LECTURES 18 AND 19

Dates of Lectures: October 22 and 24, 2019

We fix a Grothendieck topology $(\mathscr{C}, \mathscr{C}ov)$ throughout the notes for these two lectures (see [Lecture 14, 1.2.2]). As before $\mathbf{N} = \{0, 1, 2, \dots, m, \dots\}$. Rings mean commutative rings with 1.

The symbol ☡ is for flagging a cautionary comment or a tricky argument. It occurs in the margins and is Knuth's version of Bourbaki's "dangerous bend symbol".

1. Presheaves and the Čech complex

Let $\mathcal{A}b$ denote the category of abelian groups.

1.1. **Presheaves.** A *presheaf of abelian groups* $\mathscr{P}$ on $\mathscr{C}$ is a contravariant functor on $\mathscr{C}$ taking values in $\mathcal{A}b$. In other words

$$\mathscr{P} \colon \mathscr{C}^{\circ} \longrightarrow \mathcal{A}b$$

where $\mathscr{C}^{\circ}$ denotes the opposite category of $\mathscr{C}$. From now on, by a presheaf we will mean a presheaf of abelian groups, unless otherwise specified. If $\mathscr{P}$ and $\mathscr{P}'$ are presheaves, a morphism between then is a natural transformation $\mathscr{P} \to \mathscr{P}'$. This makes presheaves on $\mathscr{C}$ into a category which we denote $\mathcal{P}sh_{\mathscr{C}}$, or simply $\mathcal{P}sh$.

The topology $\mathscr{C}ov$ is irrelevant to the notion of a presheaf. If $\phi\colon \mathscr{P} \to \mathscr{Q}$ is a map of presheaves then $\ker\phi$ is the presheaf $U \mapsto \ker\phi(U)$. Similarly, define $\operatorname{coker}\phi$ as the presheaf $U \mapsto \operatorname{coker}\phi(U)$. One checks that $\ker\phi$ and $\operatorname{coker}\phi$ are indeed the kernel and cokernel of ϕ in $\mathcal{P}sh$.

For $V \in \mathscr{C}$, let $\mathcal{Z}_V$ be the presheaf of abelian groups on $\mathscr{C}$ given by

$$\mathcal{Z}_V(W) = \mathbf{Z}^{\mathrm{Hom}_{\mathscr{C}}(W,\,V)} = \bigoplus_{\phi\colon V \to W} \mathbf{Z} \qquad (W \in \mathscr{C})$$

with obvious "restriction" maps.

The following is easy to see.

1. $\mathcal{A}b$ has arbitrary direct sums.
2. If $(\mathscr{P}_i)_{i\in I}$ is a family of subobjects of $\mathscr{P} \in \mathcal{A}b$, and $\mathscr{Q}$ is another subobject of $\mathscr{P}$, then
$$\sum_{i\in I}(\mathscr{P}_i \cap \mathscr{Q}) = \sum_{i\in I}(\mathscr{P}_i) \cap \mathscr{Q}.$$
3. The collection $\{\mathcal{Z}_V\}_V$ of presheaves are a set of generators for $\mathcal{P}sh_{\mathscr{C}}$.

Because of the above properties, $\mathcal{P}sh$ has *enough injectives*, i.e., given $\mathscr{P} \in \mathcal{P}sh$, there exists an injective object $\mathscr{E}$ in $\mathcal{P}sh$ such that $\mathscr{P}$ is a subobject of $\mathscr{E}$. We will not be proving this.

1.2. **Čech theory.** Let $U \in \mathscr{C}$ and let $\mathfrak{U} = \{U_i \to U\}_{i\in I} \in \mathscr{C}ov(U)$. For $p \in \mathbf{N}$ and $\boldsymbol{i} = (i_0, \dots, i_p) \in I^{p+1}$, we write

$$U_{\boldsymbol{i}} = U_{i_0 \dots i_p} = U_{i_0} \times_U \cdots \times_U U_{i_p}.$$

For a presheaf $\mathscr{P}$ *the Čech complex* $C^\bullet(\mathfrak{U}, \mathscr{P})$ *of* $\mathscr{P}$ *over* $\mathfrak{U}$ is defined as follows. For $p \in \mathbf{N}$, the module $C^p(\mathfrak{U}, \mathscr{P})$ is

$$C^p(\mathfrak{U}, \mathscr{P}) := \prod_{\in I^{p+1}} \mathscr{P}(U_{i_0 \dots i_p}), \tag{1.2.1}$$

and the coboundary map $\partial^p : C^p(\mathfrak{U}, \mathscr{P}) \to C^{p+1}(\mathfrak{U}, \mathscr{P})$ is given by the formula

$$(\partial^p \sigma)(i_0, \dots, i_{p+1}) = \sum_{j=0}^{p+1} (-1)^j \sigma(i_0, \dots, \hat{i}_j, \dots, i_{p+1})|_{U_{i_0 \cdots i_{p+1}}}. \tag{1.2.2}$$

It is easy to see that $\partial^{p+1}\partial^p = 0$ for every p. The cohomology groups of $C^\bullet(\mathfrak{U}, \mathscr{P})$ are denoted $H^p(\mathfrak{U}, \mathscr{P})$ and are called *the Čech cohomology groups of* $\mathscr{P}$ *with respect to* $\mathfrak{U}$. For future reference we record this definition:

$$H^p(\mathfrak{U}, \mathscr{P}) := \mathrm{H}^p(C^\bullet(\mathfrak{U}, \mathscr{P})), \qquad p \in \mathbf{N}. \tag{1.2.3}$$

We also define the p^{th} *Čech cohomology of* $\mathscr{P}$ *over* U to be

$$\check{H}^p(U\, \mathscr{P}) := \varinjlim_{\mathfrak{V}} H^p(\mathfrak{V}, \mathscr{P})), \qquad (p \in \mathbf{N}). \tag{1.2.4}$$

where $\mathfrak{V}$ ranges over members of $\mathscr{C}ov(U)$ with the "partial order" being given by refinements. We pass over the existence of such direct limits, with a huge class of indices, in silence, except to note that it is a "filtered" directed system, whence usual notions of direct limits apply.

Note that we have a natural map

$$\mathscr{P}(U) \longrightarrow H^0(\mathfrak{U}, \mathscr{P}) \tag{1.2.5}$$

for if $s \in \mathscr{P}(U)$ the element $(s_\alpha) \in \prod_{\alpha \in I} \mathscr{P}(U_\alpha) = C^0(\mathfrak{U}, \mathscr{P})$ is a cocycle in $C^\bullet(\mathfrak{U}, \mathscr{P})$.

Proposition 1.2.6. *Let* $\mathscr{E}$ *be an injective object in* $\mathscr{P}\!sh$. *Then*

$$H^p(\mathfrak{U}, \mathscr{E}) = \check{H}^p(U, \mathscr{E}) = 0, \qquad (p \geq 1).$$

Proof. Let $\mathcal{Z}_\bullet$ be the homology complex defined in problem (4) of HW 7. According to *loc.cit.*, $H^p(\mathfrak{U}, \mathscr{E}) = \mathrm{H}^p(\mathrm{Hom}_{\mathscr{P}\!sh}(\mathcal{Z}_\bullet, \mathscr{E}))$. Since $\mathscr{E}$ is an injective object, the latter equals $\mathrm{Hom}_{\mathscr{P}\!sh}(\mathrm{H}_p(\mathcal{Z}_\bullet), \mathscr{E})$ and this is zero for $p \geq 1$ by *loc.cit.* Thus $H^p(\mathfrak{U}, \mathscr{E}) = 0$ for $p \geq 1$. Taking direct limits over refinements we get the remaining assertion. $\square$

Proposition 1.2.7. *Given a short exact sequence of presheaves*

$$0 \to \mathscr{P}' \longrightarrow \mathscr{P} \longrightarrow \mathscr{P}'' \to 0 \tag{E}$$

there exist long exact sequences

$$\begin{aligned} &0 \longrightarrow H^0(\mathfrak{U}, \mathscr{P}') \longrightarrow H^0(\mathfrak{U}, \mathscr{P}) \longrightarrow H^0(\mathfrak{U}, \mathscr{P}'') \longrightarrow H^1(\mathfrak{U}, \mathscr{P}') \longrightarrow \dots \\ &\qquad \dots \longrightarrow H^{p-1}(\mathfrak{U}, \mathscr{P}'') \longrightarrow H^p(\mathfrak{U}, \mathscr{P}') \longrightarrow \dots \end{aligned} \tag{$\mathrm{E}_{\mathfrak{U}}$}$$

and

$$\begin{aligned} &0 \longrightarrow \check{H}^0(U, \mathscr{P}') \longrightarrow \check{H}^0(U, \mathscr{P}) \longrightarrow \check{H}^0(U, \mathscr{P}'') \longrightarrow \check{H}^1(U, \mathscr{P}') \longrightarrow \dots \\ &\qquad \dots \longrightarrow \check{H}^{p-1}(U, \mathscr{P}'') \longrightarrow \check{H}^p(U, \mathscr{P}') \longrightarrow \dots \end{aligned} \tag{$\check{\mathrm{E}}$}$$

which are functorial in short exact sequences of the form (E).

Proof. By the definition of kernels and cokernels in $\mathscr{P}sh$ it is clear that $C^\bullet(\mathfrak{U}, (\mathrm{E}))$ is a short exact sequence of complexes of abelian groups. The long exact sequence $(\mathrm{E}_{\mathfrak{U}})$ is the one induced by this. Taking direct limits we get $(\check{\mathrm{E}})$. Functoriality with respect to (E) is clear from the functoriality of $C^\bullet(\mathfrak{U}, (\mathrm{E}))$, of long exact sequences associated with short exact sequences, and of direct limits. □

Theorem 1.2.8. *For $p \in \mathbf{N}$, $H^p(\mathfrak{U}, -)$ and $\check{H}^p(U, -)$ are the p^{th} right derived functors of $H^0(\mathfrak{U}, -)$ and $\check{H}^0(U, -)$ respectively.*

Proof. This is immediate from Propositions 1.2.6 and 1.2.7. □

2. Sheaves and sheafification

2.1. **Sheaves and the functor $\mathscr{P} \mapsto \mathscr{P}^+$.** The standard notion of a separated presheaf and that of a sheaf on a classical topological space have the following generalisation.

Definition 2.1.1. A presheaf $\mathscr{F}$ on $\mathscr{C}$ is said to be *separated* on $(\mathscr{C}, \mathscr{C}ov)$ if the natural map $\mathscr{F}(U) \to H^0(\mathfrak{U}, \mathscr{F})$ of (1.2.5) is an injective map of abelian groups for every $\mathfrak{U} \in \mathscr{C}ov(U)$ and every $U \in \mathscr{C}$. It is called a *sheaf* if $\mathscr{F}(U) \to H^0(\mathfrak{U}, \mathscr{F})$ is an isomorphism or every $\mathfrak{U} \in \mathscr{C}ov(U)$ and every $U \in \mathscr{C}$. The full subcategory of $\mathscr{P}sh$ consisting of sheaves is denoted $\mathscr{S}h_{\mathscr{C}}$ or simply $\mathscr{S}h$

2.1.2. We define a functor
$$(\;)^+ \colon \mathscr{P}sh \to \mathscr{P}sh$$
by the rule
$$\mathscr{P}^+(U) = \check{H}^0(U, \mathscr{P}) \tag{2.1.2.1}$$
for $U \in \mathscr{C}$ and $\mathscr{P} \in \mathscr{P}sh$. Using (1.2.5) we see there is a natural map
$$\theta(\mathscr{P}) \colon \mathscr{P} \to \mathscr{P}^+. \tag{2.1.2.2}$$
Now, by definition of a sheaf, if $\mathscr{P}$ is a sheaf then (1.2.5) is an isomorphism for every $\mathfrak{U} \in \mathscr{C}ov(U)$ and every $U \in \mathscr{C}$, whence $\theta(\mathscr{P})$ is an isomorphism. The map $\theta(\mathscr{P})$ is clearly functorial in $\mathscr{P}$ and so our discussion can be restated as saying there is a natural transformation
$$\theta \colon \mathbf{1}_{\mathscr{P}sh} \to (\;)^+ \tag{2.1.2.3}$$
such that
$$\theta|_{\mathscr{S}h} \colon \mathbf{1}_{\mathscr{S}h} \xrightarrow{\sim} (\;)^+|_{\mathscr{S}h}. \tag{2.1.2.4}$$
When $\mathscr{F}$ is a sheaf, we do not distinguish between $\mathscr{F}$ and $\mathscr{F}^+$ in view of (2.1.2.4).

We claim $\mathscr{P}^+$ is a separated sheaf. Suppose $\mathscr{P} \in \mathscr{P}sh$, $U \in \mathscr{C}$ and $\mathfrak{U} \in \mathscr{C}ov(U)$ are given. Say $\mathfrak{U} = \{U_\alpha \to U\}$. Suppose $\bar{\xi}_1, \bar{\xi}_2 \in \mathscr{P}^+$ are such that
$$\bar{\xi}_1|_{U_\alpha} = \bar{\xi}_2|_{U_\alpha} \tag{$*$}$$
for every index α. We can find a cover $\mathfrak{V} = \{V_\nu \to U\} \in \mathscr{C}ov(U)$ such that $\bar{\xi}_1$ and $\bar{\xi}_2$ are represented by elements in $H^0(\mathfrak{V}, \mathscr{P})$, say by $\xi_1, \xi_2 \in H^0(\mathfrak{V}, \mathscr{P})$. For a fixed α, let
$$U_\alpha \times_U \mathfrak{V} := \{U_\alpha \times_U V_\nu \to U_\alpha\} \in \mathscr{C}ov(U_\alpha).$$
The image of ξ_i in $H^0(U_\alpha \times_U \mathfrak{V}, \mathscr{P})$ represents $\bar{\xi}_i|_{U_\alpha} \in \check{H}^0(U_\alpha, \mathscr{P})$. By $(*)$ we have a "refinement" $\mathfrak{W}_\alpha = \{W_{\alpha\nu} \to U_\alpha\} \in \mathscr{C}ov(U_\alpha)$ of $U_\alpha \times_U \mathfrak{V}$ such that the images of ξ_1 and ξ_2 in $H^0(\mathfrak{W}_\alpha, \mathscr{P})$ are equal. Let $\mathfrak{W} = \{W_{\alpha\nu} \to U\} \in \mathscr{C}ov(U)$.

The images of ξ_1 and ξ_2 in $H^0(\mathfrak{W}, \mathscr{P})$ are then equal, and hence $\bar{\xi}_1 = \bar{\xi}_2$. This proves that *$\mathscr{P}^+$ is separated.*

We claim more, namely, *if $\mathscr{P}$ is separated, then $\mathscr{P}^+$ is a sheaf.* To see this, we make an observation.

> *Assume $\mathscr{P}$ is separated and fix $U \in \mathscr{C}$. Let $\mathfrak{U}, \mathfrak{V} \in \mathscr{C}ov(U)$ with $\mathfrak{V}$ a refinementof $\mathfrak{U}$. Then $H^0(\mathfrak{U}, \mathscr{P}) \to H^0(\mathfrak{V}, \mathscr{P})$ is an injective map.*

We denote the specific refinement $\mathfrak{V}$ of $\mathfrak{U}$ by the shorthand $\mathfrak{V} \to \mathfrak{U}$. For concreteness, let $\mathfrak{U} = \{U_\alpha \to U\}$ and $\mathfrak{V} = \{V_\nu \to U\}$. Then

$$\mathfrak{V} \times_U \mathfrak{U} = \{V_\nu \times_U U_\alpha \to U\} \in \mathscr{C}ov(U)$$

is a common refinement of $\mathfrak{U}$ and $\mathfrak{V}$. Since $\mathscr{P}$ is separated, we have injective maps

$$\mathscr{P}(U_\alpha) \hookrightarrow H^0(\mathfrak{V} \times_U U_\alpha, \mathscr{P}) \hookrightarrow C^0(\mathfrak{V} \times_U U_\alpha, \mathscr{P})$$

for every α. Since $C^0(\mathfrak{V}\times_U U_\alpha, \mathscr{P}) = \prod_\nu \mathscr{P}(V_\nu \times_U U_\alpha)$ and $C^0(\mathfrak{U}, \mathscr{P}) = \prod_\alpha \mathscr{P}(U_\alpha)$, taking products of the displayed inclusion over α we get

$$C^0(\mathfrak{U}, \mathscr{P}) \hookrightarrow C^0(\mathfrak{U} \times_U \mathfrak{V}, \mathscr{P})$$

where the hooked arrow is the one arising from the refinement $\mathfrak{V} \to \mathfrak{U}$. Since $\mathfrak{V}\times_U\mathfrak{U} \to \mathfrak{U}$ factors as $\mathfrak{V}\times_U\mathfrak{U} \to \mathfrak{V} \to \mathfrak{U}$, therefore the map $C^0(\mathfrak{U}, \mathscr{P}) \to C^0(\mathfrak{V}, \mathscr{P})$ is injective. It then follows that $H^0(\mathfrak{U}, \mathscr{P}) \to H^0(\mathfrak{V}, \mathscr{P})$ is injective, as claimed above.

We now show that $\mathscr{P}^+$ is a sheaf (under the assumption that $\mathscr{P}$ is a separated presheaf). Let $U \in \mathscr{C}$, $\mathfrak{V} = \{V_\alpha \to U\} \in \mathscr{C}ov(U)$ and suppose $(\bar{\xi}_\alpha) \in H^0(\mathfrak{V}, \mathscr{P}^+)$. We wish to show that there exists $\bar{\xi} \in \mathscr{P}^+(U)$ such that

$$\bar{\xi}|_{U_\alpha} = \bar{\xi}_\alpha$$

for every α. Now for each α, $\bar{\xi}_\alpha \in \mathscr{P}^+(U_\alpha)$, and hence we have $\mathfrak{W}_\alpha \in \mathscr{C}ov(V_\alpha)$ and an element $\xi_\alpha \in H^0(\mathfrak{W}_\alpha, \mathscr{P})$ such that ξ_α represents $\bar{\xi}_\alpha$. For definiteness suppose $\mathfrak{W}_\alpha = \{W_{\alpha\nu} \to V_\alpha\}$ and

$$\xi_\alpha = (\xi_{\alpha\nu})_\nu. \tag{$**$}$$

For good book-keeping let us denote the refinement $\mathscr{V}$ of (the singleton) $\{U\}$ by $\rho\colon \mathscr{V} \to \{U\}$. Varying α, the $\mathfrak{W}_\alpha$ give us a refinement $\mathfrak{W} = \{W_{\alpha\nu} \to U\}$ of $\mathfrak{V}$, and denote this refinement by $\rho'\colon \mathfrak{W} \to \mathfrak{V}$. Consider the following "cartesian" diagram of covers of U via refinements:

$$\begin{tikzcd}
\mathfrak{W} \arrow[d, "\rho'"'] & \mathfrak{W} \times_U \mathfrak{V} \arrow[l, "\mathbf{1}\times\rho"'] \arrow[d, "\rho'\times\mathbf{1}"'] & \mathfrak{W} \times_U \mathfrak{W} \arrow[l, "\mathbf{1}\times\rho'"'] \arrow[d, "\rho'\times\mathbf{1}"] \\
\mathfrak{V} \arrow[d, "\rho"'] & \mathfrak{V} \times_U \mathfrak{V} \arrow[l, "\mathbf{1}\times\rho"'] \arrow[d, "\rho\times\mathbf{1}"'] & \mathfrak{V} \times_U \mathfrak{W} \arrow[l, "\mathbf{1}\times\rho'"'] \arrow[d, "\rho\times\mathbf{1}"] \\
\{U\} & \mathfrak{V} \arrow[l, "\rho"] & \mathfrak{W} \arrow[l, "\rho'"]
\end{tikzcd}$$

Let

$$V_{\alpha\beta} = V_\alpha \times_U V_\beta.$$

Recall, $\xi_\alpha \in H^0(\mathfrak{W}_\alpha, \mathscr{P})$ represents $\bar{\xi}_\alpha \in \mathscr{P}^+(V_\alpha)$. Let

$$\xi^1_{\alpha\beta} \in H^0(\mathfrak{W}_\alpha \times_U V_\beta, \mathscr{P})$$

be its image. Then $\xi^1_{\alpha\beta}$ represents $\bar{\xi}_\alpha|_{V_{\alpha\beta}} \in \mathscr{P}^+(V_{\alpha\beta})$. Similarly, if

$$\xi^2_{\alpha\beta} \in H^0(V_\alpha \times_U \mathfrak{W}_\beta, \mathscr{P})$$

is the image of $\xi_\beta \in H^0(\mathfrak{W}_\beta, \mathscr{P})$, then $\xi^2_{\alpha\beta}$ represents $\bar{\xi}_\beta|_{V_{\alpha\beta}} \in \mathscr{P}^+(V_{\alpha\beta})$. By hypothesis, $\bar{\xi}_\alpha|_{V_{\alpha\beta}} = \bar{\xi}_\beta|_{V_{\alpha\beta}}$. It follows that there is a common refinement $\mathfrak{U}$ of $\mathfrak{W}_\alpha \times_U V_\beta$ and $V_\alpha \times_U \mathfrak{W}_\beta$ in $\mathscr{C}ov(V_{\alpha\beta})$ such that the images of $\xi^1_{\alpha\beta}$ and $\xi^2_{\alpha\beta}$ in $H^0(\mathfrak{U}, \mathscr{P})$ are the same. But since $\mathscr{P}$ is separated, from our observation above, the images of $\xi^1_{\alpha\beta}$ and $\xi^2_{\alpha\beta}$ would be the same in $H^0(\mathfrak{R}, \mathscr{P})$ for any common refinement $\mathfrak{R} \in \mathscr{C}ov(V_{\alpha\beta})$ of $\mathfrak{W}_\alpha \times_U V_\beta$ and $V_\alpha \times_U \mathfrak{W}_\beta$. Now $\mathfrak{R} = \mathfrak{W}_\alpha \times_U \mathfrak{W}_\beta$ is such a common refinement. Let $\xi_{\alpha\beta}$ be the common image of $\xi^1_{\alpha\beta}$ and $\xi^2_{\alpha\beta}$ in $H^0(\mathfrak{W}_\alpha \times_U \mathfrak{W}_\beta, \mathscr{P})$. Write $W_{\alpha\nu\beta\mu} = W_{\alpha\nu} \times_U W_{\beta\mu}$. Then

$$\xi_{\alpha\beta} = (\xi_{\alpha\nu\beta\mu})_{\nu\mu}.$$

In fact,using the representation $(**)$ above we may write

$$\xi^1_{\alpha\beta}|_{W_{\alpha\nu\beta\mu}} = \xi_{\alpha\nu\beta\mu} = \xi^2_{\alpha\beta}|_{W_{\alpha\nu\beta\mu}}.$$

Let α, β, μ, ν all vary. Set

$$\xi = (\xi_{\alpha\nu\beta\mu})_{\alpha\nu\beta\mu}.$$

Then $\xi \in C^0(\mathfrak{W} \times_U \mathfrak{W}, \mathscr{P})$. In fact, clearly $\xi \in H^0(\mathfrak{W} \times_U \mathfrak{W}, \mathscr{P})$. Let its image in $\check{H}^0(U, \mathscr{P}) = \mathscr{P}^+(U)$ be $\bar{\xi}$. Then $\bar{\xi}|_{V_\alpha} = \bar{\xi}_\alpha$ for every α. This proves that $\mathscr{P}^+$ is a sheaf. We have thus proven the following.

Theorem 2.1.3. *Let $\mathscr{P} \in \mathcal{Psh}$.*

(i) *$\mathscr{P}^+$ is separated.*
(ii) *If $\mathscr{P}$ is separated then $\mathscr{P}^+$ is a sheaf.*
(iii) *If $\mathscr{P}$ is a sheaf then $\mathscr{P}^+ = \mathscr{P}$. More precisely, if $\mathscr{P}$ is a sheaf then the natural map $\theta(\mathscr{P})\colon \mathscr{P} \to \mathscr{P}^+$ of (2.1.2.2) is an isomorphism.*

2.2. **Sheafifications.** Let

(2.2.1) $$i\colon \mathcal{Sh} \longrightarrow \mathcal{Psh}$$

be the forgetful functor. For any $\mathscr{P} \in \mathcal{Psh}$ set

(2.2.2) $$\mathscr{P}^\# := \mathscr{P}^{++}.$$

Then, according to Theorem 2.1.3, $\mathscr{P}^\#$ is a sheaf. The sheaf $\mathscr{P}^\#$ is called the *sheafification* of $\mathscr{P}$. The assignment $\mathscr{P} \mapsto \mathscr{P}^\#$ gives us a functor

(2.2.3) $$(\ )^\#\colon \mathcal{Psh} \longrightarrow \mathcal{Sh},$$

the so-called *sheafification functor*. In fact, as with $(\ )^+$, we have a natural transformation (the so called *sheafication map*)

(2.2.4) $$\vartheta\colon \mathbf{1}_{\mathcal{Psh}} \longrightarrow i \circ (\ )^\#$$

of endo-functors on $\mathcal{Psh}$. Also, according to Theorem 2.1.3, we have an isomorphism

(2.2.5) $$(\ )^\# \circ i \xrightarrow{\ \sim\ } \mathbf{1}_{\mathcal{Sh}}.$$

In fact, the functor $(\;)^{\#}$ is left adjoint to i as we shall see. In more naive terms, suppose $\mathscr{P} \in \mathcal{Psh}$ and $\mathscr{F} \in \mathcal{Sh}$. Suppose we have a map of presheaves $\varphi\colon \mathscr{P} \to i(\mathscr{F})$. Applying $(\;)^{\#}$ we get a map $\varphi^{\#}\colon \mathscr{P}^{\#} \to (i(\mathscr{F}))^{\#} = \mathscr{F}$ (we are treating (2.2.5) as the identity natural transformation). One checks easily that the diagram

$$\begin{tikzcd} \mathscr{P} \arrow[r, "\vartheta(\mathscr{P})"] \arrow[dr, bend right, "\varphi"'] & i(\mathscr{P}^{\#}) \arrow[d, "i(\varphi^{\#})"] \\ & i(\mathscr{F}) \end{tikzcd}$$

commutes. Moreover, $x = i(\varphi^{\#})$ is the only solution to the equation $x \circ \vartheta(\mathscr{P}) = \varphi$. This can be seen by using (2.2.5). In other words, we have just seen the universal property of sheafifications. This can be reformulated as

$$\mathrm{Hom}_{\mathcal{Psh}}(\mathscr{P},\, i(\mathscr{F})) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{Sh}}(\mathscr{P}^{\#},\, \mathscr{F}). \tag{2.2.6}$$

The above isomorphism is *bifunctorial*, i.e. it is functorial in $\mathscr{P} \in \mathcal{Psh}$ and in $\mathscr{F} \in \mathcal{Sh}$. We thus have the following reformulation of the universal property of sheafifications:

Lemma 2.2.7. *The sheafification functor $(\;)^{\#}$ is a left adjoint to the forgetful functor i.*

2.3. **Kernels and cokernels of maps of sheaves.** Let $\varphi\colon \mathscr{F} \to \mathscr{G}$ be a map of sheaves. Let $\mathscr{K}$ and $\mathscr{P}$ be the presheaf kernel and presheaf cokernel, i.e., $\mathscr{K} = \ker i(\varphi)$ and $\mathscr{P} = \operatorname{coker} i(\varphi)$. It turns out that $\mathscr{K}$ is already a sheaf, and the proof is exactly the same as the one in the classical theory over classical topological spaces. Moreover, it is indeed a kernel in $\mathcal{Sh}$, i.e. it has the universal property of kernels. $\mathscr{P}$ on the other hand need not be a sheaf. One sets $\operatorname{coker} \varphi = \mathscr{P}^{\#}$. With this definition, using the universal property of sheafifications, one sees that $\operatorname{coker} \varphi$ is indeed a cokernel in $\mathcal{Sh}$, i.e. it has the right universal property.

One checks that $\mathcal{Sh}$ is an abelian category. In fact, for the same reasons that $\mathcal{Psh}$ does, $\mathcal{Sh}$ has enough injectives. Finally from our description of kernels and cokernels, it is clear that $(\;)^{\#}$ is an exact functor. We record these statements below.

Lemma 2.3.1. *$\mathcal{Sh}$ is an abelian category with enough injectives, and $(\;)^{\#}\colon \mathcal{Psh} \to \mathcal{Sh}$ is an exact functor.*

An immediate consequence is the following:

Proposition 2.3.2. *Let $\mathscr{E}$ be an injective object in $\mathcal{Sh}$. Then $i(\mathscr{E})$ is an injective object in $\mathcal{Psh}$.*

Proof. We have

$$\mathrm{Hom}_{\mathcal{Psh}}(-,\, i(\mathscr{E})) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{Sh}}(-^{\#},\, \mathscr{E}) = \mathrm{Hom}_{\mathcal{Sh}}(-,\, \mathscr{E}) \circ (\;)^{\#}.$$

On the extreme right we have a composite of exact functors, $\mathscr{E}$ being injective, and $(\;)^{\#}$ being exact. Thus $\mathrm{Hom}_{\mathcal{Psh}}(-,\, i(\mathscr{E}))$ is exact, whence $i(\mathscr{E})$ is an injective object in $\mathcal{Psh}$. $\square$