HW 7

Due on Nov 5, 2019 (in class).

Covariant Yoneda.

(1) Let $\mathscr{C}$ be a category and Z_1 and Z_2 be objects in $\mathscr{C}$. Suppose we have a natural transformation $T\colon \mathrm{Hom}_{\mathscr{C}}(Z_1, -) \to \mathrm{Hom}_{\mathscr{C}}(Z_2, -)$ of **Sets**-valued functors on $\mathscr{C}$. Show that there is a unique map $\tau\colon Z_2 \to Z_1$ in $\mathscr{C}$ such that for any object C in $\mathscr{C}$, the map $T(C)\colon \mathrm{Hom}_{\mathscr{C}}(Z_1, C) \to \mathrm{Hom}_{\mathscr{C}}(Z_2, C)$ is given by $\phi \mapsto \phi \circ \tau$, $\phi \in \mathrm{Hom}_{\mathscr{C}}(Z_1, C)$.

Čech cohomology. Suppose $(\mathscr{C}, \mathscr{C}ov)$ is a Grothendieck topology on a category $\mathscr{C}$. Let $\mathcal{Psh} = \mathcal{Psh}_{\mathscr{C}}$ denote the category of pre-sheaves on $\mathscr{C}$.

For $V \in \mathscr{C}$, let $\mathcal{Z}_V$ be the presheaf of abelian groups on $\mathscr{C}$ given by

$$\mathcal{Z}_V(W) = \mathbf{Z}^{\mathrm{Hom}_{\mathscr{C}}(W,\, V)} = \bigoplus_{\phi\colon V \to W} \mathbf{Z} \qquad (W \in \mathscr{C})$$

with obvious "restriction" maps.

In the problems that follow, fix $U \in \mathscr{C}$ and $\mathfrak{U} = \{U_\alpha \to U\}_{\alpha \in I} \in \mathscr{C}ov(U)$. If is $\mathscr{P}$ a presheaf on $\mathscr{C}$ then (as usual) $C^\bullet(\mathfrak{U}, \mathscr{P})$ will denote Čech complex associated with $\mathfrak{U}$ and $\mathscr{P}$. If $(i_0, \dots, i_p) \in I^{p+1}$, then set

$$U_{i_0 \dots i_p} = U_{i_0} \times_U U_{i_1} \times_U \cdots \times_U U_{i_p}.$$

Finally for $p \in \mathbf{N}$ we write

$$\mathcal{Z}_p = \bigoplus_{\boldsymbol{i} \in I^{p+1}} \mathcal{Z}_{U_{i_0 \dots i_p}}.$$

(2) Let $V \in \mathscr{C}$. Show that

$$\mathrm{Hom}_{\mathcal{Psh}}(\mathcal{Z}_V, -) = \Gamma(V, -).$$

(3) (a) Show that $C^p(\mathfrak{U},\, \mathscr{P})$ is functorial in $\mathscr{P} \in \mathcal{Psh}$.

(b) Fix $p \in \mathbf{N}$. Show that

$$C^p(\mathfrak{U},\, \mathscr{P}) = \mathrm{Hom}_{\mathcal{Psh}}(\mathcal{Z}_p, \mathscr{P}).$$

for every $\mathscr{P} \in \mathcal{Psh}$. Show that this is a functorial identification.

(4) (a) Show (using problem (1) and problem (3)) that we have a homology complex

$$0 \longleftarrow \mathcal{Z}_0 \longleftarrow \mathcal{Z}_1 \longleftarrow \dots \longleftarrow \mathcal{Z}_{p-1} \longleftarrow \mathcal{Z}_p \longleftarrow \dots$$

such that $\mathrm{Hom}_{\mathcal{Psh}}(\mathcal{Z}_\bullet, \mathscr{P}) = C^\bullet(\mathfrak{U},\, \mathscr{P})$ for $\mathscr{P} \in \mathcal{Psh}$.

(b) Give an explicit formula for the boundary map $\mathcal{Z}_p \to \mathcal{Z}_{p-1}$.

(c) Show that $\mathrm{H}_p(\mathcal{Z}_\bullet) = 0$ for $p \geq 1$.