LECTURE 16

Date of Lecture: October 15, 2019

Unless otherwise specified, K is a complete non-archimedean field, and to avoid annoying trivialities we assume the absolute value $|\ |$ on K is non-trivial. As before $\mathbf{N} = \{0, 1, 2, \dots, m, \dots\}$. Rings mean commutative rings with 1.

The symbol ☡ is for flagging a cautionary comment or a tricky argument. It occurs in the margins and is Knuth's version of Bourbaki's "dangerous bend symbol".

1. Restricted power series

1.1. **The rings $A\langle\boldsymbol{\zeta}\rangle$ and $A\langle\boldsymbol{f}\rangle$.** Suppose $(R, \|\ \|)$ is a Banach ring with $\|x+y\| \leq \max\{\|x\|, \|y\|\}$ for $x, y \in R$. Then one defines the ring of *restricted power series* $R\langle\zeta_1, \dots, \zeta_n\rangle (= R\langle\boldsymbol{\zeta}\rangle)$ in essentially the same way as the Tate algebra T_n over $(K, |\ |)$ was defined, namely $R\langle\boldsymbol{\zeta}\rangle$ consists of formal power series $\sum_{\boldsymbol{\nu}\in\mathbf{N}^n} c_{\boldsymbol{\nu}}\boldsymbol{\zeta}^{\boldsymbol{\nu}}$ with coefficients $c_{\boldsymbol{\nu}}$ in R such that $\|c_{\boldsymbol{\nu}}\| \to 0$ as $|\boldsymbol{\nu}| \to \infty$. And one can give $R\langle\boldsymbol{\zeta}\rangle$ the Gauss norm arising from $\|\ \|$.

If A is an affinoid K-algebra, since all residue norms $\|\ \|_\alpha$ arising from surjective maps α from Tate algebras to A are equivalent, the algebra $A\langle\boldsymbol{\zeta}\rangle$ makes sense independent of $\|\ \|_\alpha$. Moreover there is an equivalent description of $A\langle\boldsymbol{\zeta}\rangle$, namely the following. Suppose $\alpha\colon K\langle\xi_1, \dots, \xi_r\rangle \twoheadrightarrow A$ is a surjection from a Tate algebra T_r to A with kernel $\mathfrak{a}$. Now $T_{n+r} = K\langle\xi_1, \dots, \xi_r, \zeta_1, \dots, \zeta_n\rangle$ and $T_r = K\langle\boldsymbol{\xi}\rangle$ is a sub-algebra of $T_{n+r} = K\langle\boldsymbol{\xi}, \boldsymbol{\zeta}\rangle$ in an obvious way. Then clearly, by definition of $\|\ \|_\alpha$ and of restricted power series we have

$$A\langle\zeta_1, \dots, \zeta_n\rangle = \frac{T_{n+r}}{\mathfrak{a}T_{n+r}}. \tag{1.1.1}$$

Seen this way $A\langle\boldsymbol{\zeta}\rangle$ is clearly an affinoid K-algebra.

For $f_1, \dots, f_n \in A$, we define $A\langle f_1, \dots, f_n\rangle = A\langle\boldsymbol{f}\rangle$ as the A-algebra:

$$A\langle f_1, \dots, f_n\rangle := \frac{A\langle\zeta_1, \dots, \zeta_n\rangle}{(\zeta_1 - f_1, \dots, \zeta_n - f_n)}. \tag{1.1.2}$$

Note that $A\langle\boldsymbol{f}\rangle$ is an affinoid K-algebra,[1] since it is the quotient of one by an ideal. We will see later that $\mathrm{Sp}(A\langle\boldsymbol{f}\rangle)$ can be identified with the Weierstrass domain $X(\boldsymbol{f})$.

For sequences $\boldsymbol{f} = (f_i)_{i=1}^r$, $\boldsymbol{g} = (g_j)_{j=1}^s$ in A, the we define an A-algebra which "inverts" the g_j's, denoted $A\langle f_1, \dots, f_r, 1/g_1, \dots, 1/g_s\rangle$, or in a more compact niotation $A\langle\mathbf{f}, \mathbf{1}/\boldsymbol{g}\rangle$, as follows:

$$A\langle\boldsymbol{f}, \mathbf{1}/\boldsymbol{g}\rangle := \frac{A\langle\zeta_1, \dots, \zeta_r, \xi_1, \dots, \xi_s\rangle}{(\zeta_1 - f_1, \dots, \zeta_r - f_r, 1 - \xi_1 g_1, \dots, 1 - \xi_s g_s)}. \tag{1.1.3}$$

Once again, note that $A\langle\mathbf{f}, \mathbf{1}/\boldsymbol{g}\rangle$ is an affinoid K-algebra. It turns out that $\mathrm{Sp}(A\langle\boldsymbol{f}, \mathbf{1}/\boldsymbol{g}\rangle)$ can be identified with the Laurent domain $X(\boldsymbol{f}, \mathbf{1}/\boldsymbol{g})$.

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[1](1.1.2) is actually a slight abuse of notation. We should write $A\langle\sigma(f_1), \dots, \sigma(f_n)\rangle$ for $A\langle\zeta_1, \dots, \zeta_n\rangle/(\zeta_1 - f_1, \dots, \zeta_n - f_n)$ where σ is the natural map from A to the right side of (1.1.2).

Finally the affinoid algebras which (as we will see) corresponds to rational domains are defined as follows. Let $f_0, f_1, \ldots, f_n$ be elements in A such that the f_j's generate the unit ideal in A. Then $A\langle \mathbf{f}/f_0\rangle = A\langle f_1/f_0, \ldots, f_n/f_0\rangle$ is defined as an A-algebra by the formula

$$A\langle \mathbf{f}/f_0\rangle := \frac{A\langle \zeta_0, \zeta_1, \ldots, \zeta_n\rangle}{(\zeta_1 f_0 - \zeta_0 f_1, \ldots, \zeta_n f_0 - \zeta_0 f_n)}. \tag{1.1.4}$$

$A\langle \mathbf{f}/f_0\rangle$ is clearly an affinoid K-algebra. As one can guess, it turns out that $\mathrm{Sp}(A\langle \mathbf{f}/f_0\rangle)$ can be identified with the rational domain $X(f_1/f_0, \ldots, f_n/f_0)$.

2. Affinoid subdomains

2.1. In the last lecture, we defined certain special open subsets of $\mathrm{Sp}(A)$, where A is an affinoid K-algebra, namely the Weierstrass, Laurent, and rational subdomains. These turn out to be affinoid subdomains in the sense of the following definition.

Definition 2.1.1. Let $X = \mathrm{Sp}(A)$, where A is an affinoid K-algebra. An *affinoid subdomain of X* is a subset U of X together with a map of affinoid K-spaces $\imath\colon X' = \mathrm{Sp}(A') \to X$ having the following properties:

(i) $\imath(X') \subset U$.

(ii) Given a map $\varphi\colon \mathrm{Sp}(B) \to X$ of affinoid K-spaces with $\varphi(\mathrm{Sp}(B)) \subset U$, there exists a unique map of affinoid spaces $\psi\colon \mathrm{Sp}(B) \to X'$ such that the broken arrow in the diagram below can be filled to make it commute.

$$\begin{tikzcd} \mathrm{Sp}(B) \arrow[r] \arrow[d, dashed, "\exists!\,\psi"'] \arrow[dr, "\varphi"] & U \arrow[d, hook] \\ X' \arrow[r, "\imath"'] \arrow[ur] & X \end{tikzcd}$$

We say that an affinoid subdomain $(U, \imath)$ is an *open affinoid subdomain of X* if U is open in X. (We will show later in the course that all affinoid subdomains are in fact open affinoid subdomains.)

Proposition 2.1.2. *Let $X = \mathrm{Sp}(A)$ be an affinoid K-space $(U, \imath\colon \mathrm{Sp}(A') \to X)$ an affinoid subdomain of X and $\imath^*\colon A \to A'$ the resulting map of K-affinoid algebras. Let $X' = \mathrm{Sp}(A')$, $x \in U$, and $\mathfrak{m} = \mathfrak{m}_x$. Then:*

(i) *The fibre $\imath^{-1}(x)$ consists of a single point $x' \in X'$. Thus $\imath(X') = U$, and $\imath\colon X' \xrightarrow{\sim} U$ is a set-theoretic bijection.*

(ii) *Let x' be as in* (i). *Then $\mathfrak{m}_{x'} = \mathfrak{m}A'$.*

(iii) *The map $\imath^*\colon A \to A'$ induces isomorphisms $A/\mathfrak{m}^n \xrightarrow{\sim} A'/\mathfrak{m}^n A'$, one for each $n \in \mathbf{N}$.*

Proof. For $n \in \mathbf{N}$ we have a commutative diagram

$$\begin{tikzcd} A \arrow[r, "\imath^*"] \arrow[d, "\pi"'] & A' \arrow[d, "\pi'"] \\ A/\mathfrak{m}^n \arrow[r, "\sigma"'] & A'/\mathfrak{m}^n A' \end{tikzcd}$$

with π and π' being the natural surjections and σ the map induced by $\imath$. Since ${}^a\pi(\mathrm{Sp}(A/\mathfrak{m}^n)) = \{x\} \subset U$, the universal property of $(\imath, X')$ gives us a map

$$\alpha\colon A' \to A/\mathfrak{m}^n$$

such that $\pi = \alpha \circ \imath^*$. Consider the diagram

$$\begin{tikzcd} A \arrow[r, "\imath^*"] \arrow[d, "\pi"'] & A' \arrow[dl, "\alpha"'] \arrow[d, "\pi'"] \\ A/\mathfrak{m}^n \arrow[r, "\sigma"'] & A'/\mathfrak{m}^n A' \end{tikzcd}$$

Since $\pi = \alpha \circ \imath^*$, the top triangle commutes. The outer rectangle commutes. We claim that the bottom triangle also commutes. By the universal property of $(\imath, X')$, $p = \pi'$ is the only solution of the equation

$$(*) \qquad \sigma \circ \pi = p \circ \imath^*.$$

On the other hand $\sigma \circ \pi = \sigma \circ (\alpha \circ \imath^*) = (\sigma \circ \alpha) \circ \imath^*$, and hence $p = \sigma \circ \alpha$ is also a solution to $(*)$. By the uniqueness of solutions to $(*)$, we have $\pi' = \sigma \circ \alpha$, and hence the lower triangle commutes. Note that α is surjective since π is, and σ is surjective since π' is.

Since $\pi = \alpha \circ \imath^*$, $\imath^*(\mathfrak{m}^n) \subset \ker \alpha$ and this in turn implies that $\mathfrak{m}^n A' \subset \ker \alpha$. On the other hand, the relation $\pi' = \sigma \circ \alpha$ implies that $\ker \alpha \subset \mathfrak{m}^n A'$. Thus $\mathfrak{m}^n A' = \ker \alpha$, implying (since α and π' are surjective) that σ is an isomorphism. This proves (iii). Moroever, this shows that $A'/\mathfrak{m}^n A'$ is an artin local ring, whence $\imath^{-1}(x)$ consists of exactly one point, say x'. This proves (i). Setting $n = 1$, we see that $\mathfrak{m}A'$ is a maximal ideal. Since $\mathfrak{m}_{x'} \supset \mathfrak{m}A'$, we therefore have $\mathfrak{m}_{x'} = \mathfrak{m}A'$. This proves (ii) and we are done. $\square$

Proposition 2.1.3. *Let A be a K-affinoid algebra. Then Weierstrass, Laurent, and rational domains in $X = \mathrm{Sp}(A)$ are open affinoid subdomains.*

Proof. By Lemma 2.1.1 of Lecture 15, Weierstrass, Laurent, and rational domains are open in the canonical topology. It remains to prove the universal property of affinoid subdomains for them.

We will prove the result for Weierstrass domains. The proofs for other domains are similar and we indicate briefly how one proceeds in those cases. So suppose $U = X(f_1, \ldots, f_n)(= X(\boldsymbol{f}))$.

Let $A' = A\langle f_1, \ldots, f_n\rangle$ where $A\langle f_1, \ldots, f_n\rangle(= A\langle \boldsymbol{f}\rangle)$ is as in (1.1.2).

Let $\imath^*\colon A \to A'$ be the natural map, $X' = \mathrm{Sp}(A')$, and $\imath\colon X' \to X$ the resulting map of affinoid spaces.

We have to show that $\imath(X') \subset X(\boldsymbol{f})$ and that $\imath$ has the required universal property. To that end suppose $\varphi^*\colon A \to B$ is a map of affinoid algebras, $Y = \mathrm{Sp}(B)$, and $\varphi\colon Y \to X$ the resulting map of afinoid spaces. Since $K(\varphi(y)) = A/\mathfrak{m}_{\varphi(y)} \hookrightarrow B/\mathfrak{m}_y = K(y)$ is a finite extension of fields for every $y \in Y$, we have

$$|\varphi^*(f_i)(y)| = |f_i(\varphi(y))| \qquad (i = 1, \ldots, n,\ y \in Y).$$

It follows that $\varphi(Y) \subset X(\boldsymbol{f})$ if and only if $\|\varphi^*(f_i)\|_{\sup} \leq 1$ for $i = 1, \ldots, n$. Since $\imath^*(f_i)$ is the residue of ζ_i under the identification $A' = A\langle \boldsymbol{\zeta} \rangle / (\zeta_i - f_i \mid i = 1, \ldots, n)$ of (1.1.2), it follows that $\|\imath^*(f_i)\|_{\sup} \leq \|\zeta_i\| = 1$. Hence $\imath^*(Y) \subset X(\boldsymbol{f})$.

Let Y, B, φ be as above. Suppose $\varphi(Y) \subset X(\boldsymbol{f})$. Let $b_i = \varphi^*(f_i)$, $i = 1, \ldots, n$. Then $\|b_i\| \leq 1$ from our discussion above. It follows that the series $\sum_{\boldsymbol{\nu} \in \mathbf{N}^r} c_{\boldsymbol{\nu}} \boldsymbol{b}^{\boldsymbol{\nu}}$ converges whenever $\sum_{\boldsymbol{\nu} \in \mathbf{N}^r} c_{\boldsymbol{\nu}} \boldsymbol{\zeta}^{\boldsymbol{\nu}} \in A\langle \boldsymbol{\zeta} \rangle$. This gives a homomorphism of K-algebras $A\langle \boldsymbol{\zeta} \rangle \to B$, namely $\sum_{\boldsymbol{\nu} \in \mathbf{N}^r} c_{\boldsymbol{\nu}} \boldsymbol{\zeta}^{\boldsymbol{\nu}} \mapsto \sum_{\boldsymbol{\nu} \in \mathbf{N}^r} c_{\boldsymbol{\nu}} \boldsymbol{b}^{\boldsymbol{\nu}}$, and since $\zeta_i - f_i$ maps to zero for every i under this map, we get a unique K-algebra map $\lambda \colon A' \to B$ such that $\lambda \circ \imath = \varphi^*$. This proves the result for Weierstrass domains.

For a Laurent domain $X(\boldsymbol{f}, \mathbf{1}/\boldsymbol{g})$ one does something similar, replacing, in the proof above, the affinoid algebra $A\langle \boldsymbol{f} \rangle$ by the affinoid algebra $A\langle \boldsymbol{f}, \mathbf{1}/\boldsymbol{g} \rangle$ defined in (1.1.3). For rational domains $X(\mathbf{f}/f_0)$ associated with unit ideals $(f_0, f_1, \ldots, f_r)$ one uses the algebra $A\langle \mathbf{f}/f_0 \rangle$ of (1.1.4) instead of $A\langle \boldsymbol{f} \rangle$. The details are left to you. $\square$