LECTURE 12

Date of Lecture: September 19, 2019

K is a complete non-archimedean field, and to avoid annoying trivialities we assume the absolute value $| \ |$ on K is non-trivial.

As before $\mathbf{N} = \{0, 1, 2, \dots, m, \dots\}$. Rings mean commutative rings with 1.

The symbol ☡ is for flagging a cautionary comment or a tricky argument. It occurs in the margins and is Knuth's version of Bourbaki's "dangerous bend symbol".

1. Finitely generated modules over affinoid algebras

1.1. **Finite modules.** Let A be an affinoid algebra, and let $\| \ \|_A$ be one of the equivalent norms of the form $\| \ \|_\alpha$ on A, where $\alpha\colon T_n \twoheadrightarrow A$ is a surjective K-algebra homomorphism. Let $(M, \| \ \|_M)$ be a Banach module over $(A, \| \ \|_A)$.

Proposition 1.1.1. *If M is finitely generated as an A-module, then every submodule of M is closed in M.*

Proof. Let $N \subset M$ be a sub-module and X its closure in M. Since A is noetherian, X is finitely generated over A, say by $x_1, \dots, x_r$. We have a surjective A-module map

$$\pi\colon A^r \longrightarrow\!\!\!\!\!\rightarrow X$$

given by $(a_1, \dots, a_r) \mapsto \sum a_i x_i$. The free module A^r has the obvious norm $\| \ \|'$ namely

$$\|(a_1, \dots, a_r)\|' = \max_{i=1\dots r} \|a_i\|_A.$$

Let $D = \max_i \|x_i\|_M$. Then

$$\Big\|\sum_i a_i x_i\Big\|_M \leq \max_i \|a_i x_i\|_M \leq \max_i \|a_i\|_A \max_i \|x_i\|_M \leq D\|(a_1, \dots, a_r)\|',$$

i.e. π is continuous. It follows that $\ker\pi$ is closed in A^r, and we have, by the open mapping theorem, an isomorphism of Banach spaces $\bar{\pi}\colon A^r/\ker\pi \xrightarrow{\sim} X$ such that π is the composite $A^r \twoheadrightarrow A^r/\ker\pi \xrightarrow{\bar{\pi}} X$. By the definition of the norm on $A^r/\ker\pi$ and the continuity of $\bar{\pi}^{-1}$ we can find $C > 1$ such that

$$\inf_{\boldsymbol{a}\in\pi^{-1}(x)} \|\boldsymbol{a}\|' < C\|x\|_M \qquad (x \in X).$$

Set $c := \frac{1}{C}$ and note that $0 < c < 1$. We have shown that for every $x \in X$ there exists $(a_1, \dots, a_r) \in A^r$ with $x = \sum_{i=1}^r a_i x_i$ and such that

$$c\|(a_1, \dots, a_r)\|' \leq \|x\|_M.$$

Pick $n_1, \dots, n_r \in N$ such that $\|n_i - x_i\|_M \leq c^2$, $i = 1, \dots, r$. We claim that $n_1, \dots, n_r$ also generate X. This will prove that $N = X$, whence N is closed.

Let $y \in X$. Pick $\boldsymbol{a}^{(0)} = (a_1^{(0)}, \dots, a_r^{(0)}) \in A^r$ such that that $y = \sum_i a_i^{(0)} x_i$ and such that $c\|\boldsymbol{a}^{(0)}\|' \leq \|y\|_M$. Then

$$y = \sum_{i=1}^{r} a_i^{(0)} n_i + y_1$$

where $y_1 = \sum_i a_i^{(0)}(x_i - n_i)$. Now $\|y_1\|_M \leq \|\boldsymbol{a}^{(0)}\|' \cdot \max_i \|x_i - n_i\|_M \leq c^{-1}\|y\|_M c^2 = c\|y\|_M$, i.e.,

$$\|y_1\|_M \leq c\|y\|_M.$$

For y_1 we can find $\boldsymbol{a}^{(1)} = (a_1^{(1)}, \dots, a_r^{(1)})$ such that $y_1 = \sum_{i=1}^r a_i^{(1)} x_i$ and $c\|\boldsymbol{a}^{(1)}\|' \leq \|y_1\|_M$. Then reasoning as before we have $y_1 = \sum_{i=1}^r a_i^{(1)} n_i + y_2$ with $y_2 = \sum_i a_i^{(1)}(x_i - n_i)$ and $\|y_2\|_M \leq c\|y_1\|_M$. Continuing this process, for each $m \in \mathbf{N}$ we can find elements $y_m \in X$, $\boldsymbol{a}^{(m)} = (a_1^{(m)}, \dots, a_r^{(m)}) \in A^r$ such that $y_m = \sum_{i=1}^r a_i^{(m)} n_i + y_{m+1}$ with $c\|\boldsymbol{a}^{(m)}\|' \leq \|y_m\|_M$ and $\|y_{m+1}\|_M \leq c\|y_m\|_M$.

Since $\|y_m\|_M \leq c^m \|y\|_M$ and since $\|a_i^{(m)}\|_A \leq c^{-1}\|y_m\|_M \leq c^{m-1}\|y\|_M$, we see that the infinite sums $\sum_{m\geq 1} y_m$ and $\sum_{m \geq 0} a_i^{(m)}$ converge for $i = 1, \dots r$. From the construction of y_m and $a_i^{(m)}$ we see that

$$y + \sum_{m\geq 1} y_m = \sum_{i=1}^{r} \left(\sum_{m \geq 0} a_i^{(m)} \right) n_i + \sum_{m \geq 1} y_m.$$

It follows that

$$y = \sum_{i=1}^{r} \left(\sum_{m \geq 0} a_i^{(m)} \right) n_i.$$

□

Proposition 1.1.2. *Let $(A, \| \, \|_A)$ be as above. Then every finitely generated A-module has the structure of a Banach A-module. Furthermore, any A-linear map between finitely generated Banach A-modules is continuous.*

Proof. If M is a finitely generated A-module then we have a surjective map $\varphi \colon A^r \to M$. According to Proposition 1.1.1, the kernel of φ is closed, and hence the residue norm on $A^r/\ker\varphi$ makes the latter into a Banach-module over A. Since M is isomorphic as an A-module to $A^r/\ker\varphi$, M acquires a Banach A-module structure.

Next, suppose M and N are Banach A-modules and $\phi \colon M \to N$ an A-module map. To show ϕ is continuous, it is enough to assume $M = A^r$ with the standard norm on A^r. To simplify notations, we will use the symbol $\| \, \|$ to denote all norms, it being obvious from the context on which modules the norms occur. Let $\boldsymbol{e}_1, \dots, \boldsymbol{e}_r$ be the standard basis of A^r, and let $\boldsymbol{a} = (a_1, \dots, a_r) \in A^r$. We have

$$\|\phi(\boldsymbol{a})\| \leq \max_i \|\phi(\boldsymbol{e}_i)\| \|\boldsymbol{a}\|$$

proving that ϕ is continuous. □

1.2. **Finite algebras.** Let A as before be an affinoid algebra and $A \to B$ a finite A-algebra, i.e. $A \to B$ is a K-algebra homomorphism and as an A-module, B is finitely generates, say by $b_1, \dots, b_r$. From Proposition 1.1.2, one has a norm $\| \, \|_*$ on B, which makes B a Banach A-module. By the construction of $\| \, \|_*$ as the residue norm from the surjection $A^r \twoheadrightarrow B$, we see that $\|b_i\|_* \leq 1$ for every $i = 1, \dots, r$. Let $M = \max_{i,j} \|b_i b_j\|_*$. Now let $x, y \in A$. We can find $\boldsymbol{a} = (a_1, \dots, a_r)$ and

$\boldsymbol{a}' = (a'_1, \ldots, a'_r)$ in A^r such that $x = \sum_i a_i b_i$, $y = \sum_i a'_i b_i$ and such that the inequalities

$$\|\boldsymbol{a}\| \leq 2\|x\|_* \quad \text{and} \quad \|\boldsymbol{a}'\| \leq 2\|y\|_*$$

are satisfied. This follows from the definition of the norm $\| \|_*$ on B. With this choice of representations of x and y as A-linear combinations of $b_1, \ldots, b_r$, we claim that

$$\|xy\|_* \leq 4M\|x\|_*\|y\|_*. \tag{1.2.1}$$

Indeed, we have the following chain of relations

$$\begin{aligned}
\|xy\|_* = \Big\|\sum_{i,j} a_i a'_j b_i b_j\Big\|_* &= \max_{i,j}\|a_i a'_j\|\|b_i b_j\|_* \\
&\leq M \max_{i,j}\|a_i a'_j\| \\
&\leq M \max_i\|a_j\| \max_j\|a'_j\| \\
&= M\|\boldsymbol{a}\|\|\boldsymbol{a}'\| \\
&\leq 4M\|x\|_*\|y\|_*
\end{aligned}$$

giving (1.2.1). One immediate consequence is that for fixed $b \in B$, the B-module map $\mu_b \colon B \to B$ given by $x \mapsto bx$ is continuous. If we define

$$\| \|_B \colon B \to \mathbf{R}_+$$

by the formula

$$\|b\|_B = \sup_{x \neq 0} \frac{\|bx\|_*}{\|x\|_*} \tag{1.2.2}$$

then $\| \|_B$ is a norm on B—namely $\|b\|_B$ is the operator norm of the bounded operator μ_b—and further

$$\|bc\|_B \leq \|b\|_B\|c\|_B. \tag{1.2.3}$$

By (1.2.1) $\|\mu_b\| \leq 4M\|b\|_*$. On the other hand $\|b\|_*/\|1\|_* = \|b \cdot 1\|_*/\|1\|_* \leq \|\mu_b\|$. Thus

$$\|b\|_B \leq 4M\|b\|_* \quad \text{and} \quad \|b\|_* \leq \|1\|_*\|b\|_B$$

for every $b \in B$. In other words, $\| \|_B$ and $\| \|_*$ are equivalent norms. It follows that $(B, \| \|_B)$ complete and hence, by (1.2.3), is a Banach A-algebra.[1] We have thus proven:

Lemma 1.2.4. *Let A be an affinoid K-algebra and B a finite A-algebra. Then B has a natural structure of a Banach A-algebra.*

In fact one can prove more. Recall that if A is a finitely generated algebra over K and B is a finite A-algebra (i.e. as an A-module, B is finitely generated), then B is also a finitely generated K-algebra. The analogue for affinoid algebras is the following theorem.

Theorem 1.2.5. *Let A be an affinoid K-algebra and B a finite A-algebra. Then B is an affinoid K-algebra.*

[1] It is obvious that $\|1\|_B = 1$.

Proof. Since A is affinoid, we have a surjective K-algebra map $T_n \twoheadrightarrow A$ for some $n \in \mathbf{N}$, and since this is a finite map, the composite $T_n \twoheadrightarrow A \to B$ is also a finite K-algebra homomorphism. We may therefore assume without loss of generality that $A = T_n$. In what follows $\| \|_B$ is the Banach A-algebra norm on B defined by (1.2.2).

Let $b_1, \ldots, b_r \in B$ be T_n-module generators of B. We may assume without loss of generality that $\|b_i\|_B \leq 1$ for $i = 1, \ldots, r$. Now $b_1, \ldots, b_r$ are also T_n-algebra generators and we have a natural surjective T_n-algebra homomorphism

$$\begin{aligned} T_n[\zeta_1, \ldots, \zeta_r] &\longrightarrow\!\!\!\!\rightarrow B \\ \zeta_i &\longmapsto b_i \qquad (i = 1, \ldots, r). \end{aligned} \tag{1.2.6}$$

let $T_n\langle\zeta_1, \ldots, \zeta_r\rangle$ be the T_n-subalgebra of $T_n[[\zeta_1, \ldots, \zeta_r]]$ consisting of formal power series $\sum_{\boldsymbol{\nu} \in \mathbf{N}^r} f_{\nu_1 \ldots \nu_r} \zeta_1^{\nu_1} \ldots \zeta_r^{\nu_r}$, $f_{\boldsymbol{\nu}} \in T_n$, such that $\|f_{\boldsymbol{\nu}}\| \to 0$ as $|\boldsymbol{\nu}| \to \infty$. It is immediate that

$$T_n\langle\zeta_1, \ldots, \zeta_r\rangle = T_{n+r}.$$

The polynomial ring $T_n[\zeta_1, \ldots, \zeta_r]$ is dense in $T_n\langle\zeta_1, \ldots, \zeta_r\rangle = T_{n+r}$. The map (1.2.6) is continuous. Indeed

$$\Big\|\sum_{\boldsymbol{\nu}} f_{\boldsymbol{\nu}} b_1^{\nu_1} \ldots b_r^{\nu_r}\Big\|_B \leq \max_{\boldsymbol{\nu}} \|f_{\boldsymbol{\nu}}\| = \Big\|\sum_{\boldsymbol{\nu}} f_{\boldsymbol{\nu}} \boldsymbol{\zeta}^{\boldsymbol{\nu}}\Big\|$$

for every polynomial $\sum_{\boldsymbol{\nu}} f_{\boldsymbol{\nu}} \boldsymbol{\zeta}^{\boldsymbol{\nu}} \in T_n[\boldsymbol{\zeta}]$. We have used the fact that $\|b_i\|_B \leq 1$ for $i = 1, \ldots, r$ to establish the inequality above. Since $T_n[\boldsymbol{\zeta}]$ is dense in $T_n\langle\boldsymbol{\zeta}\rangle$ and (1.2.6) is uniformly continuous (being a linear continuous map), (1.2.6) extends uniquely to a surjective map $T_{n+r} = T_n\langle\boldsymbol{\zeta}\rangle \twoheadrightarrow B$. It is easy to see this is a T_n-algebra homomorphism (in particular, a K-algebra homomorphism). Thus B is affinoid. □

1.3. **Spectral values.** Let $(A, \| \|)$ be a semi-normed K-algebra. Let $p \in A[\zeta]$ be a monic polynomial, say

$$p = \zeta^r + c_1\zeta^{r-1} + \cdots + c_r$$

with $c_i \in A$. The *spectral value* $\sigma(p)$ *of* p is defined to be

$$\sigma(p) = \max_{i=1\ldots r} \|c_i\|^{\frac{1}{i}}. \tag{1.3.1}$$

If $A = K$, there is a nice formula for the spectral value of p.

Lemma 1.3.2. *Suppose* $p = \zeta^r + c_1\zeta^{r-1} + \cdots + c_r \in K[\zeta]$ *is a polynomial which factors in* $\overline{K}[\zeta]$ *as*

$$p = \zeta^r + c_1\zeta^{r-1} + \cdots + c_r = \prod_{j=1}^{r} (\zeta - \alpha_j).$$

Then

$$\sigma(p) = \max_{j=1\ldots r} |\alpha_j|.$$

Proof. Let $\sigma_i(X_1, \ldots, X_r)$ be the i^{th}-symmetric polynomial in r variables. Then $c_i = \pm\sigma_i(\alpha_1, \ldots, \alpha_r)$. It follows that for every $i \in \{1, \ldots, r\}$ we have

$$|c_i| = |\sigma_i(\alpha_1, \ldots, \alpha_r)| \leq \max_{j=1\ldots r} |\alpha_j|^i.$$

On the other hand if $m = \max_{j=1\ldots r}|\alpha_j|$ and $\alpha_{j_1}, \ldots, \alpha_{j_i}$ are the roots of p such that $|\alpha_j| = m$, then exactly one of the summands of $\sigma_i(\alpha_r, \ldots, \alpha_r)$ (in its representation as the signed sum of monomials of degree i in the α_j) has absolute value m^i. All the rest have absolute value strictly less than m^i. It follows that

$$|c_i| = m^i.$$

□

The following result was stated, and we will prove it in the next lecture:

Lemma 1.3.3. *Let $(A, \| \|)$ be a semi-normed ring. Let $p, q \in A[\zeta]$ be polynomials. Then $\sigma(pq) \leq \max\{\sigma(p), \sigma(q)\}$.*