HW 6

Due on October 22, 2019 (in class).

As before $\mathbf{N} = \{0, 1, 2, \dots, m, \dots\}$. Also $\mathbf{R}_+ := [0, \infty)$.

$(K, |\ |)$ is a complete non-archimedean field such that $|\ |$ is non-trivial. For any complete field L, let

$$T_n(L) = L\langle \zeta_1, \dots, \zeta_n \rangle$$

with the Gauss norm. If $L = K$, we just write T_n for $T_n(L)$ as we have been doing all along.

Definition 1. *Let $(A, \|\ \|)$ be a normed K-algebra, i.e. it is a normed K-vector space such that $\|ab\| \le \|a\|\|b\|$ for $a, b \in A$. An A-module M with a K-norm $\|\ \|'$ is said to be* a normed A-module *if $\|ay\|' \le \|a\|\|y\|'$, $a \in A$, $y \in M$. It is said to be* a faithfully normed A-module *if $\|ay\|' = \|a\|\|y\|'$ for $a \in A$, $y \in M$. A normed normed A-module $(M, \|\ \|')$ is said to be* separable with respect to bounded linear maps *or simply* b-separable *if for every non-zero element $y \in M$, there exists a bounded linear map $\tau\colon M \to A$ such that $\tau(y) \neq 0$.*[1]

Spectral norms. We assume $\operatorname{char} K = p > 0$. Let $F = Q(T_n)$, the quotient field of $T_n = K\langle \zeta_1, \dots, \zeta_n \rangle$. Define the *Gauss norm*

$$\|\ \|_{\mathrm{gs}} \colon F \to \mathbf{R}_+$$

on F the way we did in [Lecture 13, (2.2.1.1)].

Fix an algebraic closure $\overline{F}$ over F. Since every non-zero element of $\overline{F}$ has a minimal polynomial over F, *the spectral norm*

$$\|\ \|_{\mathrm{sp}} \colon \overline{F} \longrightarrow \mathbf{R}_+$$

is well defined and from HW 5, it is a non-archimedean norm on $\overline{F}$.

For each $m \in \mathbf{N}$, define F_m as $F^{p^{-m}}$, i.e.

$$F_m = \{x \in \overline{F} \mid x^{p^m} \in F.\}.$$

We have a tower of fields

$$F = F_0 \subset F_1 \subset \dots \subset F_m \subset \dots \overline{F}.$$

We set

$$F_\infty := \bigcup_m F_m.$$

The field F_∞ is perfect. We have a purely inseparable extension $F \subset F_\infty$, and a separable extension $F_\infty \subset \overline{F}$. We may regard $\overline{K}$ as the subfield of $\overline{F}$ consisting of elements in $\overline{F}$ which are algebraic over K. One can define K_m inside $\overline{K}$ the way we defined F_m inside $\overline{F}$, or equivalently, set

$$K_m = F_m \cap \overline{K}.$$

[1] A map $\lambda\colon M \to A$ is bounded if there exists $C > 0$ such that $\|\lambda(y)\| \le C\|y\|'$, $\forall y \in M$.

In other words $x \in \overline{F}$ lies in K_m if and only if $x^{p^m} \in K$. One again, setting $K_\infty = \cup_m K_m$, one has $K \subset K_\infty$ purely inseparable, K_∞ perfect, and $K_\infty \subset \overline{K}$ separable.

Finally, set

$$T_{n,m} = \{x \in \overline{F} \mid x^{p^m} \in T_n\}.$$

One way to see all this is to consider the Frobenius isomorphism $\Phi_m \colon \overline{F} \xrightarrow{\sim} \overline{F}$, $x \mapsto x^{p^m}$. Then $F_m = \Phi_m^{-1}(F)$, $K_m = \Phi_m^{-1}(K)$, and $T_{n,m} = \Phi_m^{-1}(T_n)$.

(1) (a) Show that K_1 is complete with respect to the norm on $\overline{K}$ which extends uniquely from K.
(b) Show that if V is any normed finite dimensional K_1-vector space, then V is b-separable over K.

(2) Suppose every finite dimensional F-vector subspace of F_1 is b-separable over F. As always, the norm on any subspace of $\overline{F}$ is $\| \, \|_{\mathrm{sp}}$ and that on F is $\| \, \|_{\mathrm{gs}}$. Show that every finite dimensional F-vector subspace of $\overline{F}$ is also b-separable over F. [**Hint:** Let $U = \oplus_\nu F e_\nu \subset \overline{F}$ where the direct sum is a finite direct sum. Let $U' = \sum_\nu F_\infty e_\nu$. Use the fact that F_∞ is perfect and results from Lecture 13 and apply these to U'. Use the tower $F = F_0 \subset \cdots \subset F_m \subset \cdots \subset F_\infty$ to reduce to the case of finite dimensional F-vector subspaces of F_1.]

(3) Let M be a T_n-module with a K-norm $\| \, \|$ such that $(M, \| \, \|)$ *is faithfully normed over* $(T_n, \| \, \|_{\mathrm{gs}})$, i.e., $\|fm\| = \|f\|_{\mathrm{gs}} \|m\|$ for all $f \in T_n$ and $m \in M$. Let $V = M \otimes_{T_n} F$.
(a) Show that M is a torsion free module.
(b) Show that $\| \, \|$ extends from M to V so that $(V, \| \, \|)$ is faithfully normed over $(F, \| \, \|_{\mathrm{gs}})$.
(c) Suppose every finitely generated submodule N of M is b-separable over T_n. Show that every finite dimensional F-subspace U of V is b-separable over F.

(4) Suppose every finitely generated T_n-submodule M of $T_{n,1}$ is b-separable. Here the norm on $T_{n,1}$, and hence on M, is the spectral norm $\| \, \|_{\mathrm{sp}}$, and the norm on T_n is $\| \, \|_{\mathrm{gs}}$. Show that every finite dimensional F-subspace U of $\overline{F}$ is b-separable over F. Here the norm on U is the spectral norm $\| \, \|_{\mathrm{sp}}$ inherited from $\overline{F}$ and that on F the Gauss norm $\| \, \|_{\mathrm{gs}}$.

Double complexes. Recall the definitions from Lecture 3. Let $(D^{\bullet\bullet}, \partial_h, \partial_v)$ be a double complex of modules over a ring R and $T^\bullet = \mathrm{Tot}^\bullet(D)$ its total complex. *Assume for simplicity that $D^{\bullet\bullet}$ is a first quadrant double complex, i.e. its support is bounded below by the x-axis and on the left by the y-axis, these axes being possibly part of the support.* Let H_I^{ij}, $H_{II} H_I^{ij}$, H^{ij}, etc be as in HW 5.

Let $K^\bullet$ be the complex given by $K^p = \ker(D^{p,0} \to D^{p,1}) = \mathrm{H}^0(D^{p,\bullet})$ and

$$\varphi \colon K^\bullet \longrightarrow T^\bullet$$

the natural map of complexes which in degree p is the composite of inclusions $K^p \subset D^{p,0} \subset T^p$.

(5) Let $n \in \mathbf{N}$. Suppose $H^{ij} = 0$ for all (i, j) such that $i + j = n$. Show that $\mathrm{H}^n(T^\bullet) = 0$.

(6) (a) Show that $H^{i0} = \mathrm{H}^i(K^\bullet)$ for all i.
(b) Suppose $H^{ij} = 0$ for (i, j) such that $j \geq 1$ and $n-1 \leq i+j \leq n$. Show that $H^{n0} = \mathrm{H}^n(K^\bullet) \xrightarrow{\mathrm{H}^n(\varphi)} \mathrm{H}^n(T^\bullet)$ is an isomorphism.