MID-TERM EXAM FOR RIGID ANALYTIC SPACES

As always, $\mathbf{N} = \{0, 1, 2, \ldots, m, \ldots\}$. Rings mean commutative rings with 1.

Problems 5, 6, 10, and 11 are compulsory. Do any two others. You may use results from problems you haven't done provided those problems precede the one you are attempting.

Newton-Puiseaux series. For a field k, let $k^{\mathbf{Q}} = \prod_{\mathbf{Q}} k$, i.e. $k^{\mathbf{Q}}$ is the set of of maps from $\mathbf{Q}$ to k. An element $f\colon \mathbf{Q} \to k$ of $k^{\mathbf{Q}}$ is represented as (f_α), where α ranges over $\mathbf{Q}$, when thought of as an element of the product $\prod_{\mathbf{Q}} k$. In other words $f_\alpha = f(\alpha)$. For $f = (f_\alpha) \in k^{\mathbf{Q}}$, define

$$\operatorname{Supp} f = \{\alpha \in \mathbf{Q} \mid f_\alpha \neq 0\}.$$

Next set $P(k)$[1] equal to subset of $k^{\mathbf{Q}}$ consisting elements f such that

(i) $\operatorname{Supp} f \subset \mathbf{Z}\frac{1}{n}$ for some positive integer n (depending upon f),
(ii) $\operatorname{Supp} f$ is bounded below.

If $f \in P(k)$, it can be represented as a series

$$(f_\alpha) = \sum_{\alpha \in \operatorname{Supp} f} f_\alpha t^\alpha.$$

Elements of $P(k)$ are called *Newton-Puiseaux series* or often just *Puiseaux series*, though Newton's contribution is perhaps more significant.

Define $0 \in P(k)$ as the element (f_α) with $f_\alpha = 0$ for all α and $1 \in P(k)$ as the characteristic function $\chi_{\{0\}}$, i.e., the element $\sum_\alpha f_\alpha$ such that

$$f_\alpha = \begin{cases} 1 & \text{if } \alpha = 0, \\ 0 & \text{if } \alpha \neq 0. \end{cases}$$

For $f = \sum_\alpha f_\alpha t^\alpha$ and $g = \sum_\alpha g_\alpha t^\alpha$ in $P(k)$ define $f + g := \sum_\alpha (f_\alpha + g_\alpha) t^\alpha$ and $fg = \sum_\lambda c_\lambda t^\lambda$ where $c_\lambda = \sum_{\alpha+\beta=\lambda} f_\alpha g_\beta$.

Let $k((t)) \subset P(k)$ denote the field of Laurent series, i.e. the set of elements $\sum_{l \geq k} a_l t^l$, where the sum ranges over a bounded below set of integers. It is well-known that $k((t))$ is the field of fractions of the ring of power series $k[[t]]$.

Define the *order function* $\mathfrak{o}\colon P(k) \to \mathbf{Q} \cup \{\infty\}$ as follows; set $\mathfrak{o}(0) = \infty$ and if $f = \sum_\alpha f_\alpha t^\alpha$ is non-zero then $\mathfrak{o}(f)$ is the least rational number in $\operatorname{Supp} f$.

(1) Show that the product in $P(k)$ is well defined. Show also that $(P(k), +, \cdot, 0, 1)$ is a commutative ring.

(2) Show that $P(k)$ is a field. [**Hint:** Show that if $S = \{f_1, \ldots, f_r\}$ is a finite set of elements in $P(k)$, then S is contained in a field isomorphic to $k((t))$].

[1] The "P" in $P(k)$ is for the mathematician Puiseaux.

(3) For $\lambda \in \mathbf{Q} \cap (0, \infty)$ define $\eta_\lambda \colon P(k) \to P(k)$ by $\sum_\alpha f_\alpha t^\alpha \mapsto \sum_\alpha f_\alpha t^{\lambda\alpha}$, i.e. if $g = \eta_\lambda f$ then $g_\alpha = f_{\lambda^{-1}\alpha}$. Show that η_λ is an automorphism and $\mathfrak{o}(\eta_\lambda f) = \lambda \mathfrak{o}(f)$.

(4) Show that for $0 \neq f \in P(k)$, $\mathfrak{o}(f)$ is a well defined rational number. Show also that the map $|\,|\colon P(k) \to \mathbf{R}_+$ given by $|f| = 2^{-\mathfrak{o}(f)}$, defines a non-archimedean absolute value on $P(k)$ (with the understanding that $2^{-\infty} = 0$).

(5) Let $|\,|$ be as above, i.e. $|f| = 2^{-\mathfrak{o}(f)}$ for $f \in P(k)$.
   (a) Show that $|\,|$ when restricted to $k((t))$ is complete and that $\mathscr{O}_{k((t))} = k[[t]]$.

   (b) Let
   $$g = \zeta^n + f_1\zeta^{n-1} + \cdots + f_{n-1}\zeta + f_n$$
   be a monic irreducible polynomial in $k[[t]][\zeta]$ (note that $k[[t]][\zeta]$ is a UFD). Show that
   $$\bar{g} := \zeta^n + f_1(0)\zeta^{n-1} + \cdots + f_{n-1}(0)\zeta + f_n(0) \in k[\zeta]$$
   is a power of an irreducible polynomial. (This is a version of Hensel's Lemma, and you will need it below.)

(6) In the following problems, k is algebraically closed of characteristic zero

   (a) Suppose $g \in k[[t]][\zeta]$ is monic, of degree $n > 1$, say $g = \zeta^n + f_1\zeta^{n-1} + \cdots + f_{n-1}\zeta + f_n$, with $f_i \in k[[t]]$, such that $f_1 = 0$ and such that $f_i(0) \neq 0$ for at least one i. Show that g has a factor of degree m where $1 \leq m < n$. [**Hint:** Use Hensel's Lemma.]

   (b) Suppose $g \in k((t))[\zeta]$ is monic, of degree $n > 1$, say $g = \zeta^n + f_1\zeta^{n-1} + \cdots + f_{n-1}\zeta + f_n$, where $f_i \in k((t))$. Find an automorphism $\sigma = \sigma_g$ of $k((t))[\zeta]$ such that the coefficient of ζ^{n-1} in $\sigma(g)$ is zero.

   (c) Show that $P(k)$ is algebraically closed. [**Hint:** You might need to use Problem (3) to reduce to situations considered in parts (a) and (b).]

   (d) Is $P(k)$ algebraic over $k((t))$? Why or why not?

Banach rings. We use the definitions in HW 3. Norms need not be non-archimedean any more. These exercises are part of an alternative, and very successful, approach to the subject, which unfortunately we cannot explore too much in this course. Let $(\mathscr{A}, \|\,\|)$ be a commutative Banach ring with identity. A seminorm $|\,|$ on $\mathscr{A}$ is said to be *bounded* if there exists $C > 0$ such that $|f| \leq C\|f\|$ for all $f \in \mathscr{A}$. The *spectrum* $\mathscr{M}(\mathscr{A})$ is the set of all bounded multiplicative seminorms on $\mathscr{A}$.[2] For each $f \in \mathscr{A}$ we have a map

$$\Psi_f \colon \mathscr{M}(\mathscr{A}) \longrightarrow \mathbf{R}_+$$

[2] Note that we are considering seminorms and not just norms. On the other hand, these seminorms are multiplicative.

given by $|\ | \mapsto |f|$. Endow $\mathscr{M}(\mathscr{A})$ with the weakest topology such that each Ψ_f is continuous as f varies over $\mathscr{A}$. We would like to think of $\mathscr{M}(\mathscr{A})$ as a set of points and so we play the usual notational trick that we are familiar with from algebraic geometry as well as this course. Namely, if $x \in \mathscr{M}(\mathscr{A})$ and we wish to regard x as a seminorm on $\mathscr{A}$, then we write $|\ |_x$ for x.

(7) For $r > 0$, let $\mathscr{A}\langle r^{-1}T\rangle$ denote the set of power series $f = \sum_{i\in\mathbf{N}} a_i T^i$ such that $\sum_{i\in\mathbf{N}} \|a_i\| r^i < \infty$. Show that $\mathscr{A}\langle r^{-1}T\rangle$ is a Banach ring with norm $\|f\|_{r,T} = \sum_{i\in\mathbf{N}} \|a_i\| r^i < \infty$. Show also that an element $1 - aT$, $a \in \mathscr{A}$, is invertible in $\mathscr{A}\langle r^{-1}T\rangle$ if and only if $\sum_{i\in\mathbf{N}} \|a^i\| r^i < \infty$.

(8) If a bounded seminorm $|\ |$ on $\mathscr{A}$ is power-multiplicative, show that $|f| \leq \|f\|$ for every $f \in \mathscr{A}$.

(9) Let x be a point in $\mathscr{M}(\mathscr{A})$. Show that the subset $\mathfrak{p}_x$ of $\mathscr{A}$ consisting of all $f \in \mathscr{A}$ such that $|f|_x = 0$ is a closed prime ideal of $\mathscr{A}$ and that the value of $|f|_x$ depends only on the residue class of f in $\mathscr{A}/\mathfrak{p}_x$.

The aim of the next two exercises is to show that $\mathscr{M}(\mathscr{A})$ is non-empty.

(10) Suppose $\mathscr{A}$ is a field.[3] Let S be the set of nonzero bounded seminorms on $\mathscr{A}$. Since $\|\ \| \in S$, S is non-empty. We have an obvious partial order on S, namely $|\ |' \leq |\ |''$ if $|f|' \leq |f|''$ for all $f \in \mathscr{A}$.

(a) Show that S is non-empty and has a minimal element.

(b) Replacing $\mathscr{A}$ by its completion by a minimal seminorm we may assume $\mathscr{A}$ is complete and the given norm $\|\ \|$ on $\mathscr{A}$ is minimal in S. Do so. Let $0 \neq f \in \mathscr{A}$ and pick a real number r such that $0 < r < \|f\|$. Show that $f - T$ is invertible in $\mathscr{A}\langle r^{-1}T\rangle$ by showing that $\|\varphi(f)\|' < \|f\|$ if $f - T$ is not invertible, where $\varphi\colon \mathscr{A} \to \mathscr{A}\langle r^{-1}T\rangle/(f-T)$ is the natural map, and $\|\ \|'$ is the residue norm on $\mathscr{A}\langle r^{-1}T\rangle/(f - T)$. Why is this a contradiction?

(c) Assume that the given norm $\|\ \|$ is a minimal element in S. Suppose there is an $f \in \mathscr{A}$ is such that $\|f^n\| < \|f\|^n$ for some $n > 1$. Show that $f - T$ is not invertible in $\mathscr{A}\langle r^{-1}T\rangle$ for $r = \sqrt[n]{f}$. Similarly, show that if a non zero element $f \in \mathscr{A}$ is such that $\|f\|^{-1} < \|f^{-1}\|$ then $f - T$ is not invertible in $\mathscr{A}\langle r^{-1}T\rangle$ for $r = \|f^{-1}\|^{-1}$. Conclude that $\|\ \|$ is multiplicative. [**Hint:** Use Problem (7) for proving the assertions about $f - T$. You will have to appeal to a result from your homework problems to prove the multiplicativity assertion.]

(11) Show that $\mathscr{M}(\mathscr{A})$ is non-empty.

[3] Note that all seminorms are norms on $\mathscr{A}$.

(12) Show that $\mathscr{M}(\mathscr{A})$ is Hausdorff. (In fact it is compact too, but we leave that for some other time.)