HW 4

Due on October 8, 2019 (in class).

As before $\mathbf{N} = \{0, 1, 2, \ldots, m, \ldots\}$. We will use $\mathbf{N}_{>0}$ as the symbol for positive integers. Also $\mathbf{R}_+ := [0, \infty)$.

Seminorms. Look up definitions from HW 3.

(1) Suppose $\| \ \|$ and $\| \ \|'$ are power multiplicative semi-norms on a ring A which are equivalent to each other. Show that they are in fact equal.

(2) Suppose $(A, \| \ \|)$ and $(B, \| \ \|')$ are semi-normed rings, with $\| \ \|'$ power-multiplicative. Let $\varphi\colon A \to B$ be a ring homomorphism which is bounded, i.e. there is a constant $C \geq 0$ such that $\|\varphi(x)\|' \leq C\|x\|$ for every $x \in A$. Show that φ is a *contraction*, i.e.

$$\|\varphi(x)\|' \leq \|x\|$$

for every $x \in A$.

(3) Suppose $(A, \| \ \|)$ is a semi-normed ring. Define $\| \ \|'$ by the formula

$$\|x\|' = \inf_{n \geq 1} \|x^n\|^{\frac{1}{n}} \qquad (x \in A).$$

Show that
(a) $\|x\|' = \lim_{n\to\infty} \|x^n\|^{\frac{1}{n}}$ for every $x \in A$. In other words show that the limit on the right side exists and is equal to the left side.
(b) $\| \ \|'\colon A \to \mathbf{R}_+$ is a power multiplicative semi-norm on A.
(c) $\| \ \|' \leq \| \ \|$.
(d) If $a \in A$ is such that $\|a^n\| = \|a\|^n$, then $\|a\| = \|a\|'$. In particular, if $\| \ \|$ is power multiplicative, then $\| \ \| = \| \ \|'$.
(e) If $a \in A$ is such that $\|ax\| = \|a\|\|x\|$ for all $x \in A$, then $\|ax\|' = \|a\|'\|x\|'$ for all $x \in A$.