LECTURE 15

Date of Lecture: October 10, 2019

Unless otherwise specified, K is a complete non-archimedean field, and to avoid annoying trivialities we assume the absolute value $|\ |$ on K is non-trivial. As before $\mathbf{N} = \{0, 1, 2, \dots, m, \dots\}$. Rings mean commutative rings with 1.

The symbol ☡ is for flagging a cautionary comment or a tricky argument. It occurs in the margins and is Knuth's version of Bourbaki's "dangerous bend symbol".

1. Various topologies and morphisms

1.1. **The Zariski topology on** $\mathrm{Sp}(A)$**.** Let A be a affinoid algebra. For $\mathfrak{a} \subset A$ an ideal of A, set

$$\begin{aligned} V(\mathfrak{a}) &= \{x \in \mathrm{Sp}(A) \mid f(x) = 0,\ f \in \mathfrak{a}\} \\ &= \{x \in \mathrm{Sp}(A) \mid \mathfrak{a} \subset \mathfrak{m}_x\}. \end{aligned}$$

One has the following easily proved properties.

(i) $\mathfrak{a} \subset \mathfrak{b} \Longrightarrow V(\mathfrak{a}) \supset V(\mathfrak{b})$.
(ii) $V(\sum_{i \in I} \mathfrak{a}_i) = \cap_{i \in I} V(\mathfrak{a}_i)$.
(ii) $V(\mathfrak{a}\mathfrak{b}) = V(\mathfrak{a}) \cap V(\boldsymbol{f})$.

The *Zariski topology* on $\mathrm{Sp}(A)$ is the topology for which the closed subsets are the $V(\mathfrak{a})$. The above properties show that this gives a topology. For $f \in A$, let $D(f) = \{f \neq 0\} = \mathrm{Sp}(A) \smallsetminus V(f)$ is an open set. The collection $\{D(f)\}_{f \in A}$ forms a basis for the Zariski topology on $\mathrm{Sp}(A)$. For $Y \subset \mathrm{Sp}(A)$, set $I(Y) = \{f \in A \mid f(x) = 0\, x \in Y\}$. Then $I(Y)$ is an ideal and $V(I(Y)$ is the closure of Y in the Zariski topology. Since A is Jacobson, it is clear that $I(V(\mathfrak{a})) = \sqrt{\mathfrak{a}}$. Thus radical ideals of A are in one-to-one correspondence with closed subsets of $\mathrm{Sp}(A)$.

1.2. **Morphism of affinoid spaces.** We call sets of the form $\mathrm{Sp}(A)$ with A an affinoid K-algebra, an *affinoid* K*-space*, or simply an *affinoid space*.

Suppose $\sigma\colon A \to B$ is a map of affinoid algebras. If $\mathfrak{m} \in \mathrm{Max}(B) = \mathrm{Sp}(B)$, then we have $K \subset A/\sigma^1(\mathfrak{m}) \hookrightarrow B/fm$. Since $\dim_K B/\mathfrak{m} < \infty$, the integral domain $A/\sigma^{-1}(\mathfrak{m})$ is a field, and $\sigma^{-1}(\mathfrak{m})$ is a maximal ideal of A. Define

$$ {}^a\sigma\colon \mathrm{Sp}(B) \to \mathrm{Sp}(A) \tag{1.2.1}$$

to be the map $\mathfrak{m} \mapsto \sigma^{-1}(\mathfrak{m})$.

Definition 1.2.2. Let A and B be K-affinoid algebras. *A morphsim from* $\mathrm{Sp}(B)$ *to* $\mathrm{Sp}(A)$ is a pair (φ, σ) with $\varphi\colon \mathrm{Sp}(B) \to \mathrm{Sp}(A)$ a set-theoretic map, $\sigma\colon A \to B$ a map of K-affinoid algebras such that $\varphi = {}^a\sigma$.

It is clear that φ is redundant information in the definition of a morphism of affinoid spaces, and really we are defining category of affinoid K-spaces as the opposite category of the category of affinoid algebras. Be that as it may, for psychological reasons, we keep φ in the definition. And then abuse notation and write φ for

(φ, σ). In this case we write $\sigma = \varphi^*$. Thus $(\varphi, \sigma) = ({}^a\sigma, \sigma) = (\varphi, \varphi^*)$. Note that with this convention (of writing φ as a short hand for (φ, σ) etc.), $\varphi = \psi$ if and only if $\varphi^* = \psi^*$.

1.3. **The canonical topology.** Let $X = \mathrm{Sp}(A)$, with A an affinoid algebra. As in [Lecture 14, §§1.1], define the *canonical topology* on X to be the topology generated by sets of the form

$$X(f, \epsilon) = \{x \in X \mid |f(x)| \leq \epsilon\},$$

with $f \in A$ and $\epsilon > 0$. Note that if U is open in the canonical topology, it is a union of sets of the form $X(f_1, \epsilon_1) \cap \cdots \cap X(f_n, \epsilon_n)$.

For $f, f_1, \ldots, f_r \in A$ we write

$$X(f) = X(f, 1)$$

and

$$X(f_1, \ldots, f_r) = X(f_1) \cap \cdots \cap X(f_r).$$

Proposition 1.3.1. *The canonical topology on X is generated by sets of the form $X(f)$ $f \in A$. In particular, U is open for the canonical topology if and only if it a union of sets of the form $X(f_1, \ldots, f_r)$, $f_1, \ldots, f_r \in A$, $r \in \mathbf{N}$.*

Proof. First note that if $\epsilon \in |\overline{K}^*|$ then there is a $c \in K^*$ and a positive integer s such that $\epsilon^s = |c|$. Indeed, suppose $\epsilon = |\theta|$ with $\theta \in \overline{K}^*$, and suppose g is the minimal polynomial of θ over K. Set $s = \deg g$. Then $g = \zeta^s + c_1\zeta^{s-1} + \cdots + c_s$ with $c_i \in K$. If $\alpha_1, \ldots, \alpha_s$ are the roots of s (counted with appropriate repetition), then the α_i are conjugates of θ and hence $|\theta| = |\alpha_i|$ for every i. It follows that $|c_s| = \epsilon^s$.

Let $f \in A$ and $\epsilon > 0$ be given. We have

$$X(f,\, \epsilon) = \bigcup_{\substack{\delta \in |\overline{K}^*| \\ \delta \leq \epsilon}} X(f,\, \delta).$$

Fix $\delta \in (0,\, \epsilon] \cap |\overline{K}^*|$. We have $c \in K^*$ and a positive integer s such that $|c| = \delta^s$. Then

$$X(f,\, \delta) = X(f^s, \delta^s) = X(c^{-1}f^s).$$

□

Lemma 1.3.2. *Let $f \in A$ and $x \in X$, and suppose $|f(x)| = \epsilon > 0$. Then there exists $g \in A$ such that $g(x) = 0$ and such that $|f| = \epsilon$ for every $y \in X(g)$. In particular $\{y \in X \mid |f(y)| = \epsilon\}$ is open in X.*

Proof. Let $L = A/\mathfrak{m}_x$ and $\overline{K}$ an algebraic closure of K. Let

$$P = \zeta^n + c_1\zeta^{n-1} + \cdots + c_n \in K[\zeta]$$

be the minimal polynomial of $f(x)$ over K. Set $g = P(f)$. Then $g(x) = 0$.

Let $\alpha_1, \ldots, \alpha_n \in \overline{K}$ be the roots of P so that

$$P(\zeta) = \prod_{i=1}^{n} (\zeta - \alpha_i).$$

Here the roots are repeated as many times as is necessary for the above equality of polynomials to hold in $\overline{K}[\zeta]$. Now

$$(*) \qquad g(y) = \prod_{i=1}^{n} \Big(f(y) - \alpha_i \Big) \quad (y \in X).$$

Since $P(f(x)) = 0$, we must have $|\alpha_i| = |f(x)| = \epsilon$ for $i = 1, \ldots, n$. Suppose $y \in X$ is such that $|f(y)| \neq \epsilon$. The following chain of relations follow from $(*)$ and the fact that $|f(y)| \neq |\alpha_i|$ for any i.

$$\begin{aligned} |g(y)| = \prod_{i=1}^{n} |f(y) - \alpha_i| &= \prod_{i=1}^{n} \max\{|f(x)|, |\alpha_i|\} \\ &\geq \prod_{i=1}^{n} |\alpha_i| \\ &= \epsilon^n \end{aligned}$$

It follows that

$$|g(y)| < \epsilon^n \Longrightarrow |f(y)| = \epsilon.$$

Since $|\ |$ is non-trivial on K, we can find $c \in K^*$ such that $|c| < \epsilon^n$ and hence $|f(y)| = \epsilon$ for $y \in X(c^{-1}g)$. □

Proposition 1.3.3. *For $f \in A$ and $\epsilon > 0$, the sets $\{f \neq 0\}$, $\{|f| \leq \epsilon\}$, $\{|f| = \epsilon\}$, $\{|f| \geq \epsilon\}$, $\{|f| < \epsilon\}$, $\{|f| > \epsilon\}$ are all open in the canonical topology.*

Proof. By definition of the canonical topology, the set $\{|f| \leq \epsilon\}$ is open in the canonical topology. By Lemma 1.3.2, $\{|f| = \epsilon\}$ is open. Since

$$\begin{aligned} \{f \neq 0\} &= \bigcup_{\delta > 0} \{|f| = \delta\}, \\ \{|f| \geq \epsilon\} &= \bigcup_{\delta \geq \epsilon} \{|f| = \delta\}, \\ \{|f| < \epsilon\} &= \bigcup_{\delta < \epsilon} \{|f| = \delta\}, \\ \{|f| > \epsilon\} &= \bigcup_{\delta > \epsilon} \{|f| = \delta\}, \end{aligned}$$

the Proposition follows from Lemma 1.3.2. □

The following two results are fairly obvious, and we state them without proof.

Proposition 1.3.4. *Let $x \in X$. Then $\{X(f_1, \ldots, f_r) \mid f_1, \ldots, f_r \in \mathfrak{m}_x, r \in \mathbf{N}\}$ forms a neighbourhood basis for x.*

Proposition 1.3.5. *Let $\varphi^* \colon A \to B$ be a map of affinoid algbras, and $\varphi \colon \mathrm{Sp}(B) \to \mathrm{Sp}(A)$ the corresponding map of affinoid spaces. Then for $f_1, \ldots, f_r \in A$,*

$$\varphi^{-1}(\mathrm{Sp}(A)(f_1, \ldots, f_r)) = \mathrm{Sp}(B)(\varphi^*(f_1), \ldots, \varphi^*(f_r)).$$

2. Special affinoid domains

Fix $X = \mathrm{Sp}(A)$, where A is an affinoid K-algebra.

2.1. **Weierstrass, Laurent, and rational domains.** For $g \in A$, set $X(1/g) = \{x \in X \mid |g(x)| \geq 1\}$. By Proposition 1.3.3, $X(1/g)$ is open in the canonical topology. For $f_1, \ldots, f_r, g_1, \ldots, g_s \in A$, set

$$X\Big(f_1, \ldots, f_r, \frac{1}{g_1}, \ldots, \frac{1}{g_s}\Big) = X(f_1, \ldots, f_r) \cap \bigcap_{j=1}^{s} X\Big(\frac{1}{g_j}\Big).$$

The following three classes of domains are called *special affinoid subdomians* of X. The term affinoid subdomain has a technical meaning which we will give later in Lecture 16, and it will turn out that the doamins we defined are indeed affinoid subdomains.

(1) *Weierstrass domains.* These are domains of the form $X(f_1, \ldots, f_r)$ with $f_1, \ldots, f_r \in A$.
(2) *Laurent domains.* These are domains of the form $X(f_1, \ldots, f_r, 1/g_1, \ldots, 1/g_s)$ for $f_1, \ldots, f_r, g_1, \ldots, g_s \in A$.
(3) *Rational domains.* These are the domains defined in [Lecture 14, (1.1.1)], i.e. domains of the form

$$X\Big(\frac{f_1}{f_0}, \ldots, \frac{f_n}{f_0}\Big) = \{x \in X \,\Big|\, |f_i(x)| \leq |f_0(x)|, i = 1, \ldots, n\}$$

for $f_0, \ldots, f_r \in A$ such that the f_i have no common zero.

Lemma 2.1.1. *Weierstrass, Laurent, and rational domains are open in the canonical topology. Moroever the Weierstrass domains form a basis for the canonical topology.*

Proof. Weierstrass domains are open by definition, and Laurent by Proposition 1.3.3. Moreover, by Lemma 1.3.2, Weierstrass domains form a basis for the canonical topology. It remains to show that given $f_0, \ldots, f_r \in A$ which generate the unit ideal in A, the set $X(\mathbf{f}/f_0)$ is open in the canonical topology. First note that if $x \in X(\mathbf{f}/f_0)$ then $f_0(x) \neq 0$, since $f_0, \ldots, f_r$ have no common zero. Hence

$$X\Big(\frac{f_1}{f_0}, \ldots, \frac{f_n}{f_0}\Big) = \bigcup_{\epsilon > 0} \Big[X(f_1, \epsilon) \cap \cdots \cap X(f_r, \epsilon) \cap \{x \mid |f_0(x)| = \epsilon\} \Big],$$

i.e. $X(\mathbf{f}/f_0)$ is open in the canonical topology. $\square$