HW 3

Due on September 5, 2019 (in class).

As before $\mathbf{N} = \{0, 1, 2, \ldots, m, \ldots\}$. We will use $\mathbf{N}_{>0}$ as the symbol for positive integers. Also $\mathbf{R}_+ := [0, \infty)$.

For a richer theory of analytic spaces, which makes many things easier to handle than Tate's theory does, one needs to work in greater generality. While we won't have time to explore that in detail, these exercises (and some others during this semester) will hopefully make it easier for you to get at the higher theory. Like all good things, the beginnings are elementary, and here are the first steps into that world.

Seminorms. Let G be an abelian group. A *seminorm* on G is a function

$$\| \,\| \colon G \longrightarrow \mathbf{R}_+$$

such that $\|0\| = 0$ and $\|x - y\| \leq \|x\| + \|y\|$ for $x, y \in G$. It is *non-archimedean* if $\|x - y\| \leq \max\{\|x\|, \|y\|\}$. If $\|\cdot\|$ is a seminorm on G then the collection of sets $\{U_\epsilon\}_{\epsilon>0}$ defined for each $\epsilon > 0$ by

$$U_\epsilon = \{x \in G | \, \|x\| < \epsilon\}$$

forms a fundamental system of neighbourhoods at 0 giving rise to a unique topology on G.

A seminorm $\| \,\|$ on G is called a *norm* if $\|x\| = 0$ implies $x = 0$.

Two seminorms $\| \,\|$ and $\| \,\|'$ are said to be *equivalent* if there exist positive real numbers C and C' such that $\|x\| \leq C\|x\|'$ and $\|x\|' \leq C'\|x\|$ for every $x \in G$.

There is an obvious definition of a Cauchy sequence on a seminormed group $(G, \| \,\|)$. The *separated completion* $\widehat{G}$ *of* G is defined as the space of usual equivalence classes of Cauchy sequences. Then the usual theory of completing pseudometric spaces gives us a seminorm on $\widehat{G}$ and a continuous map $\kappa = \kappa_G \colon G \to \widehat{G}$ with $\kappa(G)$ dense in G.

In the problems in this section, we fix an abelian group G and a seminorm $\| \,\|$ on it.

(1) (a) Show that $\widehat{M} = 0$ if and only if $\| \,\| = 0$.
    (b) Show that the seminorm on $\widehat{G}$ is a norm.

(2) Show that the following are equivalent
    (a) The topology on G is Hausdorff.
    (b) $\| \,\|$ is a norm.
    (c) The canonical map $\kappa \colon G \to \widehat{G}$ is injective.
    (d) The map $G \to \kappa(G)$ induced by κ is a homeomorphism.

(3) Show that a seminorm $\| \,\|'$ is equivalent to $\| \,\|$ only if the topology induced by the two seminorms on G are the same.

(4) Let H be a subgroup of G. Define a seminorm on G/H in the usual way, viz., if $\pi\colon G \to G/H$ is the canonical surjective homomorphism, then

$$\|y\| := \inf_{x \in \pi^{-1}(y)} \|x\| \qquad (y \in G/H).$$

(a) Show that $\|\,\|\colon G/H \to \mathbf{R}_+$ is indeed a seminorm. It is called the *residue seminorm* on G/H.

(b) Show that the residue seminorm on G/H is a norm if and only if H is closed in G.

Banach rings. Let A be a ring with 1 (not necessarily commutative!). A *seminorm* on A is a seminorm on $\|\,\|$ on $(A, +)$ such that $\|1\| = 1$, and $\|ab\| \leq \|a\|\|b\|$ for $a, b \in A$.

A seminorm on a ring A is called *power multiplicative* if $\|a^n\| = \|a\|^n$ for $a \in A$ and $n \geq 1$. A seminorm on A is called *multiplicative* if $\|ab\| = \|a\|\|b\|$ for $a, b \in A$.

$(A, \|\,\|)$ is called a *seminormed ring* if $\|\,\|$ is a seminorm on the ring A. It is a *normed ring* if $\|\,\|$ is a norm.

A *Banach ring* is a normed ring A that is complete with respect to its norm.

In the exercises in this section A is a Banach ring with norm $\|\,\|$.

(5) (a) Let $\mathfrak{a}$ be a closed two sided of A. Show that $A/\mathfrak{a}$ is a Banach ring with respect to the residue norm.

(b) Show that if $\mathfrak{a}$ is a maximal two sided ideal of A, it is closed.

(6) Let $\{A_i |\ i \in I\}$ be a finite family of Banach rings indexed by a set I. Show that $\prod A_i$ is a Banach ring with norm given by $\|(a_i)_{i \in I}\| = \sup_{i \in I} \|a_i\|$.

(7) Let A be a division ring. Show that $\|\,\|$ is multiplicative if and only if $\|a^{-1}\| = \|a\|^{-1}$ for every non-zero element a of A.