Aug 8, 2018

# Lecture 1

**Example**: Let $\tau \in \mathbb{C}$ be s.t. $\text{im}(\tau) > 0$. Set

$$\Lambda = \mathbb{Z} \oplus \mathbb{Z}\tau.$$

Then we have an elliptic curve $E$ s.t.

$$E(\mathbb{C}) \xrightarrow{\sim} \mathbb{C}/\Lambda.$$

$$\begin{array}{c} \mathbb{C} \\ \Big\downarrow \text{universal cover} \\ \mathbb{C}/\Lambda \cong E(\mathbb{C}) \end{array} \qquad (E = E^{\tau}).$$

Set

$$q = e^{2\pi i \tau} \quad \text{and} \quad q^{\mathbb{Z}} = \{ q^k \mid k \in \mathbb{Z} \}$$

We have a complex analytic isomorphism

$$\mathbb{C}/\Lambda \xrightarrow{\sim} \mathbb{C}^{\times}/q^{\mathbb{Z}} \qquad (\mathbb{C}^{\times} = \mathbb{C} \setminus \{0\})$$

$$z \longmapsto e^{2\pi i z}.$$

**Note**: Since $\text{im}(\tau) > 0$, therefore $|q| < 1$, i.e., $q \in \Delta$, where $\Delta = \{ z \mid |z| < 1 \}$ is the unit disc centred at $0$ in $\mathbb{C}$.

Here is one way to re-construct $E$ from our knowledge of $q \in \Delta$.

First, define

$$s_k(q) = \sum_{n \geq 1} \frac{n^k q^n}{1 - q^n}. \qquad k \geq 0.$$

Since $|q| < 1$, $s_k(q)$ is a power series in $q$ (convergent) with integer coefficients. Thus

$$s_k(q) \in \mathbb{Z}[[q]].$$

Set $\quad a_4(q) = -5s_3(q)\ , \quad a_6(q) = -\dfrac{5s_3(q) + 7s_5(q)}{12}$

$$\left.\begin{aligned} X(u,q) &= \sum_{n\in\mathbb{Z}} \frac{q^n u}{(1-q^n u)^2} - 2s_1(q) \\ Y(u,q) &= \sum_{n\in\mathbb{Z}} \frac{(q^n u)^2}{(1-q^n u)^3} + s_1(q) \end{aligned}\right\} u \in \mathbb{C}^\times$$

Let $E_q$ be the elliptic curve whose affine eqn is

$$y^2 + xy = x^3 + a_4(q)x + a_6(q)$$

Then

$$\mathbb{C}^\times / q^{\mathbb{Z}} \longrightarrow E_q(\mathbb{C})$$

$$u \longmapsto \begin{cases} (X(u,q), Y(u,q)) & \text{if } u \notin q^{\mathbb{Z}} \\ 0 & \text{if } u \in q^{\mathbb{Z}} \end{cases}$$

is a complex analytic isomorphism.

Observe that $\mathbb{C}^\times$ is not simply connected. So $\mathbb{C}^\times \longrightarrow E_q(\mathbb{C})$ is strictly speaking not a uniformisation.

## The p-adic analogue (Tate):

Recall that an absolute value or a valuation on a field $K$ is a map

$$|\cdot| : K \longrightarrow \mathbb{R}$$

such that for any $a, b \in K$ the following conditions hold:

(a) $|a| \geq 0$ and $|a| = 0$ iff $a = 0$

(b) $|ab| = |a||b|$

(c) $|a+b| \leq |a| + |b|$.

If the (c) can be replaced by the stronger condition

(c)' $|a+b| \leq \max\{|a|, |b|\}$

then we say the valuation is non-archimedean; otherwise, it is archimedean.

The valuation $|\cdot|$ is trivial if $|a| = 1 \ \forall \ a \neq 0$.

Clearly $(a,b) \longmapsto |a-b|$ is a distance function on $K$. $K$ is complete if it is complete as a metric space.

We will be more formal and precise starting from the next lecture. For the course we will focus on complete non-trivial non-archimedean fields.

One can, using the non-archimedean metric on $K$, talk about convergent power series in many variables, topological $K$-algebras, and most importantly affinoid $K$-algebras, which are topological $K$-algebras $A$ of the type

$$A \cong \frac{K\langle T_1, \dots, T_n \rangle}{I}$$

(top. $K$-alg isomorphism with $K\langle T_1, \dots T_n \rangle$ the ring of convergent power series)

These are Tate's analogues of affine schemes. The corresponding geometric "space" is an affinoid space. One glues them together (using a Grothendieck Topology!) to form a rigid analytic space and a huge chunk of this course is about defining/constructing them and then studying their first properties. Given an algebraic scheme $X$ over $k$ (recall, this means a scheme of finite type over $k$), there exists a rigid analytic space $X^{an}$ corresponding to $X$ as well as GAGA theorems concerning them.

The uniformization of an elliptic curve

$$E(\mathbb{C}) \cong \mathbb{C}/\Lambda \qquad \left( \mathbb{C} \xrightarrow{\text{univ. cover}} \mathbb{C}/\Lambda \cong E(\mathbb{C}) \right)$$

does not have a satisfactory analogue in rigid analytic geometry. One could replace $\mathbb{C}$ by $(\mathbb{A}^1_k)^{an}$. But the lattice $\Lambda$ creates issues and one does not quite get a quotient which gives an elliptic curve (more on all that later). However the isomorphism

$$E(\mathbb{C}) = \mathbb{C}^\times/q^{\mathbb{Z}}$$

does have a rigid analytic analogue. But note $\mathbb{C}^\times \to E(\mathbb{C})$ is not the universal cover. Nevertheless, here is the story.

Recall, we have a $k$-scheme

$$\mathbb{G}_{m,k} = \mathbb{G}_m = \operatorname{Spec}(k[T, T^{-1}])$$

which is the scheme version of $k^\times = k - \{0\}$. If $k = \mathbb{C}$, this is the analogue of $\mathbb{C}^\times$.

One can consider the rigid analytic space

$$\boxed{\mathrm{Tate}_q = \mathbb{G}_m^{an}/q^{\mathbb{Z}}} \longleftarrow \text{Tate curve (Defn)}.$$

for $q \in k$ with $0 < |q| < 1$. Note the absolute value - it is non-archimedean. We are sweeping a number of things under the carpet now, but in the course we will make sense of this.

**Theorem:** (i) $\mathrm{Tate}_{q_1} \cong \mathrm{Tate}_{q_2} \iff q_1 = q_2$.

(ii) S'pse char $K = 0$, and $E$ elliptic curve $/K$ with $|j(E)| > 1$. Then $\exists$ a Galois extn $L$ of $K$ of deg $\leq 2$ s.t., $E_L^{an}$ is isomorphic to a Tate curve. Moreover, we can take $K = L$ if the minimal Weierstrass eqn for $E$ has an integral model over $K$ with split multiplicative reduction.

To be defined as the course progresses.

Finally, even though the topology on rigid spaces is weird and they are not path connected, nevertheless one can talk about a rigid space being simply connected using Galois theory, and it turns out that $\mathbb{G}_m^{an}$ is in fact simply connected even though $\mathbb{C}^\times$ is not. Thus we do have a uniformization.

**Higher genera (Mumford):** Over $\mathbb{C}$ and classical topology one can uniformize a (smooth, projective) curve by the upper half-plane or disc (Koebe). Again that does not work over $K$. There is another description of such a curve. The space $S^2 = \mathbb{P}^1(\mathbb{C})$ is acted upon by a Schottky group (these are subgroups of Kleinian groups). This Schottky group, call it $\Gamma$, defines a limiting locus $Z$ on $S^2$. $\Gamma$ acts on $S^2-Z$ in a properly discontinuous way, and

$$X = \frac{S^2-Z}{\Gamma}$$

is a compact Riemann surface (?). Every curve over $\mathbb{C}$ can be so realised. This works in the non-archimedean (i.e., rigid analytic) case also, as shown by Mumford. As in the elliptic curve case $S^2-Z$ is not simply connected in the classical topology and hence $X=(S^2-Z)/\Gamma$ is not a uniformization of $X$. However, in the rigid analytic world, the "numerator" is simply connected and we have a uniformization.

History: Tate did his calculations for elliptic curves in 1959 (he hadn't yet come up with rigid analytic spaces). He wondered if Grothendieck's ideas (on topologies) could give him a uniformization in the non-archimedean case and wrote to Grothendieck. Grothendieck was skeptical. He wrote to Serre:

"Tate has written to me about his elliptic curve stuff, and has asked me if I had any ideas for a global definition of analytic varieties over complete valuation fields. I must admit that I have absolutely not understood why his results might suggest the existence of such a definition, and I remain skeptical. Nor do I have the impression of having understood his theorem at all; it does nothing more than exhibit, via brute formulas, a certain isomorphism of analytic groups; one could conceive that other equally explicit formulas might give another one which would be no worse than his (until proof to the contrary!)".