## LECTURE 17

## Date of Lecture: October 17, 2019

Unless otherwise specified, K is a complete non-archimedean field, and to avoid annoying trivialities we assume the absolute value | | on K is non-trivial. As before  $\mathbf{N} = \{0, 1, 2, \dots, m, \dots\}$ . Rings mean commutative rings with 1.

The symbol 2 is for flagging a cautionary comment or a tricky argument. It occurs in the margins and is Knuth's version of Bourbaki's "dangerous bend symbol".

## 1. Affinoid subdomains are open

We will be using freely the results from the supplement to Lecture 17. In particular, as mentioned in §§3.1 of the supplement, if X = Sp(A) is an affinoid K-space and  $(U, i: \text{Sp}(A') \to X)$  an affinoid subdomain of X, we simply say U is an affinoid subdomain of X and write  $\mathscr{O}_X(U)$  for A'.

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1.1. Intersection of the special affinoid subdomains. Let X = Sp(A) be an K-affinoid space. The Weierstrass, Laurent, and rational subdomains of X are called special affinoid subdomains, and as we saw in Proposition 2.1.3 of Lecture 16, these are open affinoid subdomains of X. The following is a useful result.

**Lemma 1.1.1.** The intersection of two Weierstrass (resp. Laurent, resp. rational) affinoid subdomains of X is again a Weierstrass (resp. Laurent, resp. rational) affinoid subdomain of X.

*Proof.* It is more or less straightforward from the definitions that the intersection of two Weierstrass subdomains is a Weirstrass subdomain and that the intersection of two Laurent subdomains is a Laurent subdomain. The same is true for rational subdomains of X. Indeed, suppose  $f_0, f_1, \ldots, f_n$  generate the unit ideal in A and say, so do  $g_0, g_1, \ldots, g_m$ . Then  $(f_i g_j)$  generates the unit ideal in A, and it is easy to see that  $A\langle \mathbf{f}/f_0 \rangle \widehat{\otimes}_A A \langle \mathbf{g}/g_0 \rangle = A\langle (\frac{f_i g_j}{f_0 g_0})_{(i,j) \neq (0,0)} \rangle$ . From a point set perspective, if  $x \in X(\mathbf{f}/f_0) \cap X(\mathbf{g}/g_0)$  then  $f_i(x)g_j(x) \leq f_0(x)g_0(x)$  for all  $(i,j) \neq (0,0)$ . Conversely, suppse  $x \in X$  is such  $f_i(x)g_j(x) \leq f_0(x)g_0(x)$ , for all  $(i,j) \neq (0,0)$ . Note that  $(f_i g_j)_{i,j \geq 0}$  generates the unit ideal in A. If  $f_0(x) = 0$ , then  $f_i(x)g_j(x) = 0$ for all i and j, which is not possible from what we just observed. Hence  $f_0(x) \neq 0$ . Similarly  $g_0(x) \neq 0$ . The inequalities  $f_i(x)g_0(x) \leq f_0(x)g_0(x), 1 \leq i \leq n$  and  $f_0(x)g_j(x) \leq f_0(x)g_0(x), 1 \leq j \leq m$  then give  $x \in X(\mathbf{f}/f_0) \cap X(\mathbf{g}/g_0)$ . From either point of view (fibre products or set-theoretic) we are done. □

**1.2.** The first theorem below is one relating "punctual" behaviour with local behaviour.

**Theorem 1.2.1.** Let  $\sigma: A \to B$  be a map of K-affinoid algebras, X = Sp(A), Y = Sp(B) and  $\varphi(= {}^{a}\sigma): Y \to X$  the corresponding map of K-affinoid spaces. Let  $x \in X$  be a point of X and let  $\mathfrak{m} = \mathfrak{m}_{x}$ . Assume that  $\sigma$  induces

(i) A surjective homomorphism  $A/\mathfrak{m} \to B/\mathfrak{m}B$ , or

(ii) isomorphisms  $A/\mathfrak{m}^n \xrightarrow{\sim} B/\mathfrak{m}^n B$ ,  $n \in \mathbb{N}$ .

Then there exists an open affinoid subdomain  $U = \operatorname{Sp}(A')$  of A such that  $x \in U$  and the K-algebra map  $\sigma' \colon A' \to A' \widehat{\otimes}_A B$  induced by  $\sigma$  is surjective, or is an isomorphism respectively. For hypothesis (ii), the conclusion can be restated as saying that

$$U \times_X Y \xrightarrow{\sim} U$$

where the arrow is the natural projection, or that

$$\varphi|_{\varphi^{-1}(U)} \colon \varphi^{-1}(U) \xrightarrow{\sim} U$$

*Proof.* Let  $\mathfrak{m} = (m_1, \ldots, m_s)$ .

Let us prove (i). There are two possibilities. Either  $\mathfrak{m}B = B$  or, since  $A/\mathfrak{m}$  is a field,  $A/\mathfrak{m} \xrightarrow{\sim} B/\mathfrak{m}B$ . Suppose  $\mathfrak{m}B = B$ . We have to show the existence of an open affinoid subdomain U such that  $x \in U$  and  $\varphi^{-1}(U) = \emptyset$ . Since  $\mathfrak{m}B = B$ , there exist  $g_1, \ldots, g_s \in B$  such that

(\*) 
$$\sigma(m_1)g_1 + \dots + \sigma(m_s)g_s = 1.$$

Let  $c \in K^*$  be such that  $|c|^{-1} > \max_j ||g_j||$ . If we set  $U = X(m_1/c, \ldots, m_s/c)$ then  $x \in U$  and  $m_i(x) < c$  for  $i = 1, \ldots, s$ . We claim  $\varphi^{-1}(U) = \emptyset$ . Indeed, if  $y \in \varphi^{-1}(U)$ , then

Now (\*) and (\*\*) contradict each other, and hence  $\varphi^{-1}(U) = \emptyset$ .

Now suppose  $A/\mathfrak{m} \longrightarrow B/\mathfrak{m}B$ . Then the fibre  $\varphi^{-1}(x)$  is non-empty. In fact it consists of exactly one point since  $B/\mathfrak{m}B$  has only one maximal ideal. Say  $\varphi^{-1}(x) = \{y\}$ . Let  $b_1, \ldots, b_n \in B$  be "topological" K-algebra generators of B, i.e.,  $b_i$  are such that  $B = K\langle b_1, \ldots, b_n \rangle$  (see Remark 1.1.7 of the supplement). Such  $b_i$ always exist (see [Lecture 16, (1.1.2)]). Since  $A/\mathfrak{m} \longrightarrow B/\mathfrak{m}B$ , we can find  $a_i \in A$ and  $\beta_{ij} \in B$ ,  $i = 1, \ldots, n$ ,  $j = 1, \ldots, s$  such that

(1.2.1.1) 
$$b_i - \sigma(a_i) \in \sum_{j=1}^s \beta_{ij} \sigma(m_j), \quad (i = 1, \dots, n, j = 1, \dots, s),$$

where, as in the beginning of this proof,  $(m_1, \ldots, m_s) = \mathfrak{m}$ .

To go further, we need to make a couple of observations.

1) Suppose  $X' = \operatorname{Sp}(A')$  is an open affinoid subdomain of X such that  $x \in X'$ . Let  $Y' = Y \times_X X'$  and  $\varphi' \colon Y' \to X'$  the base change of  $\varphi$ . Then  $\varphi' \colon Y' \to X'$  satisfies the same hypotheses as  $\varphi$ , and the fibre of  $\varphi'$  over x is the same as the fibre of  $\varphi$  over x. This means we may replace  $X, Y, \varphi$  by  $X', Y', \varphi'$  if it is convenient to. We will need to do this.

2) Let  $Z = \operatorname{Sp}(C)$  be an affinoid K-space. Set

$$\mathbb{B}^n = \operatorname{Sp}(T_n)$$
 and  $\mathbb{B}^n_Z = Z \times_{\operatorname{Sp}(K)} \mathbb{B}^n = \operatorname{Sp}(C\langle \zeta_1, \dots, \zeta_n \rangle).$ 

Note that since B is the homomorphic image of  $T_n = K\langle \zeta_1, \ldots, \zeta_n \rangle$  with  $\zeta_i$  mapping to  $b_i$ , we have  $||b_i||_{\sup} \leq 1$ . According to Theorem 2.1.4 of the supplement to this lecture, we have a map  $\sigma: A\langle \zeta_1, \ldots, \zeta_n \rangle \to B$ . Moreover, this map is surjective. Indeed, since the  $b_i$  are topological K-algebra generators of B, they are topological A-algebra generators, and hence  $\sigma$  is surjective. In other words  $B = A\langle b_1, \ldots, b_n \rangle$ using the notation in (2.1.5.1) of the supplement to this lecture. We therefore have a commutative diagram, with the horizontal hooked arrow being the closed immersion associated to the surjective map  $A\langle \boldsymbol{\zeta} \rangle \twoheadrightarrow A\langle \boldsymbol{b} \rangle = B$ :



The point is that the above diagram behaves well with respect to open affinoid base changes. In greater detail, suppose,  $X' = \operatorname{Sp}(A')$  is an open affinoid subdomain of X, and Y',  $\varphi'$  are as in item 1) above (so X' is an open affinoid subdomain containing x). Then we have the following commutative diagram with the triangle in the foreground being the "base change" of (1.2.1.2) (i.e. of the triangle in the background) via the open immersion  $X' \subset X$ .



The rectangle on the right (east) face, the rectangle on the inclined plane, and the rectangle on top are all obviously cartesian. It is immediate that relations such as (1.2.1.1) survive base changes by open affinoid subdomains which contain x. We will use this implicitly.

And example of such an open affinoid subdomain containing x is the Weierstrass domain  $X' = X(a_1, \ldots, a_n)$ . We have already seen that  $||b_i||_{\sup} \leq 1$ . Since  $a_i(x) = \sigma(a_i)(y) = b_i(y)$ , and since  $||b_i||_{\sup} \leq 1$ , we therefore have  $|a_i(x)| \leq 1$  for  $i = 1, \ldots, n$ . In other words  $x \in X(a_1, \ldots, a_n)$ . Let us replace X by  $X(a_1, \ldots, a_n)$ , and thereby assume that

(1.2.1.4) 
$$||a_i||_{\sup} \le 1, \quad (i = 1, \dots, n).$$

Then, using Theorem 1.1.6 of the supplement to this lecture, we can find a Tate algebra mapping surjectively onto A with  $a_1, \ldots, a_n$  images of subset of the free affinoid variables in this Tate algebra. We give A the residue norm from this surjective map and thereby conclude that with this reisude norm

(1.2.1.5) 
$$||a_i|| \le 1, \quad (i = 1, \dots, n).$$

We give  $A\langle \boldsymbol{\zeta} \rangle$  the norm from A, namely  $\|\sum_{\boldsymbol{\nu}} a_{\boldsymbol{\nu}} \boldsymbol{\zeta}^{\boldsymbol{\nu}}\| = \max_{\boldsymbol{\nu}} \|a_{\boldsymbol{\nu}}\|$ , and B the residue norm from  $A\langle \boldsymbol{\zeta} \rangle$ . Then

(1.2.1.6) 
$$||b_i|| \le 1, \quad (i = 1, \dots, n).$$

One more replacement of X by an X' is needed. First, by rescaling the generators  $m_1, \ldots, m_s$  of  $\mathfrak{m}$ , we may assume that the  $\beta_{ij}$  occurring in (1.2.1.1) satisfy

(1.2.1.7) 
$$\|\beta_{ij}\| \le 1, \quad (i = 1, \dots, n, j = 1, \dots, s).$$

Pick  $c \in K^*$  such that 0 < |c| < 1 and let  $X' = X(c^{-1}m_1, \ldots, c^{-1}m_s)$ . If we replace X be X', then relations (1.2.1.4), (1.2.1.5), and (1.2.1.6) all survive. The latter two by (2.1.5.2) of the supplement to this lecture (*loc.cit.* applies because  $X(c^{-1}m_1, \ldots, c^{-1}m_s) = \operatorname{Sp}(A\langle c^{-1}m_1, \ldots, c^{-1}m_s \rangle)$ ). Finally, since  $A\langle c^{-1}m_1, \ldots, c^{-1}m_s \rangle$  can be given the residue norm from  $A\langle \xi_1, \ldots, \xi_s \rangle$  where the  $\xi_i$  are topological free variables for A, we if we replace X by X' (and implicitly, A by  $A' = A\langle c^{-1}m \rangle$ ) we get

(1.2.1.8) 
$$||m_j|| \le |c| < 1, \qquad (j = 1, \dots, s).$$

Let  $\mu \in \mathbf{N}$ . We have

$$\begin{aligned} \|b_{i}^{\mu} - \sigma(a_{i})^{\mu}\| &\leq \|b_{i} - \sigma(a_{i})\| \left\| \sum_{j=0}^{\mu-1} b_{i}^{\mu-j-1} \sigma(a_{i}^{j}) \right\| \\ &\leq \|b_{i} - \sigma(a_{i})\| \max_{0 \leq j \leq \mu-1} \{ \|b_{i}^{\mu-1-j}\| \|a_{i}^{j}\| \} \\ &\leq \|b_{i} - \sigma(a_{i})\| \quad (\text{via } (1.2.1.5) \text{ and } (1.2.1.6)) \\ &\leq \max_{1 \leq j \leq s} \{ \|\beta_{ij}\| \|m_{j}\| \} \quad (\text{via } (1.2.1.1)) \\ &\leq |c| \quad (\text{via } (1.2.1.8)). \end{aligned}$$

More generally, using the above repeatedly, for  $(\mu_1, \ldots, \mu_n) \in \mathbf{N}^n$  one has:

(1.2.1.9) 
$$\|b_1^{\mu_1} \dots b_n^{\mu_n} - \sigma(a_1)^{\mu_1} \dots \sigma(a_n)^{\mu_n}\| \le |c|.$$

If  $b \in B$  is such that  $\|b\| < 1$ , since the norm on B is the infimum over Gauss norms of preimages of b in  $A\langle \zeta \rangle$ , there is a representation  $b = \sum_{\mu \in \mathbf{N}^n} \alpha_{\mu} b_1^{\mu_1} \dots b_n^{\mu_n}$ ,  $\alpha_{\mu} \in A$ , such that  $\max_{\mu \in \mathbf{N}^n} \|\alpha_{\mu}\| < 1$ . In this case, the series  $\sum_{\mu \in \mathbf{N}^n} \alpha_{\mu} a_1^{\mu_1} \dots a_n^{\mu_n}$ converges since  $\|a_i\| \le 1$ . Let  $a = \sum_{\mu \in \mathbf{N}^n} \alpha_{\mu} a_1^{\mu_1} \dots a_n^{\mu_n}$ . Then  $\|a\| < 1$ . Moreover by (1.2.1.9) we get  $\|b - \sigma(a)\| < |c|$ . To summarize, for each  $b \in B$  with  $\|b\| < 1$ we have  $a \in A$  such that

$$||a|| < 1$$
, and  $||b - \sigma(a)|| < |c|$ .

This means if  $||b|| < |c|^{\nu}$  for some  $\nu \in \mathbf{N}$  then we can find  $a \in A$  such that

(1.2.1.10) 
$$||a|| < |c|^{\nu}$$
, and  $||b - \sigma(a)|| < |c|^{\nu+1}$ .

From here it is easy to see that  $\sigma: A \to B$  is surjective. Indeed, if  $b \in B$  is such that ||b|| < 1, then from (1.2.1.10) we can find a sequence  $\{a(\mu)\}_{\mu \in \mathbb{N}}$  in A such that the following two inequalities hold

(1.2.1.11) 
$$\|a(\mu)\| < |c|^{\mu}, \quad (\mu \in \mathbf{N}),$$
$$\|b - \sum_{\mu=0}^{\nu} \sigma(a(\mu))\| \le |c|^{\nu+1}, \quad (\nu \in \mathbf{N}).$$

The series  $\sum_{\mu} a(\mu)$  clearly converges. Let  $a = \sum_{\mu \in \mathbf{N}} a(\mu)$ . We have the string of equalities  $\sigma(a) = \sum_{\mu} \sigma(a(\mu)) = b$ , where the first equality follows from the continuity of  $\sigma$  and the second equality from the second inequality in (1.2.1.11). Thus

$$||b|| < 1 \implies b \in \sigma(A).$$

In general, we can find  $\gamma \in K^*$  such that  $\|\gamma b\| < 1$ , and so  $\gamma b \in \sigma(A)$ , which means  $b \in \sigma(A)$ . We have therefore proven (i).

We now prove (ii). In view of (i) and the nature of the assertion (ii), we may assume without loss of generality that  $\sigma: A \to B$  is surjective. Let  $\mathfrak{a} = \ker \sigma$ and write  $\bar{A} = A/\mathfrak{a}$ . Then  $\mathfrak{a} \subset \cap_{\mu} \mathfrak{m}^{\mu}$ , and by Krull's intersection theorem we get  $\mathfrak{a}_{\mathfrak{m}} = (0) \subset A_{\mathfrak{m}}$ . We can therefore find  $m \in \mathfrak{m}$  such that f = 1 - m annihilates  $\mathfrak{a}$ . Hence one has a map  $A/\mathfrak{a} \to A[f^{-1}]$  such that the composite  $A \twoheadrightarrow \bar{A} \to A[f^{-1}]$  is the localisation map  $A \to A[f^{-1}]$ . In particular, the map  $A \to A\langle f^{-1} \rangle$  factors as  $A \twoheadrightarrow \bar{A} \to A\langle f^{-1} \rangle$ . Thus

$$\bar{A}\langle \bar{f}^{-1}\rangle = A\langle f^{-1}\rangle \widehat{\otimes}_A \bar{A} = A\langle f^{-1}\rangle.$$

In greater detail, since  $\mathfrak{a}A\langle f^{-1}\rangle = 0$  we have  $A\langle f^{-1}\rangle \otimes_A \overline{A} = A\langle f^{-1}\rangle$  which is already norm complete, giving the above equality. Since  $\overline{A} \xrightarrow{\sim} B$  (via the surjective map  $\sigma$ ), we have shown that we have an isomorphism

$$\sigma' \colon A\langle f^{-1} \rangle \xrightarrow{\sim} B\langle \sigma(f)^{-1} \rangle$$

where  $\sigma'$  is the map induced by  $\sigma$ . This proves (ii).

**2.** Weak and strong topologies on Sp(A)

Let X = Sp(A) be an affinoid space. It has the canonical topology on it. In Lecture 14, §§1.2 we defined the notion of a *G*-topology on a topological space. There are three *G*-topologies that can be defined on *X* with its canonical topology, the one coming from the canonical topology, the *weak G-topology*, and the *strong G-topology*.

In the discussion that follows we fix X = Sp(A).

**2.1.** We set  $(\widetilde{\mathcal{T}}, \widetilde{\mathscr{C}ov})$  equal to the canonical topology on X. In greater detail

 $\widetilde{\mathcal{T}} = \{ U \mid U \text{ is a canonical open subset of } X \}$ 

and for  $U \in \widetilde{\mathcal{T}}$ 

$$\widetilde{\mathscr{C}ov}(U) = \Big\{ \{U_{\alpha}\}_{\alpha \in I} \ \Big| \ U_{\alpha} \in \widetilde{\mathcal{T}} \text{ for every } \alpha \in I \text{ and } \bigcup_{\alpha \in I} U_{\alpha} = U \Big\}.$$

## **2.2. The weak** *G*-topology on Sp(A). Set

 $\mathcal{T}_w = \{ U \mid U \text{ is an affinoid subdomain of } X \}.$ 

Note that by (ii) of Theorem 1.2.1, if  $U \in \mathcal{T}_w$  then U is open in the canonical topology. For each  $U \in \mathcal{T}_w$  let

$$\mathscr{C}ov_w(U) = \Big\{ \{U_\alpha\}_{\alpha \in I} \in \widetilde{\mathscr{C}ov}(U) \, \Big| \, U_\alpha \in \mathcal{T}_w \text{ for every } \alpha \in I \text{ and } I \text{ is a finite set} \Big\}.$$

One checks easily that  $(\mathcal{T}_w, \mathscr{C}ov_w)$  is a *G*-topology on *X*. It is called *the weak G*-topology or simply the weak topology on *X*.

**2.3.** The strong *G*-topology on Sp(A). This is the topology  $(\mathcal{T}, \mathscr{C}ov)$  defined as follows:  $U \in \mathcal{T}$  if

- $U \in \widetilde{\mathcal{T}}$ , and
- there exists  $\mathscr{U} = \{U_{\alpha}\}_{\alpha \in I} \in \widetilde{\mathscr{Cov}}(U)$  (*I* not necessarily finite) such that  $U_{\alpha} \in \mathcal{T}_w$  for each  $\alpha \in I$ , and whenever  $\varphi \colon Z \to X$  is a morphism of affinoid *K*-spaces with  $\varphi(Z) \subset U$ , there is a refinement  $\mathscr{V}$  of  $\varphi^{-1}(\mathscr{U})$  with  $\mathscr{V} \in \mathscr{Cov}_w(\varphi^{-1}(U)).$

For  $U \in \mathcal{T}$ ,  $\mathscr{U} = \{U_{\alpha}\}_{\alpha \in I} \in \mathscr{C}ov(U)$  if

- $\mathscr{U} \in \widetilde{\mathscr{C}ov}(U)$ , and
- $U_{\alpha} \in \mathcal{T}$  for each  $\alpha \in I$ , and
- whenever  $\varphi \colon Z \to X$  is a morphism of affinoid K-spaces with  $\varphi(Z) \subset U$ , there is a refinement  $\mathscr{V}$  of  $\varphi^{-1}(\mathscr{U})$  with  $\mathscr{V} \in \mathscr{C}ov_w(\varphi^{-1}(U))$ .

Once again, one checks easily that  $(\mathcal{T}, \mathscr{C}ov)$  is a *G*-topology on *X*. It is called the strong *G*-topology or simply the strong topology on *X*.

One can show more. It turns out that sheaves on  $(\mathcal{T}_w, \mathscr{C}ov_w)$  have a unique extension to sheaves on  $(\mathcal{T}, \mathscr{C}ov)$ , and further the strong topology enjoys certain completeness properties. More on this in later lectures.