HW 5

Due on October 15, 2019 (in class).

As before $\mathbf{N} = \{0, 1, 2, \ldots, m, \ldots\}$. Also $\mathbf{R}_+ := [0, \infty)$.

Spectral norms. Let $|\ |$ be a non-trivial non-archimedean absolute value on a field F. We do not assume $|\ |$ is complete.

(1) Let $F \to E$ be a finite normal field extension (not necessarily separable). Let $|\ |'$ be any absolute value on E extending $|\ |$. By Problem (3) of HW 1, we know there exists at least one such absolute value. For $x \in E$, define

$$\|x\| = \max_{g \in \mathrm{Gal}(E/F)} |gx|'.$$

Note that $\|\ \|$ need not be multiplicative, i.e. it may not define an absolute value on E. Nevertheless it defines a non-archimedean F-vector space norm, being a maximum.

(a) Show that $\|\ \|$ is a power multiplicative norm.

(b) If, as in Lecture 13, $\|\ \|_{\mathrm{sp}}$ denotes the spectral norm on E for the extension $F \to E$, with the norm on F being $|\ |$, show that $\|\ \| = \|\ \|_{\mathrm{sp}}$.

(2) Let $(F, |\ |)$ be as above. Let Q be a finite extension of F. Show that the spectral norm $\|\ \|_{\mathrm{sp}}$ on Q with respect to F is an F-vector space norm.

Double complexes. Recall the definitions from Lecture 3. Let $(D^{\bullet\bullet}, \partial_h, \partial_v)$ be a double complex of modules over a ring R and $T^\bullet = \mathrm{Tot}^\bullet(D)$ its total complex. Define

$$H_I^{ij} = \frac{\ker \partial_v^{ij}}{\mathrm{im}\, \partial_v^{i,j-1}}.$$

The morphisms ∂_h^{ij} induce maps $\delta_h^{ij} \colon H_I^{ij} \to H_I^{i+1,j}$, and it is easy to see that $\delta_h^{i+1,j} \circ \delta_h^{ij} = 0$. Set

$$H_{II} H_I^{ij} = \frac{\ker \delta_h^{ij}}{\mathrm{im}\, \delta_h^{i-1,j}}.$$

Next define Z^{ij} to be the sub-module of D^{ij} consisting of elements x_{ij} such that

$$(A) \qquad \partial_v x^{ij} = 0 \quad \text{and} \quad d_h x_{ij} = d_v x_{i+1,j-1}$$

for some element $x_{i+1,j-1} \in D^{i+1,j-1}$. Also define B^{ij} to be the submodule of D^{ij} consisting of all elements x_{ij} such that

$$(B) \qquad x_{ij} = \partial_v x_{i,j-1} + \partial_h x_{i-1,j} \quad \text{where} \quad \partial_v x_{i-1,j} = 0.$$

for some $x_{i,j-1} \in D^{i,j-1}$ and $x_{i-1,j} \in D^{i-1,j}$.

In the problems below, *assume for simplicity that $D^{\bullet\bullet}$ is a first quadrant double complex, i.e. its support is bounded below by the x-axis and on the left by the y-axis, these axes being possibly part of the support.*

(3) Show that $B^{ij} \subset Z^{ij}$. Set $H^{ij} = Z^{ij}/B^{ij}$. Show that

$$H^{ij} \xrightarrow{\sim} H_{II} H_I^{ij}.$$

(This does not require boundeness hypotheses on $D^{\bullet\bullet}$.)

(4) Fix $(p,q) \in \mathbf{N} \times \mathbf{N}$ and let $n = p + q$. Suppose $H^{ij} = 0$ for the following (i,j): (a) $i + j = n$, $(i,j) \neq (p,q)$, (b) $j > q$, $i + j = n - 1$, and (c) $j < q$ amd $i + j = n + 1$. Show that when these three conditions are satisfied, there is a natural isomorphism $H^{pq} \xrightarrow{\sim} H^n(\mathrm{Tot}^\bullet(D))$.