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Lecture 2

Let $(K, |\cdot|)$ be a normed field. If K is non-archimedean this is the same as a field with a valuation.

Two metrics $|\cdot|_1$ and $|\cdot|_2$ are said to be equivalent if they induce the same topology on K.

Recall that a metric is said to be trivial if $|K^\times| = \{1\}$. This is clearly non-archimedean.

Remark: The only metric equivalent to the trivial metric is the trivial metric. To see this, if $|\cdot|$ is trivial then the topology on K is discrete. If on the other hand $|\cdot|$ is non-trivial, we can find $a \in K$ with $0 < |a| < 1$. In this case $a^n \longrightarrow 0$ as $n \longrightarrow \infty$ since $|a^n| = |a|^n \longrightarrow 0$ as $n \longrightarrow \infty$. Thus the topology on K is not discrete.

Examples:

1. $K = \mathbb{R}$ or $\mathbb{C}$ with usual absolute value. These are complete and archimedean. In fact these are the only complete archimedian normed fields.
2. $K = \mathbb{Q}$. Let p be a prime. For $x \in \mathbb{Q}$ we have a unique $n \in \mathbb{Z}$ such that

$$x = p^n \frac{a}{b} \qquad (a,b) = 1, \quad (p, ab) = 1.$$

Then the formula

$$|x|_p := p^{-n}$$

defines an absolute value on $\mathbb{Q}$. This is non-archimedean.
We also have the usual absolute value $|\cdot|_\infty$.

Theorem (Ostrowski): If p, q are distinct primes then $|\cdot|_p$ and $|\cdot|_q$ are not equivalent. Further, every non-trivial absolute value on $\mathbb{Q}$ is equivalent to $|\cdot|_p$ for some $p \leq \infty$.

Proof: Will not be doing it. Look up Basic Alg II, Jacobson (or any other source). Might develop it as a series of HW exercises if I have the time.

Proposition: $|\cdot|_1$ and $|\cdot|_2$ are equivalent if and only if $\exists\ s > 0$ such that $|\cdot|_1 = |\cdot|_2^s$.

Proof:

If $|\cdot|_1 = |\cdot|_2^s$ for some $s > 0$, clearly $|\cdot|_1$ and $|\cdot|_2$ are equivalent.

To prove the other way it is enough to assume neither $|\cdot|_1$ nor $|\cdot|_2$ are trivial, since we have already adressed the "trivial case" in one of the remarks made above.

So suppose $|\cdot|_1$ & $|\cdot|_2$ are non-trivial and equivalent. Let $x \in K$ be s.t. $|x|_1 < 1$. Then $|x^n|_1 = |x|_1^n \longrightarrow 0$ as $n \longrightarrow \infty$. Since $|\cdot|_1$ and $|\cdot|_2$ are equivalent, this means $x_2^n \longrightarrow 0$ as $n \longrightarrow \infty$, i.e., $|x|_2 < 1$. By symmetry and the fact that for $a \neq 0$, $\left|\frac{1}{a}\right|_j = \frac{1}{|a|_j}$, $j = 1, 2$ we see that:

$$\left.\begin{aligned} |x|_1 < 1 &\iff |x|_2 < 1 \\ |x|_1 > 1 &\iff |x|_2 > 1 \\ |x|_1 = 1 &\iff |x|_2 = 1 \end{aligned}\right\} (*)$$

Fix $a_0 \in K$ with $|a_0| > 1$.

Set $\quad s = \dfrac{\log |a_0|_1}{\log |a_0|_2}$.

We claim that

$$|x|_1 = |x|_2^{\,s} \qquad \forall\ x \in K \quad \text{———} \ (**)$$

It is enough to prove $(**)$ for $|x|_1 > 1$, and we will assume that x satisfies this. Let

$$t = \frac{\log |x|_1}{\log |a_0|_1} \quad \text{and} \quad t' = \frac{\log |x|_2}{\log |a_0|_2}.$$

Then $\quad |a_0|_1^{\,t} = |x|_1 \quad$ and $\quad |a_0|_2^{\,t'} = |x|_2$. ——— $(***)$

We claim $t = t'$. Suppose not. Say $t < t'$. Let r be a rational number s.t. $t < r < t'$. Let $r = \frac{m}{n}$, where $m, n \in \mathbb{Z}$. From $(***)$ and the fact that $|a_0|_j > 1$, $|x|_j > 1$, we have

$$|x|_1 = |a_0|_1^{\,t} < |a_0|^{m/n} \quad \text{and} \quad |x|_2 = |a_0|_2^{\,t'} > |a_0|^{m/n}.$$

This means

$$\left|\frac{x^n}{a_0^m}\right|_1 < 1 \quad \text{and} \quad \left|\frac{x^n}{a_0^m}\right|_2 > 1.$$

This contradicts $(*)$. So $t \geq t'$. By symmetry $t' \geq t$, whence $t = t'$. This gives the following:

$$\frac{\log |x|_1}{\log |a_0|_1} = \frac{\log |x|_2}{\log |a_0|_2},$$

whence

$$\log |x|_1 = s \cdot \log |x|_2$$

i.e.,
$$|x|_1 = |x|_2^{\,s}$$

as required.

May use $|\cdot|_K$ for the absolute value on $|\cdot|$

Remarks: In what follows K is non-archimedean with a non-trivial absolute values (the latter is probably not needed, but I didn't want to deal with annoying exceptions which might creep up). $\hat{K}$ = completion of K.

1. If L/K is a finite extension, the number of extensions of $|\cdot|_K$ to L is equal to the number of prime ideals in $L \otimes_K \hat{K}$.

2. If K is complete and L/K is algebraic then the extension of $|\cdot|$ to L is unique — this is clear from considering finite extensions and using 1. (since $K = \hat{K}$).
If further L/K is finite, then L is complete w.r.t. absolute value induced by $|\cdot|_K$.

3. Let $\bar{K}$ be an algebraic closure of K, where K is complete. Then $\hat{\bar{K}}$ is also algebraically closed. (Lovely theorem! We won't be proving it.)

4. Suppose $|\cdot|_1, \ldots, |\cdot|_n$ are the extensions of $|\cdot|_K$ to L. Let L_i be the completion of L w.r.t. $|\cdot|_i$. Then there is a natural map $L \otimes_K \hat{K} \longrightarrow \prod_{i=1}^{n} L_i$ which is surjective and

such that the kernel is the Jacobson radical of $L\otimes_K \hat{K}$. In fact that is how 1. is proved.

equal to the nilradical since $L\otimes_K \hat{K}$ is a finite $\hat{K}$-algebra, whence a product of artin local rings.

Important Example:

Let $\mathbb{Q}_p$ be the completion of $\mathbb{Q}$ w.r.t. $|\cdot|_p$, $\bar{\mathbb{Q}}_p$ its algebraic closure, and let us continue to denote the unique extension of $|\cdot|_p$ to $\bar{\mathbb{Q}}_p$ by $|\cdot|_p$.

$$\mathbb{Q} \longrightarrow \mathbb{Q}_p \longrightarrow \overline{\mathbb{Q}}_p$$

$|\cdot|_p$ extends uniquely since $\mathbb{Q}_p$ is the completion of $\mathbb{Q}$ w.r.t. $\overline{\mathbb{Q}}_p$

$|\cdot|_p$ extends uniquely since $\mathbb{Q}_p$ is complete & $\bar{\mathbb{Q}}_p/\mathbb{Q}_p$ is algebraic.

Let $\mathbb{C}_p$ be completion of $\bar{\mathbb{Q}}_p$ (w.r.t. $|\cdot|_p$ of course). Then from our remarks above, $\mathbb{C}_p$ is algebraically closed. In fact it is (non-canonically) isomorphic to $\mathbb{C}$. It is often considered the ideal analogue of $\mathbb{C}$ in the p-adic world. However it has an important drawback. It is not spherically complete a notion we now define. Spherical completeness also goes by the name of maximally complete.

Definition: $(K, |\cdot|)$ is spherically complete or maximally complete if each nested sequence of balls $B_1 \supset B_2 \supset \dots \supset B_n \supset \dots$ has a non-empty intersection.

The Hahn-Banach theorem need not be true in non-archimedean functional analysis — in fact it fails for Banach spaces over $\mathbb{C}_p$. However if K is spherically complete (note that K has to be then complete) then the usual proof of Hahn-Banach works.

To make sense of all this we introduce an important topic. Before we do let us make the following

Convention: From now onwards K is complete, $|\cdot|$ non-trivial, non-archimedean, unless otherwise stated.

Normed spaces and Banach spaces over K

Definition: (i) A **normed space** over K is a vector space E over K equipped with a map (the "norm") $\|\cdot\| : E \longrightarrow K$ such that

Don't really need K to be complete or non-trivial. But keeping those ...

- $\|x\| \geq 0$ and $\|x\| = 0$ iff $x = 0$;
- $\|ax\| = |a| \cdot \|x\|$ for $a \in K$ and $x \in E$;
- $\|x+y\| \leq \max\{\|x\|, \|y\|\}$

(ii) A normed space E over K is a **Banach space** if it is complete with respect to its norm.

If $\|\cdot\|_E$ and $\|\cdot\|_F$ are the norms on the normed spaces E and F then on $E \times F$, $(x,y) \longmapsto \|x\|_E + \|y\|_F$, gives a norm $\|\cdot\|_{E\times F}$ on $E\times F$ which gives the product topology on $E\times F$.

If E and F are Banach then so is $(E \times F, \|\cdot\|_{E\times F})$.

Classical theorem on Banach spaces hold over K (except Hahn-Banach).

Theorem: Let E, F be K-Banach spaces, and

$$f : E \longrightarrow F$$

a linear map.

Open Mapping Thm

(a) If f is continuous and onto then f is an open map. In particular if f is bijective and continuous then its inverse is also continuous, i.e. $f : E \xrightarrow{\sim} F$.

Closed Graph Thm

(b) The map f is continuous if and only if the graph $\Gamma_f = \{(x, fx) \mid x \in E\}$ is closed in $E \times F$.

(c) Let $\mathcal{C}$ be a collection of continuous linear operators $T : E \longrightarrow F$ s.t. for each $x \in E$ one has

$$\sup_{T \in \mathcal{C}} \|Tx\| < \infty.$$

Then

$$\sup_{T \in \mathcal{C}} \|T\| < \infty$$

where, for any cts operator T, $\|T\| = \sup_{\|x\|=1} \|Tx\|$.

Uniform boundedness Principle.

Caution: Tian also says that if f is one-to-one and continuous then $f(E)$ is closed in F and $E \xrightarrow{\sim} f(E)$ as a Banach space. This is clearly not true in the classical case, and I very much doubt it is true in the non-archimedean case either. In any case the classical proof could not possibly apply as Tian claims, since there is no classical proof. He offers no proof.

Why the terminology? It turns out that $|\cdot|$ is non-archimedean if and only if $\forall\ a \in K$, $|a| \le 1$, $|na| \le 1$ $\forall\ n \in \mathbb{N}$. Using $|a+b| = \max\{|a|, |b|\}$ one way is clear, namely if $|\cdot|$ is non-archimedean then this property holds. Conversely suppose

(*) $|na| \le 1$ $\forall\ n \in \mathbb{N}$ and $\forall\ a$ s.t. $|a| \le 1$.

If $a, b \in K$ with $|a| \le |b|$, then $\forall\ n \in \mathbb{N}$ we have

$$|a+b|^n \le \sum_{j=0}^{n} \left| \binom{n}{j} a^j b^{n-j} \right|$$

$$= \sum_{j=0}^{n} |b^{n-j}| \cdot \left| \binom{n}{j} a^j \right|$$

$$\le \sum_{j=0}^{n} |b^{n-j}| \cdot |b^j| \qquad \text{since } \left| \binom{n}{j} \frac{a^j}{b^j} \right| \le \left| \frac{a^j}{b^j} \right| \text{ by } (*).$$

$$= (n+1)\, |b|^n$$

Thus

$$|a+b| \le (n+1)^{1/n} |b| \qquad \forall\ n \in \mathbb{N}.$$

Letting $n \to \infty$ we get $|a+b| \le |b|$. It follows that $|\cdot|$ is non-archimedean.

Recall that for $\mathbb{R}$, the property $|nx| > |x|$ for some $n \in \mathbb{N}$ is called the archimedean property.