LECTURE 11

Date of Lecture: September 17, 2019

K is a complete non-archimedean field, and to avoid annoying trivialities we assume the absolute value $|\cdot|$ on K is non-trivial.

As before $\mathbf{N} = \{0, 1, 2, \dots, m, \dots\}$. Rings mean commutative rings with 1.

The symbol ☡ is for flagging a cautionary comment or a tricky argument. It occurs in the margins and is Knuth's version of Bourbaki's "dangerous bend symbol".

1. Finite dimensional K-vector spaces

1.1. **All norms on a finite dimensional space are equivalent.** In this subsection we prove a non-archimedean analogue of a well known result on finite dimensional vectorspaces over $\mathbf{R}$ and $\mathbf{C}$, namely that all norms on such spaces are equivalent and they are complete with respect to any norm on them.

Let $(X, \|\cdot\|)$ be a *finite dimensional* normed linear space over K. Let $B = \{\boldsymbol{e}_1, \dots, \boldsymbol{e}_n\}$ be a basis for X. Define $\|\cdot\|_B \colon X \to \mathbf{R}_+$ by the formula

$$\|\alpha_1\boldsymbol{e}_1 + \cdots + \alpha_n\boldsymbol{e}_n\|_B := \max_i |\alpha_i|.$$

It is easy to see that $(X, \|\cdot\|_B)$ is a K-Banach space using the completeness of K.

If $M = \max_i\|\boldsymbol{e}_i\|$, then for $\boldsymbol{x} = \alpha_1\boldsymbol{e}_1 + \cdots + \alpha_n\boldsymbol{e}_n \in X$ we have

$$\|\boldsymbol{x}\| \le \max_i |\alpha_i|\|\boldsymbol{e}_i\| \le M\|\boldsymbol{x}\|_B. \tag{1.1.1}$$

We will prove that there exists $c > 0$ such that

$$\|\boldsymbol{x}\|_B \le c\|\boldsymbol{x}\| \qquad (\boldsymbol{x} \in X). \tag{1.1.2}$$

This will prove that $\|\cdot\|$ is equivalent to $\|\cdot\|_B$ and also prove that all norms on X are equivalent and that $(X, \|\cdot\|)$ is a Banach space over K (since $(X, \|\cdot\|_B)$ is). We will prove the existence of $c > 0$ satisfying (1.1.2) by induction on n, the dimension of X.

If $n = 1$, set $c = \|\boldsymbol{e}_1\|^{-1}$. Then, for $\boldsymbol{x} = \alpha\boldsymbol{e}_1$ we have

$$\|\boldsymbol{x}\|_B = |\alpha| = |\alpha|\|\boldsymbol{e}_1\|/\|e\|_1 = c\|\boldsymbol{x}\|.$$

Now assume $n > 1$ and set $V = \operatorname{span}\{\boldsymbol{e}_1, \dots, \boldsymbol{e}_{n-1}\}$. By way of induction we may (and do) assume that there exists $c_1 > 0$ such that

$$\|\boldsymbol{v}\|_B \le c_1\boldsymbol{v} \qquad (\boldsymbol{v} \in V). \tag{1.1.3}$$

In view of the above and (1.1.1), $\|\cdot\|_B$ and $\|\cdot\|$ are equivalent on V. Since $(V, \|\cdot\|_B)$ is a Banach space, it follows that so is $(V, \|\cdot\|)$. Hence $(V, \|\cdot\|)$ is closed in $(X, \|\cdot\|)$. In particular we have

$$\inf_{\boldsymbol{v}\in V}\|\boldsymbol{e}_n - \boldsymbol{v}\| > 0.$$

Let

$$c_2 = \frac{\|\boldsymbol{e}_n\|}{\inf_{\boldsymbol{v}\in V}\|\boldsymbol{e}_n - \boldsymbol{v}\|}.$$

Then $c_2 \geq 1$. Set

$$c := \max\left\{c_1 c_2, \frac{c_2}{\|\boldsymbol{e}_n\|}\right\}. \tag{1.1.4}$$

We claim that c as defined in (1.1.4) satisfies (1.1.2). Note that since $c_2 \geq 1$, we have $c \geq c_1$. Let $\boldsymbol{x} \in X$. We wish to show $\|\boldsymbol{x}\|_B \leq c\|\boldsymbol{x}\|$. If $\boldsymbol{x} \in V$, this is true since $c_1 \leq c$, and the because of (1.1.3). So assume $\boldsymbol{x} \notin V$. Then there exist a unique $\boldsymbol{v} \in V$ and $0 \neq b \in K$ such that

$$\boldsymbol{x} = \boldsymbol{v} + b\boldsymbol{e}_n.$$

Now

$$\|\boldsymbol{x}\| = |b|\|b^{-1}\boldsymbol{v} + \boldsymbol{e}_n\| \geq |b| \inf_{\boldsymbol{w}\in V} \|\boldsymbol{e}_n - \boldsymbol{w}\| = c_2^{-1}\|b\boldsymbol{e}_n\|.$$

This can be re-written as

$$\|b\boldsymbol{e}_n\| \leq c_2\|\boldsymbol{x}\|. \tag{1.1.5}$$

Moreover,

$$\|\boldsymbol{v}\| = \|\boldsymbol{x} - b\boldsymbol{e}_n\| \leq \max\{\|\boldsymbol{x}\|, \|b\boldsymbol{e}_n\|\} \leq \max\{\|\boldsymbol{x}\|, c_2\|\boldsymbol{x}\|\} = c_2\|\boldsymbol{x}\|.$$

We have used (1.1.5) in the second inequality in the chain above, and the fact that $c_2 \geq 1$ for the last equality. Thus

$$\|\boldsymbol{v}\| \leq c_2\|\boldsymbol{x}\|. \tag{1.1.6}$$

Now

$$\begin{aligned}
\|\boldsymbol{x}\|_B &= \|\boldsymbol{v} + b\boldsymbol{e}_n\|_B \\
&= \max\{\|\boldsymbol{v}\|_B, |b|\} \\
&\leq \max\left\{c_1\|\boldsymbol{v}\|, \frac{\|b\boldsymbol{e}_n\|}{\|\boldsymbol{e}_n\|}\right\} && \text{(by (1.1.3))} \\
&\leq \max\left\{c_1 c_2\|\boldsymbol{x}\|, \frac{c_2}{\|\boldsymbol{e}_n\|}\|\boldsymbol{x}\|\right\} && \text{(by (1.1.5) and (1.1.6))} \\
&= c\|\boldsymbol{x}\|.
\end{aligned}$$

This establishes (1.1.2) with c as in (1.1.4).

We have proved the following (with $\|\cdot\|_B$ showing that the set of norms on X is non-empty):

Theorem 1.1.7. *Let X be a finite dimensional K-vector space. The set of norms on X is non-empty and any two norms on X are equivalent. X is a K-Banach space with respect to each of these norms.*

An immediate corollary is the following:

Corollary 1.1.8. *Any K-linear map from X to a normed vector space over K is bounded.*

Proof. It is clear from the theorem that an injective map from a finite dimensional normed space to another normed space is continuous. Now suppose $T\colon X \to Y$ is K linear with Y a normed linear space. Then $\ker T$ is closed (being complete according to the theorem). Endowing $X/\ker T$ with the resulting residue norm it is clear that the canonical map $X \twoheadrightarrow X/\ker T$ is continuous. Since T is the composite $X \twoheadrightarrow X/\ker T \hookrightarrow Y$ and $X/\ker T$ is finite dimensional, we are done. $\square$

Remark 1.1.9. As an example of an annoying triviality alluded to at the beginning of this document, consider the standard method of showing that a linear transformation T is continuous if and only if only if it is bounded (i.e. if and only if there exists $M \geq 0$ such that $\|T\boldsymbol{x}\| \leq M\|\boldsymbol{x}\|$) given in a standard functional analysis course. The proof requires one to"re-scale" balls centred at the origin so that they fit into another such ball. We cannot do that if $|\cdot|$ is the trivial absolute value on K, i.e. one for which all non-zero elements have absolute value 1. If $|\cdot|$ is non-trivial the standard proof goes through.

2. K-algebra homomorphisms between affinoid algebras

2.1. **Krull's intersection theorem.** The Krull intersection theorem says that if $(R, \mathfrak{m})$ is a noetherian local ring then

$$\bigcap_{l \geq 1} \mathfrak{m}^l = 0.$$

One consequence is the following:

Lemma 2.1.1. *Let A be a noetherian ring. Then*

$$\bigcap_{\mathfrak{m} \in \mathrm{Max}(A)} \bigcap_{l \geq 1} \mathfrak{m}^l = 0.$$

Proof. Suppose f lies in the intersection on the left side. It is enough to show that the annihilator of f, $\mathrm{ann}(f)$, equals A, where $\mathrm{ann}(f)$ is the collection of elements $t \in A$ such that $tf = 0$. Pick any $\mathfrak{m} \in \mathrm{Max}(A)$. Since $f \in \cap_l \mathfrak{m}^l$, by Krull's intersection theorem $\frac{f}{1} = 0$ in $A_{\mathfrak{m}}$. This means there exists $t \notin \mathfrak{m}$ such that $tf = 0$. It follows that $\mathrm{ann}(f)$ is not contained in $\mathfrak{m}$. Since $\mathrm{ann}(f)$ is not contained in any maximal ideal of A, it must be all of A. $\square$

2.2. **Affinoid algebras.** In rigid analytic geometry, affinoid algebras (or more precisely, affinoid K-algebras) play the role that finitely generated rings over a fixed field play in algebraic geometry.

Definition 2.2.1. A K-algebra A is called an *affinoid K-algebra*, or simply an *affinoid algebra* if the context is clear, if there is a surjective K-algebra homomorphism

$$\alpha \colon T_n \longrightarrow\!\!\!\!\!\rightarrow A$$

on to A from some Tate algebra T_n.

Given an affinoid algebra A, each isomorphism $A \cong T_n/\mathfrak{a}$ gives us a residue norm on A, since all ideals in T_n are closed (see [Lecture 8, Theorem 2.2.1]). If $\alpha \colon T_n \twoheadrightarrow A$ is a surjective K-algebra hmomprphism, we denote the resulting residue norm on A by $\|\cdot\|_\alpha$. In other words

$$\|f\|_\alpha := \inf\left\{\|g\| \,\middle|\, g \in \alpha^{-1}(f)\right\} \qquad (f \in A). \tag{2.2.2}$$

We will show that all the residue norms $\|\cdot\|_\alpha$ on A, as α varies over surjective homomorphisms to A from Tate algebras, are equivalent. Towards that end we first prove

Lemma 2.2.3. *Let B be an affinoid algebra and $\mathfrak{m}$ a maximal ideal of B. Then $B/\mathfrak{m}^l$ is a finite K-alegbra for every $l \geq 1$.*

Proof. Fix $l \geq 1$ and $\mathfrak{m} \in \operatorname{Max}(B)$. Since $B/\mathfrak{m}^l$ has Krull dimension zero we have a noether normalisation $K = T_0 \hookrightarrow B/\mathfrak{m}^l$ proving the lemma. $\square$

Theorem 2.2.4. *Let*

$$\varphi\colon A \longrightarrow B$$

be a K-algebra homomorphism between affinoid K-algebras. Endow A with a residue norm $\|\cdot\|_\alpha$ arising from some surjective map α from a Tate algebra onto A. Let $\|\cdot\|$ be any K-algebra norm on B making B into a K-Banach space and such that $\mathfrak{m}^l$ is closed in B for every $l \geq 1$ and every $\mathfrak{m} \in \operatorname{Max}(B)$. Then φ is continuous.

Proof. By the closed graph theorem we have to prove that if $\{a_n\}$ is a sequence in A with $a_n \to 0$ as $n \to \infty$, and $\lim_{n\to\infty} \varphi(a_n) = b$, then $b = 0$. Fix $\mathfrak{m} \in \operatorname{Max}(B)$ and $l \geq 1$. Let $\nu\colon B \twoheadrightarrow B/\mathfrak{m}^l$ be the canonical surjection and

$$\bar{\varphi}\colon A \to B/\mathfrak{m}^l$$

the composite $A \xrightarrow{\varphi} B \xrightarrow{\nu} B/\mathfrak{m}^l$. Since $\mathfrak{m}^l$ is closed in B, $B/\mathfrak{m}^l$ acquires a residue norm and with this norm the map ν is continuous. Let

$$\mu\colon A \longrightarrow A/\ker\bar{\varphi}$$

be the canonical surjection and

$$\psi\colon A/\ker\bar{\varphi} \longrightarrow B/\mathfrak{m}^l$$

be the induced map, and endow $A/\ker\bar{\varphi}$ with the residue norm from $\|\cdot\|_\alpha$. Note that μ is continuous. The data can be arranged in a commutative diagram as below:

$$\begin{tikzcd} A \arrow[r, "\varphi"] \arrow[d, "\mu"'] \arrow[dr, "\bar{\varphi}"] & B \arrow[d, "\nu"] \\ A/\ker\bar{\varphi} \arrow[r, hook, "\psi"'] & B/\mathfrak{m}^l \end{tikzcd}$$

Note that ψ is injective, and hence $A/\ker\bar{\varphi}$ is finite dimensional as a K-vector space since $B/\mathfrak{m}^l$ is by Lemma 2.2.3. By Corollary 1.1.8, ψ is continuous, and hence so is $\bar{\varphi} = \psi \circ \mu$. Thus

$$\begin{aligned} \mu(b) &= \nu\Big(\lim_{n\to\infty} \varphi(a_n)\Big) && \text{(by definition of } b) \\ &= \lim_{n\to\infty} \nu(\varphi(a_n)) && \text{(since } \nu \text{ is continuous)} \\ &= \lim_{n\to\infty} \bar{\varphi}(a_n) && \text{(since } \bar{\varphi} = \nu\circ\varphi) \\ &= \bar{\varphi}\Big(\lim_{n\to\infty} a_n\Big) && \text{(since } \bar{\varphi} \text{ is continuous)} \\ &= 0. \end{aligned}$$

It follows that $b \in \mathfrak{m}^l$. Since $\mathfrak{m} \in \operatorname{Max}(B)$ and $l \geq 1$ were arbitrary, Lemma 2.1.1 shows that $b = 0$. $\square$

An immediate corollary is:

Corollary 2.2.5. *Let A and B be affinoid algebras endowed with residue norms arising from surjective maps from Tate algebras, and let $\varphi\colon A \to B$ be a K-algebra homomorphism. Then φ is continuous. In particular if we have two surjective homomorphisms from Tate algebras to A, say $\alpha\colon T_n \twoheadrightarrow A$ and $\beta\colon T_m \twoheadrightarrow A$, then $\|\cdot\|_\alpha$ and $\|\cdot\|_\beta$ are equivalent.*

2.3. **The supremum "norm" on an affinoid algebra.** Recall from (3.2.1) of Lecture 8, the Gauss norm on T_n can also be computed by the fomula:

$$\|f\| = \sup_{\boldsymbol{x}\in \mathrm{Max}(T_n)} |f(\boldsymbol{x})| \qquad (f \in T_n)$$

where $f(\boldsymbol{x})$ is the image of f in the field $K(\boldsymbol{x}) := T_n/\mathfrak{m}_{\boldsymbol{x}}$.[1] With this in mind we define the *sup norm* $\|\cdot\|_{\mathrm{sup}}$ on an affinoid algebra A by the formula

$$\|f\|_{\mathrm{sup}} := \sup_{\boldsymbol{x}\in \mathrm{Max}(A)} |f(\boldsymbol{x})| \qquad (f \in A). \tag{2.3.1}$$

As before we write $\mathfrak{m}_{\boldsymbol{x}}$ for $\boldsymbol{x} \in \mathrm{Max}(A)$ when we think of it as a maximal ideal, and $f(\boldsymbol{x})$ is the image of f in $K(\boldsymbol{x}) := A/\mathfrak{m}_{\boldsymbol{x}}$. Recall from noether normalisation that $K(\boldsymbol{x})$ is finite over K and hence $|\cdot|$ extends uniquely from K to $K(\boldsymbol{x})$.

It should be remarked that the sup norm need not be a norm. Indeed, it is not hard to see that $\|f^n\|_{\mathrm{sup}} = \|f\|_{\mathrm{sup}}^n$, and hence if $f \neq 0$ is nilpotent, we have $\|f\|_{\mathrm{sup}} = 0$. If A is reduced then the sup norm is indeed a norm.

We record this and other fairly obvious facts about the sup norm below.

Proposition 2.3.2. *Let A be an affinoid K-algebra. Then*

(a) *The sup norm $\|\cdot\|_{\mathrm{sup}}$ on A is a semi-norm.*
(b) *$\|f^n\|_{\mathrm{sup}} = \|f\|_{\mathrm{sup}}^n$ for $f \in A$.*
(c) *If $\alpha\colon T_n \twoheadrightarrow A$ is a surjective K-algebra homomorphism, then*

$$\|f\|_{\mathrm{sup}} \leq \|f\|_\alpha \qquad (f \in A).$$

In particular the map $(A, \|\cdot\|_\alpha) \to (A, \|\cdot\|_{\mathrm{sup}})$ which is the identity on the underlying sets, is continuous.
(d) *$\|f\|_{\mathrm{sup}} = 0$ if and only if f is nilpotent.*
(e) *$\|\cdot\|_{\mathrm{sup}}$ is a norm if and only if A is reduced.*

Proof. Only part (d) needs elaboration. The rest follow more or less from the defining formula (2.3.1). Now $\|f\|_{\mathrm{sup}} = 0$ if and only if $f(\boldsymbol{x}) = 0$ for every $\boldsymbol{x} \in \mathrm{Max}(A)$, i.e. if and only if $f \in \mathfrak{m}$ for every maximal ideal $\mathfrak{m}$ of A. Since A is Jacobson by [Lecture 8, Theorem 2.1.1] we have $\cap_{\mathfrak{m}}\mathfrak{m} = \sqrt{(0)}$. This proves (d). □

[1] $\mathfrak{m}_{\boldsymbol{x}}$ is $\boldsymbol{x}$ when we wish to think of it as a maximal ideal rather than as a point of $\mathrm{Max}(T_n)$.