LECTURE 17 SUPPLEMENT

Date of Lecture: October 17, 2019

This supplements Lecture 17. Unless otherwise specified, K is a complete nonarchimedean field, and to avoid annoying trivialities we assume the absolute value | |on K is non-trivial. As before $\mathbf{N} = \{0, 1, 2, ..., m, ...\}$. Rings mean commutative rings with 1.

The symbol \bigotimes is for flagging a cautionary comment or a tricky argument. It occurs in the margins and is Knuth's version of Bourbaki's "dangerous bend symbol".

1. Finite maps between affinoid algebras

This is section is preliminary to answering questions about maps $T_n \to A$ such that the "free affinoid algebra variables" ζ_1, \ldots, ζ_n map to a specified set of elements a_1, \ldots, a_n in A. If instead of T_n we have the polynomial ring $K[\boldsymbol{\zeta}]$, then of course this can always be done.

1.1. Let $T_n = K\langle \zeta_1, \ldots, \zeta_n \rangle$. Let A be an affinoid K-algebra and $\| \|_{\alpha}$ some residue norm from a Gauss norm on a Tate algebra. Define (as before)

(1.1.1)
$$A^{\circ} := \{ x \in A \mid ||x||_{\alpha} \le 1 \}$$

Also set

(1.1.2)
$$\overset{\circ}{A} := \{ x \in A \mid ||x||_{\sup} \le 1 \}.$$

Since $\| \|_{\sup} \leq \| \|_{\alpha}$, we have

 $A^{\circ} \subset \overset{\circ}{A}.$

Lemma 1.1.3. $\stackrel{\circ}{A}$ is integral over A° .

Proof. Suppose $f \in A$. Let $T = T_r$ be the Tate algebra giving $\| \|_{\alpha}$, i.e., we have a surjective map $\alpha \colon T \to A$ and $\| \|_{\alpha}$ is the residue norm of the Gauss norm $\| \|$ on T. According to [Lecture 13, Lemma 1.2.4], for each $a \in A$ we have an integral equation

(*)
$$a^n + c_1 a^{n-1} + \dots + c_n = 0$$

with $c_i \in T$ such that $||a||_{\sup} = \max_i ||c_i||^{\frac{1}{i}}$. We are using the fact that $|| || = || ||_{\sup}$ on T. By definition of the residue norm, if b_i is the image of c_i in A, then $||b_i||_{\alpha} \leq ||c_i||$. Now suppose $a \in A$. Then $||c_i|| \leq 1$ for $i = 1, \ldots, n$ (since $1 \geq ||a||_{\sup} = \max_i ||c_i||^{\frac{1}{i}}$) and hence $||b_i||_{\alpha} \leq 1$ for all i, i.e. $b_i \in A^\circ$. Thus the integral relation (*), remains true with b_i in place of c_i and this shows that a is integral over A° .

Definition 1.1.4. Let $(A, || ||_{\alpha})$ be as above. An element $a \in A$ is said to be *power* bounded if $\{||a^n||_{\alpha} | n \in \mathbf{N}\}$ is bounded.

Theorem 1.1.5. Let $(A, || ||_{\alpha})$ be as above and let $a \in A$. The following are equivalent

(i) $a \in A$,

(ii) a is integral over A° ,

(iii) a is power bounded.

In particular, being power bounded does not depend on the choice of the residue norm $\| \|_{\alpha}$.

Proof. We have already seen that (i) implies (ii). Now suppose (ii) is true. Then $A^{\circ}[a]$ is a finite module over A° . If b_1, \ldots, b_m are any A° -module generators of $A^{\circ}[a]$, then $\|x\|_{\alpha} \leq \max_{1 \leq j \leq m} \|b_j\|$ for every $x \in A^{\circ}[a]$. In particular $\{\|a^n\|_{\alpha}\}_{n \in \mathbb{N}}$ is bounded, giving us (iii).

Now suppose (iii) is true. Say $||a^n||_{\alpha} \leq M$ for all $n \in \mathbf{N}$. For any $x \in \mathrm{Sp}(A)$ we have $|a(x)|^n = |a^n(x)| \leq ||a^n||_{\sup} \leq ||a^n||_{\alpha} \leq M$ for $n \in \mathbf{N}$ and hence $|a(x)| \leq 1$. It follows that $||a||_{\sup} \leq 1$ giving us (i).

Theorem 1.1.6. Let B be an affinoid K-algebra and a K-algebra map $\sigma: T_n \to B$. Then $\sigma(\zeta_i) \in \overset{\circ}{B}$ for i = 1, ..., n. Conversely, suppose $b_1, ..., b_n \in \overset{\circ}{B}$. Then there is a unique K-algebra map $\sigma: T_n = K\langle \zeta_1, ..., \zeta_n \rangle \to B$ such that $\sigma(\zeta_i) = b_i$, i = 1, ..., n.

Proof. The first part is obvious since $|\sigma(\zeta_i)(x)| = |\zeta_i({}^a\sigma(x))| \le ||\zeta_i||_{\sup} = 1$.

We now prove the converse. Uniqueness follows from the fact that $K[\boldsymbol{\zeta}]$ is dense in $K\langle \boldsymbol{\zeta} \rangle$ and all K-algebra maps between affinoid algebras are continuous. It remains to show existence. Endow B with any residue norm $\| \|_{\alpha}$. The b_i are power bounded according to Theorem 1.1.5. Hence there exists M > 0 such that for $\| b_1^{\nu_1} \dots b_n^{\nu_n} \|_{\alpha} < M$ for every $\boldsymbol{\nu} \in \mathbf{N}^n$. If $\sum_{\boldsymbol{\nu} \in \mathbf{N}^n} c_{\boldsymbol{\nu}} \boldsymbol{\zeta}^{\boldsymbol{\nu}} \in T_n$, it follows that $\sum_{\boldsymbol{\nu}} \| c_{\boldsymbol{\nu}} \boldsymbol{b}^{\boldsymbol{\nu}} \| \leq M \sum_{\boldsymbol{\nu}} |c_{\boldsymbol{\nu}}| < \infty$, whence $\sum_{\boldsymbol{\nu}} c_{\boldsymbol{\nu}} \boldsymbol{b}^{\boldsymbol{\nu}}$ is convergent, being absolutely convergent in the Banach algebra B. Set $\sigma(\sum_{\boldsymbol{\nu} \in \mathbf{N}^n} c_{\boldsymbol{\nu}} \boldsymbol{\zeta}^{\boldsymbol{\nu}}) = \sum_{\boldsymbol{\nu} \in \mathbf{N}^n} c_{\boldsymbol{\nu}} \boldsymbol{b}^{\boldsymbol{\nu}}$. This gives $\sigma: T_n \to B$ with the required properties. \Box

Remark 1.1.7. Theorem 1.1.6 gives us the reason why ζ_i are called free affinoid algebra generators or free affinoid algebra variables. The Tate algebra T_n should be viewed as free algebra in n generators for affinoid K-algebras. If $\sigma: T_n = K\langle \zeta_1, \ldots, \zeta_n \rangle \to B$ is a surjective map, and $b_i = \sigma(\zeta_i)$ for $i = 1, \ldots, n$, then we often write

$$(1.1.7.1) B = K\langle b_1, \dots, b_n \rangle.$$

In such a case the b_i are called topological K-algebra generators of B.

2. Inverse images of affinoid domains

2.1. Complete tensor products. We briefly touched on this in §§1.1 of Lecture 14. Suppose (A, || ||) is a Banach ring, $(M, || ||_M)$ and $(N, || ||_N)$ be Banach A-modules. Let $x \in M \otimes_A N$. Set

(2.1.1)
$$||x||' = \inf_{x=\sum_i m_i \otimes n_i} \max_i ||m_i||_M ||n_i||_N$$

where the sum runs through all representations of x as a finite sum $x = \sum_{i} m_i \otimes n_i$. One checks that $(M \otimes_A N, || ||')$ is a normed A-module. **Definition 2.1.2.** Let $(M, || ||_M)$ and $(N, || ||_N)$ be Banach A-modules. The complete tensor product $M \otimes_A N$ of M and N over A is the completion $M \otimes_A N$ with respect to the norm || ||' defined in (2.1.1).

2.1.3. Let M, N, E be Banach A-modules, and R, S, B, Banach A-algebras. The following properties are not hard to establish. The proofs are left to you.

- (1) $R \widehat{\otimes}_A S$ is a Banach A-algebra.
- (2) If $\Psi: M \times N \to E$ is a continuous A-bilinear map then there is a unique continuous A-module map $\Phi: M \widehat{\otimes}_A N \to E$ such that $\Phi(m \otimes n) = \Psi(m, n)$ for $(m, n) \in M \times N$.
- (3) If $\Phi: R \times S \to B$ is a continuous bilinear map of A-algebras, then the resulting continuous bilinear a module map $\Phi: R \widehat{\otimes}_A S \to B$ is a map of Banach A-algebras.
- (4) $(-)\widehat{\otimes}_A A = (-) \otimes_A A = (-).$
- (5) The functor $(-)\widehat{\otimes}_A M$ is right exact on Banach A-modules.
- (6) Let $R\langle \zeta_1, \ldots, \zeta_n \rangle$ be as in [Lecture 16, §§1.1]. Then

$$R\langle\zeta_1,\ldots,\zeta_n\rangle = A\langle\zeta_1,\ldots,\zeta_n\rangle\widehat{\otimes}_A R$$

The natural norm of the left side is equivalent to the norm on the right side arising from (2.1.1). In particular, if A if a K-affinoid algebra, then

$$A\langle\zeta_1,\ldots,\zeta_n\rangle=T_n\widehat{\otimes}_K A.$$

- (7) $T_n \widehat{\otimes}_K T_r = T_{n+r}$. The norm on the complete tensor product on the left arising from the formula in (2.1.1)—is equivalent to the Gauss norm on T_{n+r} .
- (8) If A, R, S are K-affinoid algebras, then so is $R \widehat{\otimes}_A S$. The norm on $R \widehat{\otimes}_A S$ arising from the norms on R and S as in (2.1.1) is equivalent to a residue norm from a Tate algebra.
- (9) If A, R, S are K-affinoid then $\operatorname{Sp}(R \widehat{\otimes}_A S)$ is the fibre product of $\operatorname{Sp}(R) \to \operatorname{Sp}(A)$ and $\operatorname{Sp}(S) \to \operatorname{Sp}(A)$ in the category of K-affinoid spaces. If $X = \operatorname{Sp}(A), U = \operatorname{Sp}(R), V = \operatorname{Sp}(S)$, and $W = \operatorname{Sp}(R \widehat{\otimes}_A S)$, then we represent this situation by the cartesian square

$$W \longrightarrow V$$

$$\downarrow \qquad \Box \qquad \downarrow$$

$$U \longrightarrow X$$

and write $W = U \times_X V$.

(10) Let A and R be affinoid K-algebras, $\varphi^* \colon A \to R$ a map of affinoid K algebras and $f_1, \ldots, f_r, g_1, \ldots, g_s$ elements in A. Then

$$R\langle \varphi^*(\boldsymbol{f})\rangle = A\langle \boldsymbol{f}\rangle \widehat{\otimes}_A R,$$

and

$$R\langle \varphi^*(\boldsymbol{f}), \boldsymbol{1}/\varphi^*(\boldsymbol{g}) \rangle = A\langle \boldsymbol{f}, \boldsymbol{1}/\boldsymbol{g} \rangle \widehat{\otimes}_A R.$$

If f_0, f_1, \ldots, f_r generate the unit ideal in A, then

$$R\langle \varphi^*(\mathbf{f})/\varphi^*(f_0)\rangle = A\langle \mathbf{f}/f_0\rangle \widehat{\otimes}_A R.$$

Theorem 2.1.4. Let $A \to B$ be map of affinoid K-algebras and let $\sigma: A\langle \zeta_1, \ldots, \zeta_n \rangle \to B$ an A-algebra map.. Then $\|\sigma(\zeta_i)\|_{\sup} \leq 1$ for $i = 1, \ldots, n$. Conversely, suppose

 $b_1, \ldots, b_n \in B$ are such that $||b_i||_{\sup} \leq 1$ for $i = 1, \ldots, n$. Then there is a unique A-algebra map $\sigma: A\langle \zeta_1, \ldots, \zeta_n \rangle \to B$ such that $\sigma(\zeta_i) = b_i, i = 1, \ldots, n$.

Proof. The proof is exactly that same as that given for Theorem 1.1.6.

Remark 2.1.5. This means the ζ_i can be regarded as in some sense as free variables for affinoid A-algebras. If $\sigma: A\langle \zeta_1, \ldots, \zeta_n \rangle \to B$ above is a surjective map, and $b_i = \sigma(\zeta_i)$ for $i = 1, \ldots, n$, then we often write

$$(2.1.5.1) B = A\langle b_1, \dots, b_n \rangle.$$

The b_i in this case are called *topological A-algebra generators of B*. We draw the reader's attention to the ever so slight inconsistency between this notation and the one in (1.1.2) of Lecture 16. See the footnote in *loc.cit*.

In the above situation, with $B = A\langle b_1, \ldots, b_n \rangle$, if $A\langle \boldsymbol{\zeta} \rangle$ is given the norm from A, i.e. $\|\sum_{\boldsymbol{\nu} \in \mathbf{N}^n} a_{\boldsymbol{\nu}} \boldsymbol{\zeta}^{\boldsymbol{\nu}}\| = \sup_{\boldsymbol{\nu}} \|a_{\boldsymbol{\nu}}\|$, and B the residue norm from $A\langle \boldsymbol{\zeta} \rangle$, then the A-algebra map $\tau \colon A \to B$ is clearly a contraction, i.e.

$$(2.1.5.2) \|\tau(a)\| \le \|a\|$$

2.2. The universal property of complete tensor products very easily yields the following.

Proposition 2.2.1. Let $\varphi^* \colon A \to B$ be a map of affinoid K-algebras, with the corresponding map of affinoid K-spaces being denoted $\varphi \colon \operatorname{Sp}(B) \to \operatorname{Sp}(A)$. Suppose U is an affinoid subdomain of $\operatorname{Sp}(A)$ and $\iota \colon \operatorname{Sp}(A') \to \operatorname{Sp}(A)$ the associated map of affinoid spaces. Let $B' = B \widehat{\otimes}_A A'$ and $j \colon \operatorname{Sp}(B') \to \operatorname{Sp}(B)$ the natural map. Then $(\varphi^{-1}(U), j)$ is an affinoid subdomain of $\operatorname{Sp}(B)$.

Proof. We have a cartesian square (with φ' being the natural map)

$$\begin{array}{c|c} \operatorname{Sp}(B') & \stackrel{\varphi'}{\longrightarrow} & \operatorname{Sp}(A') \\ \downarrow^{j} & \Box & \downarrow^{i} \\ \operatorname{Sp}(B) & \stackrel{\varphi}{\longrightarrow} & \operatorname{Sp}(A) \end{array}$$

and hence $(\varphi \circ j)(\operatorname{Sp}(B')) \subset U$. It follows that $j(\operatorname{Sp}(B')) \subset \varphi^{-1}(U)$. If $\psi^* \colon B \to C$ is a map of K-affinoid algebras such that $\psi(\operatorname{Sp}(C)) \subset \varphi^{-1}(U)$, where $\psi = {}^a\psi^*$, then applying the universal property of i to $\varphi \circ \psi$ we get a unique a map $\theta \colon \operatorname{Sp}(C) \to$ $\operatorname{Sp}(A')$ such that $i \circ \theta = \varphi \circ \psi$. Next, the universal property of the fibre product $\operatorname{Sp}(B') = \operatorname{Sp}(A') \times_{\operatorname{Sp}(A)} \operatorname{Sp}(B)$, applied to the pair of maps ψ and θ , gives us the asserted universal property for j.

3. Basic properties of affinoid subdomains

3.1. A couple of remarks tying the discussions above with matters in earlier lectures is probably worth our while. First, if $(U, i: \text{Sp}(A') \to X)$ is an affinoid subdomain of X = Sp(A), then in view of Proposition 2.1.2 of Lecture 16, especially item (i) of *loc.cit.*, we can identify U with Sp(A'). We often do this and simply say U is an affinoid subdomain of X. When we do this we write $\mathscr{O}_X(U)$ for A'. From this point of view, the main conclusion of Proposition 2.2.1 can be rephrased succinctly as:

The inverse image of an affinoid subdomain of Sp(A) is an affinoid subdomain of Sp(B).

Next, compare 2.1.3(10) with Proposition 1.3.5 of Lecture 15 and Proposition 2.1.3 of Lecture 16 to understand how inverse images of special affinoid subdomains look under maps of affinoid domains.

Proposition 3.1.1. Using the conventions above, let X be an affinoid space, U an affinoid subdomain of X, and V an affinoid subdomain of U. Then V is an affinoid subdomain of X.

Proof. Unpackaged, using the notation introduced directly above, what is being said is this: the map $\mathscr{O}_X(X) \to \mathscr{O}_U(V)$ given by the composite $\mathscr{O}_X(X) \to \mathscr{O}_X(U) = \mathscr{O}_U(U) \to \mathscr{O}_U(V)$, has the required universal property turning $(V, \mathscr{O}_X(X) \to \mathscr{O}_U(V))$ into an affnoid domain. But this is essentially a tautology. (We can therefore write $\mathscr{O}_U(V) = \mathscr{O}_X(V)$.)

Proposition 3.1.2. The intersection of two affinoid subdomains of an affinoid space is again an affinoid subdomain.

Proof. This is a direct consequence of Proposition 2.2.1. In greater detail, if U and V are affinoid subdomains of X, then $U \cap V$ can be identified with $U \times_X V$, giving the result. Note that $\mathscr{O}_A(U \cap V) = \mathscr{O}_A(U) \widehat{\otimes}_{\mathscr{O}_X(X)} \mathscr{O}_X(V)$.