HW 1

Due on August 20, 2019 (in class).

Some facts about absolute values. Let K be a field with a non-archimedean absolute value $|\cdot|$. This is sometimes called a *normed field*. To avoid annoying trivialities, we will assume that $|\cdot|$ is non-trivial, i.e., there is some element $a \in K$ such that $|a| \notin \{0, 1\}$. Recall that the map $d\colon K \times K \to [0, \infty)$, $(x, y) \mapsto |x - y|$ is a metric on K, and that $(K, |\cdot|)$ is said to be *complete* if (K, d) is a complete metric space. Recall also that a map $v\colon K \to \mathbf{R} \cup \{\infty\}$ is said to be *valuation* if for $a, b \in K$

(i) $v(a) = \infty \iff a = 0$
(ii) $v(ab) = v(a) + v(b)$
(iii) $v(a + b) \geq \min\{v(a), v(b)\}$

In what follows you may assume that

- $v(a) = -\log|a|$ sets up a 1-to-1 correspondence between non-archimedean absolute values and valuations. So we will not make a distinction between them.
- The completion (as a metric space) $\widehat{K}$ of K is a complete normed field, with absolute value the unique extension of the continuous function $|\cdot|$ on K. Note that since K is dense in $\widehat{K}$, there can be at most one extension of the absolute value on K to an absolute value on $\widehat{K}$. Uniform continuity given the rest. Alternately, let Δ be the metric on $\widehat{K}$ extending d (from the theory of metric spaces). Then for $x \in \widehat{K}$, set $|x| = \Delta(0, x)$.
- If K is complete and $K \to L$ is an algebraic extension, then the absolute value on K extends to unique absolute value on L. In particular, if we fix an algebraic closure $\overline{K}$ of K, then $|\cdot|$ extends uniquely to an absolute value on $\overline{K}$.
- If K is complete and L/K is a finite extension, then L is complete with respect to the unique extension of the absolute value on K.
- If K is algebraically closed then so is its completion $\widehat{K}$. This is a deeper result than the ones above, in the sense that it requires more. Hensel's Lemma by itself is not enough. If you are curious, read about Krasner's Lemma and Krasner's Corollary.

Extending absolute values. In the exercises below, you are once again reminded that K is a non-archimedean normed field with a non-trivial norm $|\cdot|$. The set $\mathbf{N}$ is $\{0, 1, \ldots, n, \ldots\}$ (note the inclusion of 0).

(1) Let $a, b \in K$ with $|a| \neq |b|$. Show that $|a + b| = \max\{|a|, |b|\}$.

(2) Show that if L/K is an algebraic extension then there is at least one absolute value on L which extends the absolute value on K. [Hint: Embed L

in the algebraic closure of $\widehat{K}$.]

(3) Let L/K be a finite extension. Show that the number of absolute values on L extending the given absolute value on K is the same as the number of prime ideals in $L \otimes_K \widehat{K}$.

(4) Show that the topology of K is totally disconnected, i.e., any subset of K with more than one point is necessarily disconnected. (Note that this makes notions of path connectedness and fundamental groups meaningless. However we will tease out a meaning for simple connectedness and fundamental groups using a form of Galois Theory.)

Convergent power series. One can have a notion of a convergent infinite series $\sum_{i=0}^{\infty} a_i$ in K in the usual manner, namely the partial sums forma convergent sequence.

(5) A series $\sum_{m=0}^{\infty} a_m$, $(a_m \in K, m \geq 0)$ is said to be *Cauchy* if the sequence of partial sums is Cauchy. Show that $\sum_m a_m$ is Cauchy if and only if $a_m \to 0$ as $m \to \infty$.

(6) Let K be complete and $\overline{K}$ an algebraic closure of K. Let

$$\mathbb{B}^n(\overline{K}) = \{(x_1, \dots, x_n) \in \overline{K}^n \mid |x_i| \leq 1\}.$$

Show that a formal power series

$$f = \sum_{\nu \in \mathbf{N^n}} a_{\nu_1 \dots \nu_n} \zeta_1^{\nu_1} \dots \zeta_n^{\nu_n} \in K[|\zeta_1, \dots, \zeta_n|]$$

is convergent at every point on $\mathbb{B}^n(\overline{K})$ if and only if $\lim_{|\nu| \to \infty} |a_{\nu_1 \dots \nu_n}| = 0$. Here $|\nu| := \nu_1 + \dots + \nu_n$.