LECTURE 14

Date of Lecture: October 3, 2019

Unless otherwise specified, K is a complete non-archimedean field, and to avoid annoying trivialities we assume the absolute value $|\ |$ on K is non-trivial. As before $\mathbf{N} = \{0, 1, 2, \ldots, m, \ldots\}$. Rings mean commutative rings with 1.

The symbol ☡ is for flagging a cautionary comment or a tricky argument. It occurs in the margins and is Knuth's version of Bourbaki's "dangerous bend symbol".

The lecture is an overview of how the rigid analytic spaces are constructed. Details will be in subsequent lectures.

For an affinoid K-algebra A, we often write $\operatorname{Sp} A$ for $\operatorname{Max} A$, especially if we wish to think of $\operatorname{Max} A$ as a space. As before, if $x \in \operatorname{Sp} A$, we often write $\mathfrak{m}_x$ for x when we wish to think of it as a maximal ideal of A. So while A/x makes perfect logical sense for $x \in \operatorname{Sp} A$, we will generally prefer to write $A/\mathfrak{m}_x$ for A/x.

1. Rational Subdomains and G-topologies

1.1. **Rational subdomains.** Let A be an affinoid K-algebra, endowed with any residue norm $\|\ \|_\alpha$ on A induced by a Gauss norm on a Tate algebra. Let $X = \operatorname{Sp} A$. Consider a sequence of elements $f_0, \ldots, f_n$ in A such that f_i have no common zeros in X. Set

$$X\Big(\frac{f_1}{f_0}, \ldots, \frac{f_n}{f_0}\Big) = \Big\{x \in X \,\Big|\, |f_i(x)| \le |f_0(x)|, i = 1, \ldots, n\Big\}. \tag{1.1.1}$$

There is a bijective correspondence between $X(f_1/f_0, \ldots, f_n/f_0)$ and $\operatorname{Sp}(A_{\mathbf{f}/f_0})$ where

$$A_{\mathbf{f}/f_0} = A \widehat{\otimes}_K K\langle\zeta_1, \ldots, \zeta_n\rangle/(f_1 - \zeta_1 f_0, \ldots, f_n - \zeta_n f_0).$$

The symbol $\widehat{\otimes}$ represents *the complete tensor product* obtained by completing the usual tensor product with the norm on it arising from the norms on A and $T_n = K\langle\zeta_1, \ldots, \zeta_n\rangle$.

In fact the relationship between $\operatorname{Sp}(A_{\mathbf{f}/f_0})$ and $X(\mathbf{f}/f_0) := X(f_1/f_0, \ldots, f_n/f_0)$ is quite close. First note that the natural map $\operatorname{Sp}(A_{\mathbf{f}/f_0}) \to X$ has its image in $X(\mathbf{f}/f_0)$. It turns out that one has the following universal property: *If $A \to B$ is a map of affinoid algebras, and $\varphi\colon \operatorname{Sp} B \to \operatorname{Sp} A = X$ the resulting map on spaces is such that $\varphi(\operatorname{Sp} B) \subset X(f_1/\mathbf{f}/f_0)$, then there is a unique map of affinoid algebras $A_{\mathbf{f}/f_0} \to B$ such that $A \to B$ is the composite $A \to A_{\mathbf{f}/f_0} \to B$.*

$$\begin{tikzcd}
\operatorname{Sp} B \arrow[r] \arrow[d, dashed, "\exists!"'] \arrow[dr, "\varphi"] & X(\mathbf{f}/f_0) \arrow[d, hook] \\
\operatorname{Sp} A_{\mathbf{f}/f_0} \arrow[r] \arrow[ur] & X
\end{tikzcd}$$

We identify $X(\mathbf{f}/f_0)$ with $\operatorname{Sp} A_{\mathbf{f}/f_0}$. It turns out that $X(\mathbf{f}/f_0)$ is an open set for the so-called *canonical topology* on X, namely the topology generated by sets of the form $X(f, \epsilon) = \{x \in X \mid |f(x)| \le \epsilon\}$, $f \in A$, $\epsilon > 0$.

Definition 1.1.2. Subspaces of X of the form $X(\mathbf{f}/f_0)$ are called *rational subdomains* of X.

It turns out that if U and V are two rational subdomains of X then $U \cap V$ is a rational subdomain of X. Further, a rational subdomain U of a rational subdomain V of X is a rational subdomain of X.

For any set of $m+n$ elements $f_1, \ldots, f_n$, $g_1, \ldots, g_m$ of A we write

$$X(\mathbf{f}, 1/\mathbf{g}) = X(f_1, \ldots, f_n, 1/g_1, \ldots, 1/g_m) := \{x \in X \mid |f_i| \le 1, |g_j| \ge 1\}.$$

Since

$$X(\mathbf{f}, 1/\mathbf{g}) = X(f_1/1, \ldots, f_n/1) \cap \bigcap_{j=1}^{m} X(1/g_j),$$

the set $X(\mathbf{f}, 1/\mathbf{g})$ is a rational subdomain of X. For the same reason

$$X = (1/\mathbf{g}) = X(1/g_1, \ldots, 1/g_m) := \bigcap_{j=1}^{m} X(1/g_j).$$

is a rational subdomain of X.

Examples 1.1.3. We identify $\mathbb{B}^n := \operatorname{Sp} T_n$ with the unit polydisc $\mathbb{B}^n(\overline{K})$ at least when $K = \overline{K}$. Here are some standard rational subdomains of $\mathbb{B}^n$.

1. *Closed Polydisc.* Let $\pi_1, \ldots, \pi_n \in K$ be such that $|\pi_i| \le 1$, $i = 1, \ldots, n$, i.e. $\pi \in \mathscr{O}_K$ for $i = 1, \ldots, n$. Recall $\mathbb{B}^n = \operatorname{Sp} T_n$. We have

$$\mathbb{B}^n(\zeta_1/\pi_1, \ldots, \zeta_n/\pi_n) = \operatorname{Sp}\left\{ K\left\langle \frac{\zeta_1}{\pi_1}, \ldots, \frac{\zeta_n}{\pi_n} \right\rangle \right\}.$$

We regard the "$\overline{K}$-rational points" of $\mathbb{B}^n(\zeta_1/\pi_1, \ldots, \zeta_n/\pi_n)$ as the subset of $\mathbb{B}^n(\overline{K})$ given by points $\boldsymbol{x} = (x_1, \ldots, x_n)$ such that $|x_i| \le |\pi_i|$, $i = 1, \ldots, n$.

2. *Annulus.* Let $\pi_1, \ldots, \pi_n$ and $\varpi_1, \ldots, \varpi_n$ be elements $\mathscr{O}_K$ with $|\pi_i| \le |\varpi_i|$ for every i. The annulus

$$\Big\{(x_1, \ldots, x_n) \in \mathbb{B}^n(\overline{K}) \mid |\pi_i| \le |x_i| \le |\varpi_i|, i = 1, \ldots, n\Big\}$$

is represented as a rational subdomain of $\mathbb{B}^n$ by

$$\mathbb{B}^n(\pi_1/\zeta_1, \ldots, \pi_n/\zeta_n, \zeta_1/\varpi_1, \ldots, \zeta_n/\varphi_n) = \operatorname{Sp}\left\{ K\left\langle \frac{\pi_1}{\zeta_1}, \ldots, \frac{\pi_n}{\zeta_n}, \frac{\zeta_1}{\varpi_1}, \ldots, \frac{\zeta_n}{\varpi_n} \right\rangle \right\}.$$

1.2. **G-topologies.** We need the following mild version of a Grothendieck topology. Let X be a topological space. A *G-topology* is a pair $(\mathcal{T}, \mathscr{C}ov)$ where:

- $\mathcal{T}$ is a collection of open sets of X,
- For each $U \in \mathcal{T}$, $\mathscr{C}ov(U)$ is a collection of coverings of U by members of $\mathcal{T}$,[1]

satisfying the following conditions:

(1) $\emptyset \in \mathcal{T}$; if $U, V \in \mathcal{T}$ then $U \cap V \in \mathcal{T}$.
(2) If $U \in \mathcal{T}$ then $\{U\} \in \mathscr{C}ov(U)$.
(3) If $U \in \mathcal{T}$, $\{U_\lambda\}_{\lambda \in \Lambda} \in \mathscr{C}ov(U)$, and $V \in \mathcal{T}$, then $\{U_\lambda \cap V\}_{\lambda \in \Lambda} \in \mathscr{C}ov(V)$.
(4) Coverings of a covering give a covering, i.e. if $U \in \mathcal{T}$, $\{U_\alpha\}_{\alpha \in A} \in \mathscr{C}ov(U)$, $\{V_{\alpha\beta}\}_{\beta \in B_\alpha} \in \mathscr{C}ov(U_\alpha)$, then $\{V_{\alpha\beta}\}_{\beta \in B_\alpha, \alpha \in A} \in \mathscr{C}ov(U)$.

[1] Note that $\mathscr{C}ov \colon \mathcal{T} \to \mathscr{P}(\mathcal{T})$ is a map from $\mathcal{T}$ to the power set $\mathscr{P}(\mathcal{T})$ of $\mathcal{T}$.

Definition 1.2.1. If $(\mathcal{T}, \mathscr{C}ov)$ is a G-topology on a topological space X, we call members of $\mathcal{T}$ *admissible open sets* of X, and for $U \in \mathcal{T}$, an open covering $\{U_\lambda\}$ in $\mathscr{C}ov(U)$ is called an *admissible open covering of* U.

1.2.2. For completeness, we give the general definition of a Grothendieck topology.

Let $\mathscr{C}$ be a category. A *Grothendieck topology* on $\mathscr{C}$ is an assignment, for each object U of $\mathscr{C}$, of a collection $\mathscr{C}ov(U)$ of sets of arrows[2] $\{U_i \to U\}$ called *coverings* (of U) such that:

(1) If $V \to U$ is an isomorphism in $\mathscr{C}$, then the singleton set $\{V \to U\}$ is a covering of U.
(2) If $\{U_i \to U\}$ is a covering and $V \to U$ is an arrow in $\mathscr{C}$, then the fiber products $V \times_U U_i$ exist in $\mathscr{C}$, and the set of projections $\{V \times_U U_i \to V\}$ is also a covering.
(3) If $\{U_i \to U\}$ is a covering and for each i, $\{U_{ij} \to U_i\}$ is a covering of U_i, then the set of composites $\{U_{ij} \to U_i \to U\}$ (as i and j vary) is a covering of U.

A *site* is a category $\mathscr{C}$ together with a Grothendieck topology on it. A Grothendieck topology is often represented as a pair $(\mathscr{C}, \mathscr{C}ov)$.

The standard example is that of a topological space X. Let $\widehat{X}$ be the category whose objects are the open sets of X and whose morphisms are given by

$$\mathrm{Hom}_{\widehat{X}}(U, V) = \begin{cases} \emptyset \text{ if } U \not\subseteq V \\ U \subseteq V \text{ otherwise} \end{cases}$$

for two objects U and V in $\widehat{X}$. For U an object in $\widehat{X}$ a covering is a collection $\{U_\alpha \to U\}$ where the U_α give a covering (in the usual, set theoretic, sense) of U. Note that in this case each U_α is an open subset of U, whence $\{U_\alpha\}$ is an open covering (in the usual classical sense) of U. One checks easily that this notion of coverings defines a Grothendieck topology on $\widehat{X}$. Indeed, if $U \in \widehat{X}$, then the only isomorphism in $\widehat{X}$ with target U is the identity map, and this is clearly a covering. Next note that if U and U' are open subsets of V (V an open subset of X), then $U \times_V U'$ exists. In fact $U \times_V U' = U \cap U'$. From this observation, (2) is immediate. The third axiom is equally trivial to verify.

Note that the notion of a G-topology given above is also a Grothendieck topology, but it is not the classical topology.

1.3. **The G-topology on** $\mathrm{Sp}\, A$**.** Let $X = \mathrm{Sp}\, A$ where A is an affinoid K-algebra. Give X the canonical topology. Recall from §§1.1 that this is the topology on X (in the classical sense) generated by sets of the form $X(f, \epsilon) = \{x \in X \mid |f(x)| \le \epsilon\}$, $f \in A$, $\epsilon > 0$. Enrich X with a G-topology in the sense of §§1.2 cooked up as follows.

- Admissible open sets are decreed to be the rational subdomains of X.
- Admissible coverings of a rational subdomain $X(f_1/f_0, \ldots, f_n/f_0)$ of X are finite coverings of $X(f_1/f_0, \ldots, f_n/f_0)$ by rational subdomains.

One checks easily that the above recipe does indeed give a G-topology on X.

[2] arrows = morphisms

2. **Rigid Analytic Spaces**

2.1. **The sheaf $\mathscr{O}_X$ on $X = \operatorname{Sp} A$.** Let $X = \operatorname{Sp} A$ be an affinoid K-space. If $U \subset X$ is a rational subdomain, say $U = X(f_1/f_0, \ldots, f_n/f_0)$, we write A_U for $A_{\frac{f_1}{f_0}, \ldots, \frac{f_n}{f_0}}$.

We define a pre-sheaf $\mathscr{O}_X$ on the G-topology on X by setting $\mathscr{O}_X(U) = A_U$ for each rational subdomain U of X. It turns out that $\mathscr{O}_X$ is a sheaf in the G-topology on X. This is weak form of *Tate's acyclicity theorem* which we will prove later in the course. Moroever $(X, \mathscr{O}_X)$ is a locally ringed space.

2.2. **Rigid Analytic spaces.** Here is the main object of study for this course:

Definition 2.2.1. A *rigid analytic space* is a locally ringed space $(Y, \mathscr{O}_Y)$, equipped with a G-topology (in the sense of §§1.2) on Y, such that Y admits a (possibly infinite) open covering $\{Y_i\}_{i \in I}$ (in the classical sense), with the property that for each $i \in I$, $(Y_i, \mathscr{O}_Y|_{Y_i})$ is isomorphic to $(\operatorname{Sp} A_i, \mathscr{O}_{\operatorname{Sp} A_i})$ for some affinoid K-algebra A_i.