LECTURE 13

Date of Lecture: October 1, 2019

Unless otherwise specified, K is a complete non-archimedean field, and to avoid annoying trivialities we assume the absolute value $|\ |$ on K is non-trivial. As before $\mathbf{N} = \{0, 1, 2, \ldots, m, \ldots\}$. Rings mean commutative rings with 1.

The symbol ☡ is for flagging a cautionary comment or a tricky argument. It occurs in the margins and is Knuth's version of Bourbaki's "dangerous bend symbol".

1. The supremum norm and spectral values

The definition of a spectral value was given in [Lecture 12, (1.3.1)] and we proved an important property of it in Lemma 1.3.2 of *ibid.* We recall these below in (1.1.1) and Lemma 1.1.3.

1.1. Spectral values.

Let $(A, \|\ \|)$ be a semi-normed ring. Let $p \in A[\zeta]$ be a monic polynomial, say

$$p = \zeta^r + c_1\zeta^{r-1} + \cdots + c_r$$

with $c_i \in A$. The *spectral value* $\sigma(p)$ *of* p is defined to be

$$\sigma(p) = \max_{i=1\ldots r} \|c_i\|^{\frac{1}{i}}. \tag{1.1.1}$$

Lemma 1.1.2. *Let $(A, \|\ \|)$ be a semi-normed ring. Let $p, q \in A[\zeta]$ be monic polynomials. Then $\sigma(pq) \le \max\{\sigma(p), \sigma(q)\}$.*

Proof. If $p = \zeta^r + a_1\zeta^{r-1} + \cdots + a_r$ and $q = \zeta^s + b_1\zeta^{s-1} + \cdots + b_s$ then

$$pq = \zeta^{r+s} + \sum_{\lambda=1}^{r+s} c_\lambda \zeta^{r+s-\lambda}$$

where $c_\lambda = \sum_{i+j=\lambda} a_j b_j$. Now,

$$\|c_\lambda\| \le \max_{i+j=\lambda} \|a_i\|\|b_j\| \le \max_{i+j=\lambda} \sigma(p)^i \sigma(q)^j \le \max\{\sigma(p), \sigma(q)\}^\lambda.$$

□

If $A = K$, there is a nice formula for the spectral value of p.

Lemma 1.1.3. *Suppose $p = \zeta^r + c_1\zeta^{r-1} + \cdots + c_r \in K[\zeta]$ is a polynomial which factors in $\overline{K}[\zeta]$ as*

$$p = \zeta^r + c_1\zeta^{r-1} + \cdots + c_r = \prod_{j=1}^{r} (\zeta - \alpha_j).$$

Then

$$\sigma(p) = \max_{j=1\ldots r} |\alpha_j|.$$

Proof. See [Lecture 12, Lemma 1.3.2] □

1.2. **Finite maps over Tate algebras.** Let

$$T_d \hookrightarrow B$$

be a finite monomorphism of K-algebras such that B is torsion-free as a module over T_d. Fix $b \in B$. Then b is integral over T_d. Actually more can be said. Since T_d is a UFD, an easy application of Gauss' Lemma shows that if $0 \neq b \in B$, then the element b has a *minimal polynomial* $p_b \in T_d[\zeta]$, i.e., p_b is monic and generates the kernel of the T_d-algebra map $T_d[\zeta] \to B$, $g \mapsto g(b)$.

We have a composite of finite monomorphisms $T_d \hookrightarrow T_d[b] \subset B$. We have a surjection $\operatorname{Max} B \twoheadrightarrow \operatorname{Max}(T_d[b])$ (from the "lying over" part of the Cohen-Seidenberg theorem). Moreover, if $y' \in \operatorname{Max} B$ maps to $y \in \operatorname{Max}(T_d[b])$, we have

$$|b(y)| = |b(y')|$$

since $K(y) \hookrightarrow K(y')$.[1]

Let $x \in \operatorname{Max} T_d$ and let $\bar{p}_b \in K(x)[\zeta]$ be the reduction of $p_b \in T_d[\zeta]$. If $z_1, \ldots, z_r \in \operatorname{Max}(T_d[b])$ and $y_1, \ldots, y_s \in \operatorname{Max} B$ are the points lying over x (these are non-empty collections by the "lying over" part of the Cohen-Seidenberg theorem) then our observation above and Lemma 1.1.3 shows that

$$\sigma(\bar{p}_b) = \max_{j=1\ldots r} |b(z_j)| = \max_{i=1\ldots s} |b(y_i)|. \tag{1.2.1}$$

The first equality follows from the fact that if $\varphi\colon K(z_j) \hookrightarrow \overline{K}$ is any embedding of $K(z_j)$ into an algebraic closure $\overline{K}$ of K, then all conjugates of $\varphi(b(z_j))$ have the same absolute value.

Lemma 1.2.2. *Let $T_d \hookrightarrow B$ be a finite monomorphism of K-algebras so that B is affinoid by [Lecture 12, Lemma 1.2.4], and suppose B is torsion free as an T_d-module. Let $0 \neq b \in B$ be an element and*

$$p_b = \zeta^r + f_1\zeta^{r-1} + \cdots + f_r$$

be its minimal polynomial over T_d (so that $f_i \in T_d$).

(a) *Let $x \in \operatorname{Max} T_d$ and suppose $y_1, \ldots, y_s \in \operatorname{Max} B$ are the elements lying over x. Let $\bar{p}_b \in K(x)[\zeta]$ be the reduction of p_b in $K(x)[\zeta]$. Then*

$$\max_{i=1\ldots s} |b(y_i)| = \sigma(\bar{p}_b) = \max_{j=1\ldots r} |f_j(x)|^{\frac{1}{j}}.$$

(b) *The supremum semi-norm of b is given by*

$$\|b\|_{\sup} = \sigma(p_b).$$

Proof. Part (a) is simply (1.2.1). Part (b) follows from (a) using the fact that $\| \ \| = \| \ \|_{\sup}$ on T_d, and the fact that as x varies over $\operatorname{Max} T_d$, the y_i exhaust $\operatorname{Max} B$. □

1.2.3. Let $\varphi\colon A \to B$ be a finite map of affinoid K-algebras. How far can we take the arguments in Lemma 1.2.2? Assume (temporarily) that B is an integral domain. Let $T_n \twoheadrightarrow A$ be surjective K-algebra map, so that $A/\ker\varphi \cong T_n/\mathfrak{a}$ for some ideal $\mathfrak{a}$ of T_n. The proof of Noether normalisation [Lecture 7, Theorem 1.2.4] shows that we have an inclusion $T_d \hookrightarrow T_n$ such that the composite $T_d \to A/\ker\varphi$ is

[1] Here, as always, if A is an affinoid algebra over K and $x \in \operatorname{Max} A$, then $K(x) := A/\mathfrak{m}_x$.

a finite monomorphism.[2] Note that $T_d \hookrightarrow A/\ker\varphi$ factors through $A \to A/\ker\varphi$. Now, for any $f \in T_d$, if $a \in A$ is its image, then

$$(*) \qquad \|f\|_{\sup} \geq \|a\|_{\sup}.$$

Indeed, if $x \in \operatorname{Max} A$ and if $y \in \operatorname{Max} T_d$ is its image (i.e. its contraction in T_d), then $|f(y)| = |a(x)|$, giving the above inequality.

Since B is torsion free over T_d, being a domain, it follows that every $b \in B$ has a minimal polynomial $p_b \in T_d[\zeta]$ over T_d. Suppose $p_b = \sum_{i=0}^{r} f_i \zeta^{r-i}$, with $f_0 = 1$, and $f_i \in T_d$. Let $a_i \in A$ be the images of f_i in A. Then

$$b^r + a_1 b^{r-1} + \cdots + a_{r-1} b + a_r = 0.$$

Since $\|f_i\| = \|f_i\|_{\sup} \geq \|a_i\|_{\sup}$ by $(*)$, we have

$$\|b\|_{\sup} = \sigma(p_b) = \max_{i=1\ldots r} \|f_i\|^{\frac{1}{i}} \geq \max_{i=1\ldots r} \|a_i\|_{\sup}^{\frac{1}{i}}.$$

On the other hand, since $b^r = -\sum_{i=1}^{r} a_i b^{r-i}$, we have $\|b\|_{\sup}^r = \|b^r\|_{\sup} \leq \max_{1\leq i\leq r} \|a_i b^{r-i}\|_{\sup}$. Hence there exists an index i such that

$$(1.2.3.1) \qquad \|b\|_{\sup}^r \leq \|a_i b^{r-i}\|_{\sup} \leq \|a_i\|_{\sup} \|b\|_{\sup}^{r-i}$$

giving $\|b\|_{\sup} \leq \|a_i\|_{\sup}^{\frac{1}{i}}$. Thus

$$(1.2.3.2) \qquad \|b\|_{\sup} = \max_{i=1\ldots r} \|a_i\|_{\sup}^{\frac{1}{i}}.$$

We have more or less proved the following result:

Lemma 1.2.4. *Let $\varphi\colon A \to B$ be a finite K-algebra map between affinoid algebras. For each $b \in B$ exists an integral equation*

$$b^r + a_1 b^{r-1} + \cdots + a_{r-1} b + a_r = 0$$

with $a_i \in A$ such that $\|b\|_{\sup} = \max_{i=1\ldots r} \|a_i\|_{\sup}^{\frac{1}{i}}$.

Proof. When B is an integral domain, (1.2.3.2) gives the result. Otherwise, let $\mathfrak{p}_1, \ldots, \mathfrak{p}_s$ be the minimal primes of the noetherian ring B. Let b_i be the image of b in $B/\mathfrak{p}_i$. Then $\|b\|_{\sup} = \max_{i=1\ldots s} \|b_i\|_{\sup}$. Since the rings $B/\mathfrak{p}_i$ are integral domains, we have monic polynomials $q_1, \ldots, q_s \in A[\zeta]$ such that $q_i(b_i) = 0$ and $\|b_i\|_{\sup} = \sigma(q_i)$ for every $i \in \{1, \ldots, s\}$. Since $(q_1(b)q_2(b)\ldots q_s(b))(x) = 0$ for every $x \in \operatorname{Max} B$, and since B is Jacobson, we have $q_1(b)q_2(b)\ldots q_s(b) \in \sqrt{(0)} \subset B$. Hence there exists a power q of $q_1 q_2 \ldots q_s$ such that $q(b) = 0$ in B. Moreover

$$\|b\|_{\sup} = \max_{i=1\ldots s} \|b_i\|_{\sup} = \max_{i=1\ldots s} \sigma(q_i) \geq \sigma(q).$$

Arguing as we did to obtain the inequality (1.2.3.1), we see that in fact we have an equality, namely $\|b\|_{\sup} = \sigma(q)$. $\square$

Theorem 1.2.5. (The Maximum Principle) *Let A be an affinoid K algebra, and let a be an element of A. There exists $y \in \operatorname{Max}(A)$ such that $\|a\|_{\sup} = |a(y)|$.*

[2] See also the proof of Lemma 1.1.1 in Lecture 7 and the Remark 1.1.2 that follows it.

Proof. The case $a = 0$ is trivial and so let us assume $a \neq 0$. Let $\mathfrak{p}_1, \dots, \mathfrak{p}_s$ be the minimal primes of A, and let a_j be the image of a in $A/\mathfrak{p}_j$. Since $\|a\|_{\sup} = \max_{j=1\dots s}\|a_j\|_{\sup}$ there exists an index j such that $\|a\|_{\sup} = \|a_j\|_{\sup}$. So without loss of generality we will assume A is a domain. Let $T_d \hookrightarrow A$ be a Noether normalisation and

$$p_a = \zeta^r + f_1\zeta^{r-1} + \cdots + f_r$$

the minimal polynomial of a over T_d (so that $f_i \in T_d$). Set

$$f := f_1 f_2 \dots f_r.$$

By the Maximum Principle for T_d (see [Lecture 4, Theorem 3.2.1] and [Lecture 7, Theorem 1.3.1]) we can find $x \in \operatorname{Max}(T_d)$ such that $\|f\|_{\sup} = |f(x)|$. Now $\| \, \|_{\sup}$ is multiplicative on T_d, being equal to $\| \, \|$ there. Hence

$$\|f_1\|_{\sup} \dots \|f_r\|_{\sup} = \|f\|_{\sup} = |f(x)| = |f_1(x)| \dots |f_r(x)|.$$

It follows that for each j we have $\|f_j\|_{\sup} = |f_j(x)|$. If $y_1, \dots, y_s$ are the points in $\operatorname{Max} A$ lying over $x \in \operatorname{Max}(T_d)$, then by Lemma 1.2.2 we have

$$\|a\|_{\sup} = \max_{j=1\dots r} \|f_j\|_{\sup}^{\frac{1}{j}} = \max_j |f_j(x)|^{\frac{1}{j}} = \max_{i=1\dots s} |a(y_i)|.$$

There is a $y \in \{y_1, \dots, y_s\}$ such that $|a(y)| = \max_{i=1\dots s} |a(y_i)|$, since we are dealing with a finite set of numbers. This proves the assertion. $\square$

2. Complete norms on Banach K-algebras

2.1. For a commutative ring R, we set $j(R)$ equal to $\cap_{\mathfrak{m}\in \operatorname{Max} R}\mathfrak{m}$, i.e. $j(R)$ is the Jacobson radical of R. Recall that if R is a K-algebra, then a K-algebra norm $\| \, \|$ on R is a sub-multiplicative norm on the K-vector space R and if further $\| \, \|$ is complete, then $(R, \| \, \|)$ is a Banach K-algebra.

Theorem 2.1.1. *Let B be a K-algebra such that $\dim_K(B/\mathfrak{m}) < \infty$ for every $\mathfrak{m} \in \operatorname{Max} B$ and such that $j(B) = 0$.*

(a) *Let $(A, \| \, \|_A)$ be a Banach K-algebra and $\varphi\colon A \to B$ a K-algebra homomorphism such that $\varphi^{-1}(\mathfrak{m})$ is closed in A for every $\mathfrak{m} \in \operatorname{Max} B$. Then φ is continuous for every complete K-algebra norm on B.*

(b) *Any two complete K-algebra norms on B are equivalent.*

Proof. We use the closed graph theorem for part (a). Suppose $\| \, \|$ is a complete K-algebra norm on B. Let $\{a_n\}$ be a null sequence in A, i.e. $a_n \to 0$ as $n \to \infty$, and suppose there exists $b \in B$ such that $\lim_{n\to\infty} \varphi(a_n) = b$. To prove (a), we have to show that $b = 0$. Let $\mathfrak{m} \in \operatorname{Max} B$ and let $\mathfrak{p} = \varphi^{-1}(\mathfrak{m})$. Then $\mathfrak{p}$ is closed in A, by hypothesis. Since $\| \, \|$ is complete, therefore $\mathfrak{m}$ is closed in B (by Problem 5 of HW 3). The natural surjections $\mu\colon A \twoheadrightarrow A/\mathfrak{p}$ and $\nu\colon B \twoheadrightarrow B/\mathfrak{m}$ are therefore continuous with respect to the residue norms on $A/\mathfrak{p}$ and $B/\mathfrak{m}$ respectively. Let $\psi\colon A/\mathfrak{p} \hookrightarrow B/\mathfrak{m}$ be the map induced by φ. We have a commutative diagram

$$\begin{tikzcd} A \arrow[r, "\varphi"] \arrow[d, "\mu"', two heads] & B \arrow[d, "\nu", two heads] \\ A/\mathfrak{p} \arrow[r, hook, "\psi"'] & B/\mathfrak{m} \end{tikzcd}$$

By hypothesis $B/\mathfrak{m}$ is finite-dimensional over K, whence so is $A/\mathfrak{p}$ since ψ is injective. By Corollary 1.1.8 of Lecture 11, ψ is therefore continuous. Thus

$$\nu(b) = \nu(\lim_{n\to\infty} \nu(\varphi(a_n)) = \lim_{n\to\infty} \nu(\varphi(a_n)) = \lim_{n\to\infty} \psi(\mu(a_n)) = 0.$$

We have used the continuity of μ, ν and ψ in the above chain of equalities. It follows that $b \in \mathfrak{m}$, and since $\mathfrak{m}$ was an arbitrary maximal ideal, $b \in j(B) = 0$. Thus $b = 0$, and φ is continuous.

Part (b) follows from part (a). Indeed, let $\| \; \|_1$ and $\| \; \|_2$ be two complete K-algebra norms on B. Apply part (a) to $(A, \| \; \|_A) = (B, \| \; \|_1)$ and $\varphi\colon B \to B$ the identity map, with $\| \; \|_2$ the complete norm on the target. The conclusion of part (a) shows that $\| \; \|_2 \le C \| \; \|_1$ for some $C > 0$. Reversing the roles of $\| \; \|_1$ and $\| \; \|_2$ produces the result. We are once again using the result that every maximal ideal in a Banach K-algebra is closed. □

We have an immediate and very useful corollary.

Corollary 2.1.2. *All complete K-algebra norms on a reduced affinoid K-algebra are equivalent.*

Proof. This follows from part (b) of the theorem. Indeed if A is an affinoid algebra, then it is Jacobson, and if further it is reduced, then $j(A) = 0$. Other hypotheses for applying Theorem 2.1.1 (b) are clearly satsified by A from results in earlier lectures. □

2.2. The supremum norm on a reduced affinoid algebra. Suppose A is a reduced affinoid K-algebra. Then the semi-norm $\| \; \|_{\sup}$ is a norm on A. We also know that all norms of the form $\| \; \|_\alpha$ are equivalent, where α is a surjective K-algebra map from a Tate algebra. Indeed each $\| \; \|_\alpha$ is complete, and Corollary 2.1.2 gives the result (the earlier proof of the equivalence of the various $\| \; \|_\alpha$ is more or less the proof given in the corollary above). If we show $\| \; \|_{\sup}$ is complete, then by Corollary 2.1.2, $\| \; \|_{\sup}$ would be equivalent to $\| \; \|_\alpha$ for every α.

In fact $\| \; \|_{\sup}$ is complete on the reduced algebra A. The proof is somewhat involved. The first steps are straightforward enough. One reduces readily to the case where A is a domain. Indeed, returning to a familiar argument, used for example in the proof of Theorem 1.2.5, we note that for $f \in A$, $\|f\|_{\sup} = \max_{i=1\dots s} \|f_i\|_{\sup}$, where f_i is the residue of f in $A/\mathfrak{p}_i$, with $\mathfrak{p}_1, \dots, \mathfrak{p}_s$ the minimal primes of A. If $\| \; \|_{\sup}$ is complete on each of the $A/\mathfrak{p}_i$, then clearly $\| \; \|_{\sup}$ is complete on A.

Let us the assume A is a domain. Let $T_d \hookrightarrow A$ be a Noether normalisation. On T_d the norm $\| \; \|_{\sup}$ is complete since it equals the Gauss norm there. One would like to relate $\| \; \|_{\sup}$ on T_d to $\| \; \|_{\sup}$ on A using the fact that $T_d \to A$ is a finite extension, hoping that some of the results on the extensions of norms and completeness on finite extensions of complete fields hold in this more general case, perhaps by mimicking the proofs in the field case. This does not work in a straightforward way, and one has work with fields with absolute values which are not complete. What follows is a discussion of the pitfalls along the way, and a strategy to overcome these.

2.2.1. Let

$$F = Q(T_d) \quad \text{and} \quad Q = Q(A)$$

be the quotient fields of T_d and A respectively. Recall we are assuming A is an integral domain and $T_d \hookrightarrow A$ is a Noether normalisation. Then $Q = A \otimes_{T_d} F$ and hence we have a finite field extension $F \to Q$. For good book-keeping let us write $\| \|_{\mathrm{gs}}$ for the Gauss norm (= the supremum norm) on T_d and $\| \|_{\mathrm{sp}}$ for the spectral norm (= supremum norm) on A, i.e. $\|a\|_{\mathrm{sp}} = \sigma(p_a)$ for $a \in A$, and $p_a \in T_d[\zeta]$ the minimal polynomial of a over T_d, and $\sigma\colon A[\zeta] \to \mathbf{R}_+$ the spectral value function as in (1.1.1).

Since $\| \|_{\mathrm{gs}}$ is multiplicative on T_d, one extends $\| \|_{\mathrm{gs}}$ to F in an obvious way, namely via the formula

$$\Big\|\frac{f}{g}\Big\|_{\mathrm{gs}} = \frac{\|f\|_{\mathrm{gs}}}{\|g\|_{\mathrm{gs}}} \qquad (f, g \in T_d,\ g \neq 0) \tag{2.2.1.1}$$

making $(F, \| \|_{\mathrm{gs}})$ a normed field with valuation ring T_d°.

The norm $\| \|_{\mathrm{sp}}$ is not (necessarily) multiplicative on A and therefore the analogue of (2.2.1.1) need not give us a norm on A, or for that matter be well-defined. However, the spectral norm for the field extension $F \to Q$, with $\| \|_{\mathrm{gs}}$ the norm on F, makes sense on Q and a little thought (essentially Gauss' Lemma) shows that the resulting spectral norm agrees with $\| \|_{\mathrm{sp}}$ on A. Thus we define

$$\| \|_{\mathrm{sp}}\colon Q \to \mathbf{R}_+ \tag{2.2.1.2}$$

by the formula

$$\|x\|_{\mathrm{sp}} = \sigma(p_x) \tag{2.2.1.3}$$

where p_x is the minimal polynomial for x over F, and $\sigma\colon Q[\zeta] \to \mathbf{R}_+$ the spectral value function as in (1.1.1). This extends the norm $\| \|_{\mathrm{sp}}$ on A. Note that $\|x\|_{\mathrm{sp}} = \|x\|_{\mathrm{gs}}$ if x is in the subring T_d of A and that $\|fx\|_{\mathrm{sp}} = \|f\|_{\mathrm{gs}}\|x\|_{\mathrm{sp}}$ for $f \in T_d$ and $x \in A$. It is not hard to show that $\| \|_{\mathrm{sp}}$ is a (non-archimedean) norm on Q, and in fact a F-algebra norm.

2.2.2. The fact that Q is a finite dimensional F-vector space allows us to define other norms on Q, one for each F-basis of Q. Let $e_1, \dots, e_n$ be a basis for Q. For $x = x_1e_1 + \cdots + x_ne_n \in Q$, $x_i \in F$, set $\|x\|_c = \max_{i=1\dots n}\|x_i\|_{\mathrm{gs}}$. Then $\| \|_c$ is a F-norm on Q, and it is the *cartesian norm* on Q in the following sense. The basis $e_1, \dots, e_n$ of Q gives us a canonical isomorphism $F^n \xrightarrow{\sim} Q$, and on F^n, the *product topology* is induced by the *product norm* $\| \|_p\colon F^n \to \mathbf{R}_+$ given by the standard formula $\|(x_1, \dots, x_n)\|_p = \max_{i=1\dots n}\|x_i\|_{\mathrm{gs}}$. Clearly $\| \|_c$ is simply the transplant of $\| \|_p$ under the isomorphism $F^n \xrightarrow{\sim} Q$ induced by the basis $e_1, \dots, e_n$.

While the norm $\| \|_c$ on Q depends upon the basis chosen, the induced topology doesn't. This is seen as follows. First, any F-linear map $\phi\colon F^n \to V$, where V is a F-normed space, is continuous. Here, as always, F^n has $\| \|_p$ as its norm. Indeed, if $M = \max_{i=1\dots n}\|\phi(e_i)\|_V$, where $\| \|_V$ is the norm on V, then $\|\phi(\sum_{i=1}^n x_ie_i)\|_V = \|\sum_{i=1}^n x_i\phi(e_i)\|_V = M\max_{i=1\dots n}\|x_i\|_{\mathrm{gs}} \leq M\|\sum_{i=1}^n x_ie_i\|_c$. It follows that every automorphism of F^n is continuous, and this amounts to saying that the norms of the form $\| \|_c$ on Q are all equivalent, even if they do depend on a chosen basis.

2.2.3. As a matter of fact one can choose a basis $\{e_i\}_{i=1}^n$ for Q with $e_i = \frac{a_i}{f}$, $a_i \in A$, and $0 \neq f \in T_d$ such that $A \subset \bigoplus_{i=1}^n T_d\frac{a_i}{f}$. To see this, first note that we can find a basis of Q from elements in A, say $a_1, \dots, a_n$ is a basis of Q with $a_i \in A$. Let $\alpha_1, \dots, \alpha_m$ be T_d-module generators of A. Then we have $\phi_{ij} \in Q$ such that $\alpha_i = \sum_j \phi_{ij}a_j$. We can find a "common denominator" $f \in T_d \setminus \{0\}$ for the ϕ_{ij} and write $\phi_{ij} = \frac{a_{ij}}{f}$, $a_{ij} \in A$, $0 \neq f \in T_d$. Then $\frac{a_1}{f}, \dots, \frac{a_n}{f}$ is an F-basis for Q and clearly

$$A \subset \bigoplus_{i=1}^{n} T_d \frac{a_i}{f} \subset \bigoplus_{i=1}^{n} F \frac{a_i}{f} = Q.$$

Since the norm $\| \, \|_p$ on F^n is complete on T_d^n (it need not be complete on F^n), therefore the norm $\| \, \|_c$ is complete on $\bigoplus_{i=1}^{n} T_d \frac{a_i}{f}$. By Proposition 1.1.1 of Lecture 12, A is a closed subspace of $(\bigoplus_{i=1}^{n} T_d \frac{a_i}{f}, \| \, \|_c)$ and hence $\| \, \|_c$ is complete on A.

2.2.4. If one could show that $\| \, \|_{\mathrm{sp}}$ is equivalent to $\| \, \|_c$ on A as a T_d-module norm, then $\| \, \|_{\mathrm{sp}}$ would be complete and we are done by Corollary 2.1.2. One strategy for showing this would be to show that F-norms $\| \, \|_{\mathrm{sp}}$ and $\| \, \|_c$ are equivalent on Q, and since Q is finite dimensional over F, this seems quite likely given Theorem 1.1.7 of Lecture 11. However *loc.cit.* required the underlying field to be complete. We are working with vector spaces over $(F, \| \, \|_{\mathrm{gs}})$, and $\| \, \|_{\mathrm{gs}}$ is not complete on F (even though it is on the subring T_d). We cannot assume $\| \, \|_{\mathrm{sp}}$ and $\| \, \|_c$ are equivalent on F without proof as the following example shows.

Example: Let $\widehat{F}$ be the completion of F with respect to $\| \, \|_{\mathrm{gs}}$ and pick $x \in \widehat{F} \smallsetminus F$. Consider the $V = F + Fx \subset \widehat{F}$. Then V is a two-dimensional vector space over F and it has a norm coming from the extension of $\| \, \|_{\mathrm{gs}}$ to $\widehat{F}$. One checks that the F-vector space isomorphism $V \xrightarrow{\sim} F^2$ given by $f + gx \mapsto (f, g)$ is not continuous where the norm on F^2 is $\| \, \|_p$.

2.2.5. However, it does turn out that $\| \, \|_{\mathrm{sp}}$ and $\| \, \|_c$ are equivalent on Q, despite the example in **2.2.4** (see Theorem 2.2.7 below). The key result is the following:

Theorem 2.2.6. *Given a non-zero element $x \in Q$, there exists a bounded F-linear functional $\lambda\colon (Q, \| \, \|_{\mathrm{sp}}) \to (F, \| \, \|_{\mathrm{gs}})$ such that $\lambda(x) \neq 0$.*

We will prove this a little later in this lecture when F is perfect and the proof when F is imperfect will be developed in HW problems. Here are some consequences.

Theorem 2.2.7. *Let F be given the Gauss norm $\| \, \|_{\mathrm{gs}}$ and Q the spectral norm $\| \, \|_{\mathrm{sp}}$.*

(a) *All F-functionals on Q are bounded.*
(b) *There is an F-linear homeomorphism $Q \xrightarrow{\sim} F^n$.*
(c) *Given any basis $e_1, \ldots, e_n$ of Q over F, the induced cartesian norm $\| \, \|_c$ on Q is equivalent to the spectral norm $\| \, \|_{\mathrm{sp}}$ on Q.*

Proof. To prove all functionals on Q are bounded, it is enough to prove that there exist n linearly independent bounded functionals $\lambda_1, \ldots, \lambda_n$. We do this by induction. Pick an arbitrary non-zero bounded functional λ_1. There exists at least one by Theorem 2.2.6. By way of induction, suppose we have a set of linearly independent bounded functionals $\lambda_1, \ldots, \lambda_j$ on Q with $1 \leq j < n$. Since $j < n = \dim_F Q$, therefore $\cap_{i=1}^{j} \ker \lambda_i \neq 0$. Pick $0 \neq x \in \cap_{i=1}^{j} \ker \lambda_i$. By Theorem 2.2.6 there exists a bounded linear functional λ_{j+1} on Q such that $\lambda_{j+1}(x) \neq 0$. If

$$c_1 \lambda_1 + \cdots + c_j \lambda_j + c_{j+1} \lambda_{j+1} = 0$$

with $c_i \in F$, then $c_1 \lambda_1(x) + \cdots + c_j \lambda_j(x) + c_{j+1} \lambda_{j+1}(x) = 0$. Since $x \in \cap_{i=1}^{j} \ker \lambda_i$, this means $\lambda_i(x) = 0$ for $i = 1, \ldots, j$, and hence $c_{j+1} = 0$. Since $\lambda_1, \ldots, \lambda_j$ are linearly independent, $c_i = 0$ for $1 \leq i \leq j$. It follows that $\lambda_1, \ldots, \lambda_j, \lambda_{j+1}$ is linearly independent. By induction we can find a basis of bounded linear functionals $\lambda_1, \dots, \lambda_n$ on Q. This proves that every linear functional on Q is bounded.

For the part (b) pick a linearly independent set $\lambda_1, \dots, \lambda_n$ of linear functionals on Q, i.e a basis for the dual space of Q. Each λ_j is continuous by part (a). We therefore get a continous F-linear isomorphism $(\lambda_1, \dots, \lambda_n)\colon Q \xrightarrow{\sim} F^n$. The inverse is also continuous by the discussion in §§§2.2.2.

Since the proof of part (b) involved an arbitrary basis for the dual of Q, part (c) follows from (b) by picking $\lambda_1, \dots, \lambda_n$ to be the basis dual to $e_1, \dots, e_n$, i.e. λ_i is defined by $\lambda_i(e_j) = \delta_{ij}$, $i, j \in \{1, \dots, n\}$. □

We have therefore, granting Theorem 2.2.6, proven that $\| \|_{\text{sup}}$ (which is equal to $\| \|_{\text{sp}}$ since A is a domain) is complete on A and hence is equivalent to $\| \|_\alpha$ for any K-algebra surjection α from a Tate algebra to A. In fact, we have proven the result for A reduced not merely an integral domain. We state the result for the record.

Theorem 2.2.8. *If A is a reduced K-affinoid algebra then $\| \|_{\text{sup}}$ is equivalent to every norm of the form $\| \|_\alpha$ where α is a surjective K-algebra homomorphism from a Tate algebra onto A.*

Proof. This follows from the discussion in §§§2.2.3, Theorem 2.2.7 (c), and Corollary 2.1.2. □

It remains to prove Theorem 2.2.6. As we mentioned earlier, we will only prove it when F is perfect. The case where F is imperfect will be proven by you in your homework exercises.

Proof of Theorem 2.2.6 *when F is perfect.* Since F is perfect, the finite extension $F \to Q$ is separable. It follows that the trace map

$$\text{Tr} = \text{Tr}_{Q/F}\colon Q \to F$$

is a non-zero functional. We claim that Tr is continuous. Let $x \in Q$. Let

$$\Phi = \zeta^n + f_1\zeta^{n-1} + f_2\zeta^{n-2} + \dots + f_n$$

be the characteristic polynomial of the F-linear endomorphism on Q given by $y \mapsto xy$ and let Ψ be the minimal polynomial of x over F. Then $\Phi = \Psi^m$ for some positive integer m. Let $s = \frac{n}{m}$, so that $s = \deg \Psi$. If $\Psi = \zeta^s + g_1\zeta^{s-1} + \dots + g_s$, then $f_1 = mg_1$. Now $\text{Tr}(x) = -f_1 = -mg_1$. Since K is non-archimedean $|m| \leq 1$, where we write m for the image of m in K for convenience. We have the following chain of inequalities

$$\|\text{Tr}(x)\|_{\text{gs}} = \|f_1\|_{\text{gs}} = \|mg_1\|_{\text{gs}} \leq \|g_1\|_{\text{gs}} \leq \max_{i=1\dots s} \|g_i\|_{\text{gs}}^{\frac{1}{i}} = \|x\|_{\text{sp}},$$

whence Tr is continuous. Now

$$\dim_Q \text{Hom}_F(Q, F) = 1.$$

and since $\text{Tr} \neq 0$ we must have

$$\text{Hom}_F(Q, F) = Q \cdot \text{Tr} \cong Q.$$

Let $0 \neq x \in Q$. Then $x\text{Tr} \neq 0$. Therefore there exists $y \in Q$ such that $x\text{Tr}(y) \neq 0$, i.e. $\text{Tr}(xy) \neq 0$. Let $\lambda = y\text{Tr}$. Then λ is continuous, since Tr is continuous on Q and the multiplication map $Q \times Q \to Q$ is continous. Now $\lambda(x) = \text{Tr}(xy) \neq 0$ and hence we are done. □