HW 2

Due on August 29, 2019 (in class).

Double complexes. Please refer to Lecture 3 for various definitions.

(1) Let $\varphi^\bullet\colon K^\bullet \to T^\bullet$ be the map in the Proposition on page 9 of Lecture 3. Let $\widetilde{T}^\bullet$ be the complex in the proof of that Proposition. Show that $\widetilde{T}^\bullet$ is isomorphic as a complex to the mapping cone $C^\bullet_\varphi$ of $\varphi^\bullet$. [Caution: They needn't be equal as complexes.]

Finite Krull dimension of T_n. Let K be a complete non-archimedean field. Let T_n be the n^{th} Tate algebra over K,i.e., $T_n = K < \zeta_1, \ldots, \zeta_n >$. In what follows you may assume T_n is a noetherian UFD and that for $n \geq 1$, if $0 \neq f \in T_n$, then there is a finite monomorphism of rings

$$T_{n-1} \longrightarrow T_n/(f).$$

(2) Show that T_n has finite Krull dimension which equals n.

(3) (Noether normalisation) Let $\mathfrak{a}$ be an ideal in T_n and let $d = \dim T_n/\mathfrak{a}$. Show that there is a monomorphism of K-algebras

$$T_d \hookrightarrow T_n/\mathfrak{a}$$

which is a finite morphism of rings. (We will use the Bourbaki and EGA terminology. A ring himomorphism $A \to B$ is finite if B is a finitely generated as an A-module.)

Formal power series. For a commutative ring A, and m a positive integer, as always, $A[|\xi_1, \ldots, \xi_m|] = A[|\boldsymbol{\xi}|]$ denotes the ring of formal power series in m analytically independent variables $\xi_1, \ldots, \xi_m$. A typical element $f \in A[|\boldsymbol{\xi}|]$ is denoted

$$f = \sum_{\boldsymbol{\nu} \in \mathbf{N^m}} c_{\nu_1 \ldots \nu_m} \xi_1^{\nu_1} \ldots \xi_m^{\nu_m} = \sum_{\boldsymbol{\nu} \in \mathbf{N^m}} c_{\boldsymbol{\nu}} \boldsymbol{\xi}^{\boldsymbol{\nu}}.$$

The *order* $o(f)$ of an element f as above is the smallest integer e such that $c_{\boldsymbol{\nu}} \neq 0$ for some $\boldsymbol{\nu}$ with $|\boldsymbol{\nu}| = e$, where $\boldsymbol{\nu} := \nu_1 + \cdots + \nu_m$. This is a topological ring with a system of fundamental neighbourhoods at 0 given by $\{I^n\}$ where I is the ideal generated by $\xi_1, \ldots, \xi_m$.

(4) Show that formal power series can be "substituted" into other formal power series. In greater detail, let $A[|\xi_1, \ldots, \xi_m|]$ and $A[|\eta_1, \ldots, \eta_n|]$ be two formal power series rings. Let $G_1, \ldots, G_m$ be elements of $A[|\boldsymbol{\eta}|]$ with $o(G_i) \geq 1$, $1 \leq i \leq m$ and let $F \in A[|\boldsymbol{\xi}|]$. Show that $F(G_1, \ldots, G_m)$ makes sense as an element of $A[|\boldsymbol{\eta}|]$.