{q} \arrow[u] \arrow[r, "\varepsilon^{q}"] & I^{0,q} \arrow[u] \arrow[r] & I^{1,q} \arrow[u] \arrow[r] & \cdots \\
0 \arrow[r] & C^{q-1} \arrow[u] \arrow[r, "\varepsilon^{q-1}"] & I^{0,q-1} \arrow[u] \arrow[r] & I^{1,q-1} \arrow[u] \arrow[r] & \cdots \\
 & \vdots \arrow[u] & \vdots \arrow[u] & \vdots \arrow[u] &
\end{tikzcd}$$

Such a "resolution" of $C^\bullet$ always exists. It is by no means unique. It is called a *Cartan-Eilenberg resolution* of $C^\bullet$. Here is one way of building one.

Pick arbitrary injective resolutions for B^q and for $\mathrm{H}^p(C^\bullet)$, with the caveat that injective resolutions of zero objects will be chosen to be the zero injective resolution. Call these resolutions $I_B^{\bullet,q}$ and $I_H^{\bullet,q}$ respectively. Since

$$0 \to B^q \to Z^q \to H^q(C^\bullet) \to 0 \tag{$*$}$$

is a short exact sequence of objects, one can use the *Horseshoe Lemma* to get an injective resolution of $Z^{\bullet,q}$ of I_Z^q which fits into a short exact sequence of complexes

$$0 \to I_B^{\bullet,q} \to I_Z^{\bullet,q} \to I_H^{\bullet,q} \to 0 \tag{$\dagger$}$$

lifting $(*)$. Next we have an exact sequence

$$0 \to Z^q \to C^q \to B^{q+1} \to 0. \tag{$**$}$$

Since we have injective resolutions for the two ends of the short exact sequence, another application of the Horseshoe Lemma gives us an injective resolution $I^{\bullet,q}$ which fits into a short exact sequence of complexes

$$0 \to I_Z^{\bullet,q} \to I^{\bullet,q} \to I_B^{\bullet,q+1} \to 0 \tag{$\ddagger$}$$

lifting $(**)$. Note that since we dealing with injective modules in $(\dagger)$ and $(\ddagger)$ we have decompositions.

$$I^{p,q} = I_Z^{p,q} \oplus I_B^{p,q+1} = I_B^{p,q} \oplus I_H^{p,q} \oplus I_B^{p,q+1}. \tag{1.2.1}$$

It follows that for a fixed p the composite

$$I^{p,q} \twoheadrightarrow I_B^{p,q+1} \hookrightarrow I^{p,q+1} \tag{1.2.2}$$

gives a complex $I^{p,\bullet}$. In fact, as is easily checked, $I^{\bullet,\bullet}$ forms a double-complex and the notations we have used in the construction are consistent with the notations used in the list of requirements from a Cartan-Eilenberg resolution of $C^\bullet$.

The following (easy) Lemma is what gives us the Grothendieck spectral sequence.

Lemma 1.2.3. *Let $G\colon \mathscr{A} \to \mathscr{B}$ be an additive functor. Then for every pair of integers $(p,\, q)$ we have*

$$G(I_H^{p,q}) = \mathrm{H}^q(G(I^{p,\bullet})).$$

Proof. Since G is additive, it respects direct sums. Apply G to the decompositions in (1.2.1) to obtain

$$G(I^{p,q}) = G(I_B^{p,q}) \oplus G(I_H^{p,q}) \oplus G(I_B^{p,q+1}).$$

The q-th coboundary map for the complex $G(I^{p,\bullet})$ can be computed via (1.2.2), and it is the projection $G(I^{p,q}) \twoheadrightarrow G(_B^{p,q+1})$ followed by the inclusion $G(I_B^{p,q+1}) \hookrightarrow G(I^{p,q+1})$. From here the q-cocycles and the q-coboundaries in the complex $G(I^{p,\bullet})$ are easily seen to be $G(I_B^{p,q}) \oplus G(I_H^{p,q})$ and $G(I_B^{p,q})$ respectively, giving the lemma. $\square$

An immediate corollary is the following.

Corollary 1.2.4. *Suppose F above is left exact, $\mathscr{B}$ also has enough injectives and $G\colon \mathscr{B} \to \mathscr{C}$ is a left exact functor between abelian categories such that $F(I)$ is G-acyclic for every injective object I of $\mathscr{A}$. Let $A \in \mathscr{A}$, $A \to E^\bullet$ an injective resolution of A, and $I^{\bullet\bullet}$ a Cartan-Eilenberg resolution of $C^\bullet = F(E^\bullet)$. Let $D^{\bullet\bullet} = G(I^{\bullet\bullet})$ and let $H_{II}H_I^{pq}$ be as in Problems 3, 4 in HW 5. Then*

$$H_{II}H_I^{pq} \xrightarrow{\sim} (\mathrm{R}^pG \circ \mathrm{R}^qF)(A) \qquad (p, q \in \mathbf{N}).$$

Proof. By definition/construction of Cartan-Eilenberg resolutions, $I_H^{\bullet,q}$ is an injective resolution of $\mathrm{R}^qF(A)$. Lemma 1.2.3 gives $G(I_H^{\bullet,q}) = H_I^{\bullet,q}$, whence we have

$$\mathrm{R}^pG(\mathrm{R}^qF(A)) = \mathrm{H}^p(G(I_H^{\bullet,q})) = \mathrm{H}^p(H_I^{\bullet,q}) = H_{II}H_I^{pq}.$$

$\square$

1.2.5. Consider the situation in Corollary 1.2.4. We have a natural map of complexes (in fact an injective map of complexes)

$$\varphi^\bullet\colon F(E^\bullet) \to \mathrm{Tot}\,(I^{\bullet\bullet}) \tag{1.2.5.1}$$

given at the graded level by $F(E^q) \to I^{0,q} \hookrightarrow (\mathrm{Tot}\,(I^{\bullet\bullet}))^q$. Since the q^{th}-row of $I^{\bullet\bullet}$ resolves $F(E^q)$ (via φ^q), $\varphi^\bullet$ is a quasi-isomorphism. Since it is a quasi-isomorphism between G-acyclic complexes,

$$G(\varphi^\bullet)\colon GF(I^\bullet) \longrightarrow G(\mathrm{Tot}\,(I^{\bullet\bullet})) = \mathrm{Tot}\,(D^{\bullet\bullet})$$

is also a quasi-isomorphism. This is seen as follows (in the event you haven't seen the argument before). First, if $R^\bullet$ is an exact bounded below complex of G-acyclics, then $G(R^\bullet)$ is exact, as can be seen by breaking up $R^\bullet$ into short exact sequences, and noting that if the first two terms of a short exact sequence are G-acyclic, then so is the last term. The mapping cone $C_\varphi^\bullet$ of $\varphi^\bullet$ is exact since $\varphi^\bullet$ is a quasi-isomorphism. Moroever, $C_\varphi^\bullet$ consists of G-acyclics. Hence, setting $R^\bullet = C_\varphi^\bullet$ in the above, we see that $G(C_\varphi^\bullet)$ is exact. But clearly $G(C_\varphi^\bullet) = C_{G\varphi}^\bullet$. Hence $G(\varphi^\bullet)$ is a quasi-isomorphism. There are other ways of seeing this (for example use the fact that the q^{th}-row of $D^{\bullet\bullet}$ must resolve $GF(I^q)$ since $F(I^q)$ is G-acyclic). The net result is that one has isomorphisms

$$\mathrm{R}^n(GF)(A) \xrightarrow{\sim} \mathrm{H}^n(\mathrm{Tot}\,(D^{\bullet\bullet})) \qquad (n \in \mathbf{N}). \tag{1.2.5.2}$$

The map $\varphi^\bullet$ identifies $F(E^\bullet)$ as a sub-complex of $\mathrm{Tot}\,(I^{\bullet\bullet})$. This can be regarded as a map along the "y-axis" which forms an edge of $I^{\bullet\bullet}$. There is one along the "x-axis" too. Namely, the complex $I_H^{\bullet,0}$ which is an injective resolution of

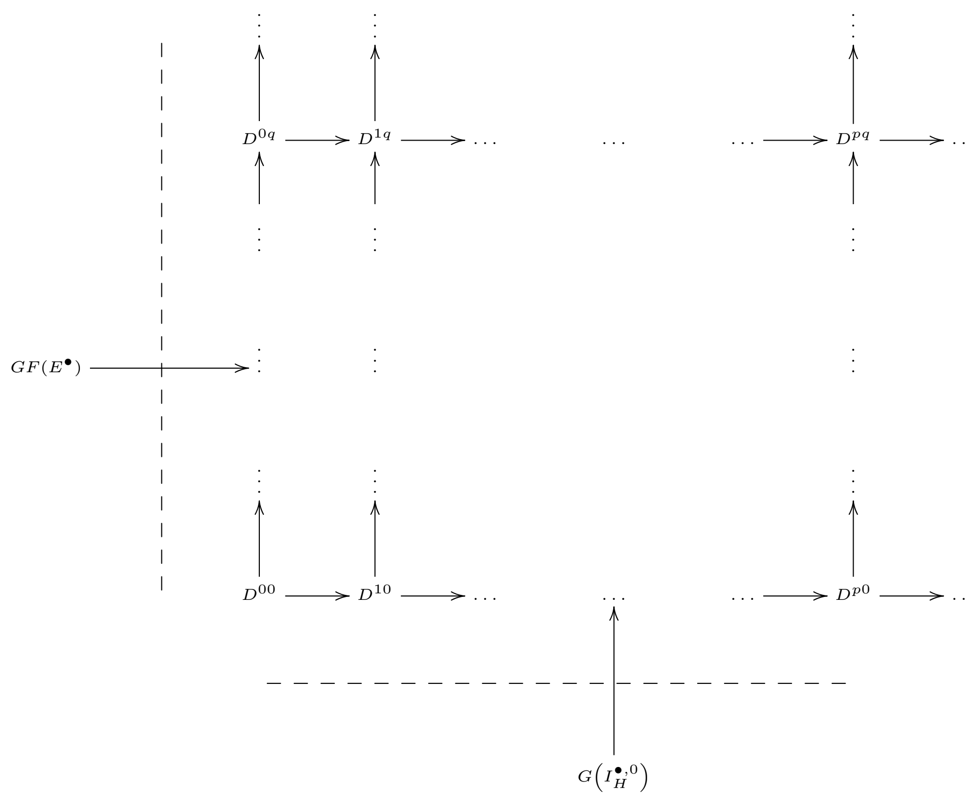
$\mathrm{H}^0(F(E^\bullet)) = F(A)$, by the definition/construction of a Cartan-Eilenberg resolution of $F(E^\bullet)$. We thus have a map of complexes

$$\psi^\bullet\colon I_H^{\bullet,0} \hookrightarrow \mathrm{Tot}\,(I^{\bullet\bullet}). \tag{1.2.5.3}$$

This need not be a quasi-isomorphism. But it gives rise to a map,

$$\mathrm{R}^n G(F(A)) \longrightarrow \mathrm{H}^n(\mathrm{Tot}\,(D^{\bullet\bullet})) \qquad (n \in \mathbf{N}). \tag{1.2.5.4}$$

The following diagram might help in keeping things clear in one's head (see comments below the diagram).



The double-complex $D^{\bullet\bullet}$ is within the dotted lines. The horizontal arrow which goes across the vertical dotted line on the left is a quasi-isomorphism and gives rise to the isomorphism (1.2.5.2). The vertical arrow which goes across the horizontal dotted line at the bottom need not be a quasi-isomorphism. It gives rise to (1.2.5.4).

Remark 1.2.6. In situation discussed in §§§1.2.5, $(1.2.5.1)^{-1} \circ (1.2.5.4)$ gives us a *edge homomorphism* (one of many which are so called)

$$R^n G(F(A)) \longrightarrow \mathrm{R}^n(GF)(A) \qquad (n \in \mathbf{N}). \tag{1.2.6.1}$$

According to Problem (6) of HW 6 and Corollary 1.2.4 above, if $\mathrm{R}^p G \mathrm{R}^q F(A) = 0$ for (p, q) such that $q \geq 1$ and $n - 1 \leq p + q \leq n$, then (1.2.6.1) is an isomorphism.