Nurture 1996-2000
Lectures on Measure Theory and Integration

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Lecture 1

1. Sigma algebras and Measures

The Riemann integral has been seen to be the limit of sums. The Riemann-Stieltjes integral with respect to an increasing function g can be seen as the limit of weighted sums, where the "weight" given to a subinterval (c, d) is roughly $g(b)-g(a)$ (this is strictly true if c and d are points of continuity of g). Once we allow ourselves the notion of weighted sums and their limit, the most natural integral one can think of is integral with respect to a measure. This leads to an abstract defintion of an integral $\int_X f\, d\mu$, where X is a fairly arbitrary space, and μ is the "weight" function, or a measure. However to lay a proper foundation, one has to introduce the idea of a σ-algebra. In this lecture, we define σ-algebras and measures and give examples. We also introduce a class of functions, the "measurable" functions, which are the functions we will attempt to integrate in later lectures.

Definition 1.1. Let X be a set. A subset $\mathcal{F}$ of 2^X is called a *σ-algebra on X* if

(1) $\emptyset, X \in \mathcal{F}$,
(2) $\mathcal{F}$ is closed under complementation, i.e. if $E \in \mathcal{F}$ then $X \setminus E \in \mathcal{F}$, and
(3) $\mathcal{F}$ is closed under countable unions, i.e. if $\{E_j\}_{j=1}^{\infty}$ is a sequence of members of $\mathcal{F}$, then $\cup_{j=1}^{\infty} E_j \in \mathcal{F}$.

A *measurable space* is a a pair $(X, \mathcal{F})$ where X is a set and $\mathcal{F}$ is a σ-algebra on X. Members of $\mathcal{F}$ are called *measurable sets*.

Using D'Morgan's Laws one checks that a σ-algebra is closed under countable intersection. Note that 2^X is itself a σ-algebra. Moreover, the *arbitrary intersection* of σ-algebras on X is again a σ-algebra on X. From this it follows that given a class $\mathcal{H} \subset 2^X$, there is a smallest σ-algebra $\mathcal{F}$ on X containing $\mathcal{H}$. $\mathcal{F}$ is the "smallest" in the sense that if $\mathcal{G} \supset \mathcal{H}$, and $\mathcal{G}$ is a σ-algebra, then $\mathcal{G} \supset \mathcal{F}$. We often say that $\mathcal{F}$ is the σ-algebra *generated* by $\mathcal{H}$.

Example 1.1. The following are some examples of σ-algebras other than 2^X.

(i) Let $\mathcal{B}$ be the σ-algebra generated by the class $\mathcal{H}$ of open sets in $\mathbb{R}$. $\mathcal{B}$ is called the *Borel* σ-algebra on $\mathbb{R}$. We will show, at some point, that $\mathcal{B}$ is not $2^{\mathbb{R}}$.
(iii) Let $\bar{\mathbb{R}}$ be the *extended real line*, i.e. $\mathbb{R} \cup \{-\infty, +\infty\}$. On $\bar{\mathbb{R}}$ define a metric given by $d(x, y) = |\arctan(x) - \arctan(y)|$. Here we use the convention that $\arctan(-\infty) = -\pi/2$ and $\arctan(+\infty) = \pi/2$. Now as a set, $\mathbb{R}$ is a subset of $\bar{\mathbb{R}}$, and hence inherits this metrics from $\bar{\mathbb{R}}$. Note that the open sets on $\mathbb{R}$ induced by this metric agree with the open sets in $\mathbb{R}$ under the usual metric. Let $\bar{\mathcal{B}}$ be the sigma algebra generated by open subsets of $\bar{\mathbb{R}}$.
(iii) Let X be any set. Let $\mathcal{F} = \{E |\, E \text{ is countable, or } X \setminus E \text{ is countable}\}$. If X is not countable, then $\mathcal{F} \neq 2^X$.

DEFINITION 1.2. Let $(X, \mathcal{F})$ be a measurable space. A *measure* on $(X, \mathcal{F})$ is a function

$$\mu : \mathcal{F} \longrightarrow [0, \infty]$$

such that

(i) $\mu(\emptyset) = 0$, and
(ii) (Countable Additivity) If $\{E_i\}$ is a countable collection of disjoint subsets of X, with each $E_i \in \mathcal{F}$, then

$$\mu\left(\cup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \mu(E_i).$$

A *measure space* is a triple $(X, \mathcal{F}, \mu)$ where $(X, \mathcal{F})$ is a measurable space and μ is a measure on $(X, \mathcal{F})$.

REMARK 1.1. Note that if $A \subset B$, $A, B \in \mathcal{F}$ then $\mu(B \setminus A) = \mu(B) - \mu(A)$. In particular $\mu(A) \leq \mu(B)$. The second property is often referred to as the *monotonicity* of μ.

EXAMPLE 1.2. The following are some examples of measure spaces.

(i) (The Lebesgue Measure on $\mathbb{R}$) On $\mathcal{B}$ we can define a unique measure m such that for every interval I, $m(I) =$ length of I. Exercises 3—15 of your Homework assigment gives the method of obtaining m. In fact, from the Exercises, one sees that m extends as a measure to a larger σ-algebra $\mathcal{M}$, and this extended measure is also denoted m. The measure m is called the *Lebesgue measure on* $\mathbb{R}$ and $\mathcal{M}$ the *Lebesgue σ-algebra*. The measure space $(\mathbb{R}, \mathcal{M}, m)$ is called the Lebesgue measure space.
(ii) The above has a natural generalisation to d-dimensions. One can talk about the Lebesgue σ-algebra $\mathcal{M}_d$ and the Lebesgue measure m_d on $\mathbb{R}^d$ (see Exercises 55—60 of your Homework assignment).
(iii) Let X be a non-empty set. $(X, \{\emptyset, X\}, \mu)$ is a measure space, where $\mu(\emptyset) = 0$ and $\mu(X) = \infty$.
(iv) (The Dirac Measure) Let $(X, \mathcal{F})$ be a measurable space, and let $x_o \in X$. The *Dirac measure at x_o* is the measure δ_{x_o} given by :

$$\delta_{x_o} = \begin{cases} 1 & \text{if } x_o \in E \\ 0 & \text{if } x_o \notin E \end{cases}$$

for $E \in \mathcal{F}$. Later, when we do integration with respect to a measure, you will see that

$$\int_X f \, d\delta_{x_o} = f(x_o).$$

(v) (The Counting Measure) Let (X, cf) be a measurable space. The *counting measure* on $(X, \mathcal{F})$ is the measure $\#_X$ given by

$$\#_X(E) = \begin{cases} \infty & \text{if } E \text{ is not a finite set} \\ \text{cardinality of } E & \text{if } E \text{ id finite} \end{cases}$$

for $E \in \mathcal{F}$.

Let X be a set and $\{E_n\}$ a sequence of subsets of X. We will use the notation $E_n \uparrow E$ to mean that $\{E_n\}$ is *increasing*, i.e. $E_n \subset E_{n+1}$ for all $n \in \mathbb{N}$, and that it *increases to* E, i.e. $E = \cup_n E_n$. Similarly $E_n \downarrow E$ means that $E_n \supset E_{n+1}$ and that $E = \cap_n E_n$, and of course, in this instance we say that $\{E_n\}$ is *decreasing* and *decreases to* E.

For the rest of this lecture, let us fix a measure space $(X, \mathcal{F}, \mu)$.

THEOREM 1.1. *Let $E_n \in \mathcal{F}$ and $E_n \uparrow E$. Then $\mu(E_n) \uparrow \mu(E)$.*

PROOF. We have

$$E = \cup_{n\geq 1} (E_n \setminus E_{n-1})$$

where we define $E_o = \emptyset$. The right side is a disjoint union, and hence

$$\begin{aligned}\mu(E) &= \sum_{j=1}^{\infty} \mu(E_j \setminus E_{j-1}) \\ &= \lim_{n\to\infty} \sum_{j=1}^{n} \mu(E_j \setminus E_{j-1}) \\ &= \lim_{n\to\infty} \mu(E_n)\end{aligned}$$

□

EXAMPLE 1.3. Consider the measure space $(\mathbb{N}, 2^{\mathbb{N}}, \#)$ where $\#$ is the counting measure on $(\mathbb{N}, 2^{\mathbb{N}})$. Let $E_n = \{n, n+1, \dots\}$. Then clearly $E_n \downarrow \emptyset$. However, $\#(E_n)$ does not decrease to zero. So the analogue of the above theorem is not true for a decreasing sequence of sets. The difficulty is that $\#(E_n) = \infty$ for all n. The next theorem shows that the analogue for decreasing sequences is true with further hypotheses.

THEOREM 1.2. *Let $E_n \downarrow E$, $E_n \in \mathcal{F}$ and $\mu(E_1) < \infty$. Then $\mu(E_n) \downarrow \mu(E)$.*

PROOF. Let $A_i = E_1 \setminus E_i$. Then $A_i \uparrow E_1 \setminus E$. Apply the previous theorem. □

2. Measurable Maps

DEFINITION 2.1. Let $(X, \mathcal{F})$ and $(Y, \mathcal{G})$ be measurable spaces. A map $f : X \longrightarrow Y$ is said to be *measurable* (or more precisely, *measurable with respect to* $(\mathcal{F}, \mathcal{G})$) if $f^{-1}(E) \in \mathcal{F}$ for every $E \in \mathcal{G}$. We will often use the notation $f : (X, \mathcal{F}) \to (Y, \mathcal{G})$ to indicate that f is a measurable map.

PROPOSITION 2.1. *Let $(X, \mathcal{F}) \xrightarrow{f} (Y, \mathcal{G}) \xrightarrow{g} (Z, \mathcal{H})$ be a pair of measurable maps. Then $g \circ f : X \to Z$ is measurable.*

PROOF. Obvious. □

PROPOSITION 2.2. *Let $f : X \to Y$ be a map of sets, and suppose $\mathcal{G}$ is a σ-algebra on Y. Then the class $f^{-1}(\mathcal{G}) \subset 2^X$ given by*

$$f^{-1}(\mathcal{G}) := \{f^{-1}(B) | \, B \in \mathcal{G}\}$$

is also a σ-algebra.

PROOF. This follows from the fact that the map

$$f^{-1} : 2^Y \to 2^X$$

is a map which preserves arbitrary unions, arbitrary intersections and complementation. □

THEOREM 2.1. *Let $f : X \to Y$ be a map of sets. Let $\mathcal{Q} \subset 2^Y$, and let $\mathcal{G}$ be the σ-algebra generated by $\mathcal{Q}$. Let $\mathcal{P} = \{f^{-1}(B) | \, B \in \mathcal{Q}\}$. Then the σ-algebra generated by $\mathcal{P}$ is $f^{-1}(\mathcal{G})$.*

Proof. Let $\mathcal{F}$ = the σ-algebra generated by $\mathcal{P}$. Since $f^{-1}(\mathcal{G}) \supset \mathcal{P}$ and $f^{-1}(\mathcal{G})$ is a σ-algebra, therefore $f^{-1}(\mathcal{G}) \supset \mathcal{F}$. On the other hand, let

$$\mathcal{G}' = \{B \in \mathcal{G} \mid f^{-1}(B) \in \mathcal{F}\}.$$

It is easy to check that $\mathcal{G}'$ is a σ-algebra. Clearly $\mathcal{Q} \subset \mathcal{G}' \subset \mathcal{G}$. Hence $\mathcal{G}' = \mathcal{G}$ (for $\mathcal{G}$ is the smallest σ-algebra containing $\mathcal{Q}$). It follows that $f^{-1}(B) \in \mathcal{F}$ for every $B \in \mathcal{G}$. This means that $f^{-1}(\mathcal{G}) \subset \mathcal{F}$. □

3. Measurable Functions

Definition 3.1. For this course, a *function* on a set X is a map $f : X \to \bar{\mathbb{R}}$. If $(X, \mathcal{F})$ is a measurable space, then an $\mathcal{F}$ *measurable function* on X is a function which is $(\mathcal{F}, \bar{\mathbb{R}})$-measurable. A *function on* $(X, \mathcal{F})$ is the same as a $\mathcal{F}$-measurable function on X. If the context is clear, we will drop the adjective $\mathcal{F}$ from the phrase $\mathcal{F}$-measurable.

Let $\mathcal{F}$ be the σ-algebra on $\mathbb{R}$ generated by sets of the form (α, ∞). Let $\alpha, \beta \in \mathbb{R}$. Then,

(a) $(-\infty, \beta] \in \mathcal{F}$.
(b) Hence $(\alpha, \beta] \in \mathcal{F}$.
(c) $(\alpha, \beta) = \cup_n (\alpha, \beta + 1/n] \in \mathcal{F}$.
(d) The above is true for $\alpha = -\infty$ also (same proof).
(e) $\{\beta\} = \cap_n (\beta - 1/n, \beta] \in \mathcal{F}$.
(f) Every open interval, and hence every open set is in $\mathcal{F}$, since every open set is a countable union of open intervals.
(g) Hence $\mathcal{F} = \mathcal{B}$.

In the same way, one can show that $\bar{\mathcal{B}}$ is the σ-algebra generated by sets of the form $(\alpha, \infty]$ where $\alpha \in \mathbb{R}$. We thus have

Theorem 3.1. *Let $(X, \mathcal{F})$ be a measurable space. Then a function f on X is measurable if and only if*

$$\{x \in X \mid f(x) > \alpha\} \in \mathcal{F}$$

for every $\alpha \in \mathbb{R}$.

Remark 3.1. Clearly there are other equivalent tests of measurabilty. For example, $f : X \to \bar{\mathbb{R}}$ is measurable if and only if each set of the form $\{f \geq a\}$ is measurable in X. Indeed, sets of the form $\{x \in \bar{\mathbb{R}} \mid x \geq a\}$ generate $\bar{\mathcal{B}}$ —as can be easily verified from arguments given earlier.

Proof. The Theorem follows from the observations above it and Theorem 2.1. □

We will use the following conventions.

$$\begin{aligned} a \cdot \infty &= \infty \qquad \text{for } a > 0,\, a \in \bar{\mathbb{R}} \\ 0 \cdot \infty &= 0 \\ \infty - \infty &= \text{undefined} \end{aligned}$$

Theorem 3.2. *Suppose f, g are functions on $(X, \mathcal{F})$. Then $f + g$, $f - g$, $f \cdot g$ are all measurable, whenever they are defined.*

PROOF. Note that

$$\{f+g>a\} = \cup_{r\in\mathbb{Q}}\left[\{f>r\}\cap\{g>a-r\}\right]$$

and hence $f+g$ is measurable. It follows that $f-g$ is measurable (why ?). Next

$$\{f^2>a\} = \{f>\sqrt{a}\}\cup\{f<-\sqrt{a}\}$$

and hence f^2 is measurable if f is. Since

$$f\cdot g = \frac{1}{4}\left\{(f+g)^2-(f-g)^2\right\}$$

we are done. Note that the proof (of measurability of fg works even if $f+g$ or $f-g$ are not defined at every point (check this !). □

EXAMPLE 3.1. In the examples that follow, we will abuse notation and treat $\mathcal{B}$ and $\mathcal{M}$ as σ-algebras on any interval (or for that matter, any Lebesgue measurable set) in $\mathbb{R}$. We will assume familiarity with the notion of Lebesgue measurability (see Exercises 3—15 of your Homework assignment).

(1) Any continous function on an interval is $\mathcal{B}$-measurable.
(2) Monotone functions on an interval are $\mathcal{B}$-measurable.
(3) Bounded Variation functions on closed bounded intervals are $\mathcal{B}$-measurable.
(4) Let $f=\chi_{\mathbb{Q}}:\mathbb{R}\to\mathbb{R}$. Then f is $\mathcal{B}$-measurable. Indeed,

$$\{f>a\} = \begin{cases}\emptyset & \text{if } a\geq 1\\ \mathbb{Q} & \text{if } 0\leq a<1\\ \mathbb{R} & \text{if } a<0\end{cases}$$

4. Signed Measures and Complex Measures

Sometimes it is useful to have a more general notion of measures. These are particularly useful in understanding the various Riesz Representation Theorems, which we will do in Lecture 8.

DEFINITION 4.1. By a *signed measure* on the measurable space $(X, \mathcal{F})$ we mean an extended real valued set function

$$\mu:\mathcal{F}\to\bar{\mathbb{R}}$$

satisfying the following conditions

(i) μ assumes at most one of the values $+\infty, -\infty$.
(ii) $\mu(\emptyset)=0$.
(iii) $\mu(\cup_{i=1}^{\infty}E_i) = \sum_{i=1}^{\infty}\mu(E_i)$ for any sequence $\{E_i\}$ of disjoint measurable sets, the equality taken to mean that the series on the right converges absolutely if $\mu(\cup_i E_i)$ is finite and that it properly diverges to $\mu(\cup_i E_i)$ otherwise.

DEFINITION 4.2. By a *complex measure* on the measurable space $(X, \mathcal{F})$ we mean a complex valued function $\mu:\mathcal{F}\to\mathbb{C}$ such that

(i) $\mu(\emptyset)=0$.
(ii) For each countable disjoint union $\cup_i E_i$ of sets in $\mathcal{F}$ we have

$$\mu(\cup_i E_i) = \sum_{i=1}^{\infty}\mu(E_i)$$

with absolute convergence on the right.

Lecture 2

In this Lecture we prove two crucial results viz., Egoroff's Theorem and Lusin's Theorem. But first we continue our discussion of measurability.

5. Measurability Continued

We begin with the following observations.

(i) Let $(X,\, \mathcal{F})$ be a measurable space. Then $E \in \mathcal{F}$ if and only if χ_E is measurable.
(ii) Consider the measurable space $(X,\, \{\emptyset, X\})$, where X is a non-empty set. Then a function on X is measurable if and only if it is a constant.
(iii) Consider the measurable space $(X,\, 2^X)$. Clearly every function on X is measurable.

EXAMPLE 5.1. We give an example of a continuous *increasing* function on $I_o = [0,\, 1]$ which sends a set of measure zero to a set of measure one ! Note that such a function must be a homemorphism between I_o and its range. The example is meant to illustrate the difficulty in resolving topology with measure. A "small" set can become a "big" set under a homeomorphism. Consider the Cantor function on $I_o = [0,\, 1]$ defined as follows: In the construction of the Cantor set, suppose $I_{1j}, \dots, I_{2^{j-1},j}$ are the "middle thirds" after $(j-1)$-steps. In other words, I_{11} is the middle third of I_o, and I_{12}, I_{22} are the middle thirds of $I_o \setminus I_{11}^o$, etc, etc, $\dots$. Let $J_o = \cup I_{ij}^o$. In other words $J_o = I_o \setminus C$ where C is the Cantor set. Define

$$f(x) = \begin{cases} 0 & \text{if } x = 0 \\ \dfrac{2i-1}{2^j} & \text{if } x \in I_{ij} \\ \sup_{y<x,\, y\in\cup I_{ij}} f(y) & \text{if } x > 0 \text{ and } x \notin J_o \end{cases}$$

One checks that f is non-decreasing and continuous on I_o and its image is I_o. Now consider $\varphi : I_o \to [0,\, 2]$ where

$$\varphi(x) = f(x) + x \qquad \text{for } x \in I_o.$$

Then φ is an homeomorphism from I_o to $[0, 2]$. One checks that $m\,(\varphi(C)) = 1$.

THEOREM 5.1. *Let $(X,\, \mathcal{F})$ be a measurable space and $\{f_n\}$ a sequence of measurable functions on X. Then $h = \sup_n f_n$ and $g = \inf_n f_n$ are also measurable.*

PROOF. We have

$$\{h > a\} = \cup_{n=1}^{\infty} \{f_n > a\}$$

and hence h is measurable. Similarly, g is measurable. □

COROLLARY 5.1. $\liminf f_n$ *and* $\limsup f_n$ *are measurable.*

PROOF. $\liminf f_n = \sup_n \inf\{f_n,\, f_{n+1}, \dots\}$. A similar description is there for $\limsup f_n$. □

COROLLARY 5.2. *If $f_n \to f$ pointwise, then f is measurable.*

6. Simple Functions

Definition 6.1. Let $(X, \mathcal{F})$ be a measurable space. A function $S : X \to \mathbb{R}$ is *simple* if

(i) S is measurable, and
(ii) $S(X)$ is a finite set.

Let S be a simple function on X, and say $S(X) = \{\alpha_1, \dots, \alpha_n\}$. Let $A_j = \{S = \alpha_j\} \in \mathcal{F}$. Then X is the disjoint union of the A_j's. Note that

$$S = \sum_{j=1}^{n} \alpha_j \chi_{A_j}.$$

Theorem 6.1. *Let $f \geq 0$ be a function on the measurable space $(X, \mathcal{F})$. Then there exists a sequence of simple functions $\{S_n\}$ on $(X, \mathcal{F})$ such that $S_n \uparrow f$ pointwise.*

Proof. Let $I_{nj} = [\frac{j-1}{2^n}, \frac{j}{2^n})$, $j = 1, 2, \dots, n2^n$. Let

$$S_n = \sum_{j=1}^{n2^n} \frac{j-1}{2^n} \chi_{f^{-1}(I_{nj})} + n\chi_{f^{-1}[n, \infty]}, \qquad n \in \mathbb{N}.$$

One can check easily that $S_n \uparrow f$ pointwise. □

7. The Theorems of Egoroff and Lusin

Theorem 7.1. [Egoroff] *Let $(X, \mathcal{F}, \mu)$ be a measure space with $\mu(X) < \infty$. Suppose $\{f_n\}$ is a sequence of measurable functions on X such that $f_n \to f$ pointwise on X. Then given $\epsilon > 0$, there exists $E \in \mathcal{F}$ such that $\mu(X \setminus E) \leq \epsilon$ and $f_n \to f$ uniformly on E.*

Proof. Let $A_{N,j} = \{|f_n - f| \leq 1/j, n \geq N\}$. Then, for *fixed* j, $\{A_{N,j}\}_N$ is an increasing sequence of measurable sets, increasing to X. Therefore, $\mu(A_{N,j}) \uparrow \mu(X)$ as $N \uparrow \infty$. Since $\mu(X) < \infty$ therefore there exists an $N_j \in \mathbb{N}$ such that

$$\mu\left(X \setminus A_{N_j,j}\right) \leq \frac{\epsilon}{2^j}.$$

Let $E = \cap_{j=1}^{\infty} A_{N_j,j}$. Then,

$$\begin{aligned} \mu(X \setminus E) &= \mu\left(\cup_j \left(X \setminus A_{N_j,j}\right)\right) \\ &\leq \sum_j \mu\left(X \setminus A_{N_j,j}\right) \\ &\leq \epsilon. \end{aligned}$$

We claim that $f_n \to f$ uniformly on E. To see this, let $\eta > 0$ be given. Let $j_o \in \mathbb{N}$ be such that $1/j_o \leq \eta$. Since $E \subset A_{N_{j_o},j_o}$, one sees that

$$|f_n(x) - f(x)| \leq \eta \qquad (x \in E; n \geq N_{j_o}).$$

This gives uniform convergence of $\{f_n\}$ on E. □

Theorem 7.2. *Consider the Lebesgue measure space $(\mathbb{R}, \mathcal{M}, m)$. Let $E \in \mathcal{M}$ be a set of finite Lebesgue measure. Let f be a measurable real-function on E. Then for every $\epsilon > 0$, there exists a continuous function*

$$g_\epsilon : \mathbb{R} \to \mathbb{R}$$

such that

$$m\left(\{x \in E|\, g_\epsilon(x) \neq f(x)\}\right) \leq \epsilon.$$

PROOF. We may assume $f \geq 0$ on E by breaking up f as the difference of $f^+ = \max\{f, 0\}$ and $f^- = \max\{-f, 0\}$. Then we have sequence of non-negative simple functions on E, $\{S_n\}$ increasing to f on E. Suppose

$$S_j = \sum_{i=1}^{k_j} \alpha_{ji}\chi_{A_{ji}}.$$

Then, by Exercise 12 (d) of your homework assignment, we can approximate each A_{ji} by closed F_{ji}, sop that

$$m\left(A_{ji} \setminus F_{ji}\right) \leq \frac{\epsilon}{k_j 2^{j+1}}.$$

Let $F_j = \cup_{i=1}^{k_j} F_{ij}$. Then F_j is a closed subset of F and

$$m(E \setminus F_j) \leq \frac{\epsilon}{2^{j+1}}.$$

Note that $S_j|_{F_j}$ is continuous on F_j. Let $F = \cap_j F_j$. Then

$$m(E \setminus F) \leq \sum_j m(E \setminus F_j) \leq \frac{\epsilon}{2}$$

and $S_j|_F$ is continuous for all j.

By Egoroff's Theorem (applied to S_j and the set F), and by Exercise 12 (d) of the Homework assignment, there is a closed set $F_o \subset F$ such that S_j converges uniformly to f on F_o and $m(E \setminus F_o) \leq \epsilon$. Since each S_j is continuous on F_o, therefore f is continuous on F_o. We can extend $f|_{F_o}$ to a continuous g_ϵ on $\mathbb{R}$, and this g_ϵ clearly does the required job. □

Lecture 3

8. Integration of non-negative functions

Let $(X, \mathcal{F}, \mu)$ be a measure space. Then any function $S = \sum_{i=1}^{m} \beta_i \chi_{B_i}$ where $\beta_i \in \mathbb{R}$, $i = 1, \dots, m$ and the B_i are mutually disjoint, and $B_i \in \mathcal{F}$ for every i, is a simple function (this can be seen easily from the definition of a simple function). There may be many ways in which S could be represented in this manner, however, the *canonical representation* is

$$S = \sum_{j=1}^{n} \alpha_i \chi_{A_j}$$

where $\alpha_1, \dots, \alpha_n$ are the distinct values of S and $A_j = S^{-1}(\alpha_j)$, $j = 1, \dots, n$.

LEMMA 8.1. *Let S be a simple function, say, $\sum_{i=1}^{m} \beta_i \chi_{B_i}$, where B_i are mutually disjoint measurable sets. Then the number*

$$\sum_{i=1}^{m} \beta_i \mu(B_i)$$

does not depend on the particular representation of S as a linear combination of characteristic functions over mutually disjoint measurable sets.

PROOF. Let $\sum_{j=1}^{n} \alpha_j \chi_{A_j}$ be the canonical representation of S. It follows that for each $i = 1, \dots, m$ there is a $\sigma(i) \in \{1, \dots, n\}$ such that $\beta_i = \alpha_{\sigma(i)}$ and $A_j = \cup_{\sigma(i)=j} B_i$. Clearly this is a disjoint union. Now

$$\begin{aligned}
\sum_{i=1}^{m} \beta_i \mu(B_i) &= \sum_{j=1}^{n} \sum_{\sigma(i)=j} \beta_i \mu(B_i) \\
&= \sum_{j=1}^{n} \sum_{\sigma(i)=j} \alpha_{\sigma(i)} \mu(B_i) \\
&= \sum_{j=1}^{n} \sum_{\sigma(i)=j} \alpha_j \mu(B_i) \\
&= \sum_{j=1}^{n} \alpha_j \sum_{\sigma(i)=j} \mu(B_i) \\
&= \sum_{j=1}^{n} \alpha_j \mu(A_j).
\end{aligned}$$

The result follows. □

Definition 8.1. Let $S \geq 0$ be a simple function, say $\sum_{i=1}^{m} \beta_i \chi_{B_i}$. Then the *integral of S over X with repect to μ* is

$$\int_X S\, d\mu = \sum_{i=1}^{m} \beta_i \mu(B_i).$$

Note that, by the previous Lemma, this is well-defined.

Definition 8.2. Let $E \in \mathcal{F}$. The *restriction of $\mathcal{F}$ to E*, written $\mathcal{F} \cap E$ is the σ-algebra on E given by

$$\mathcal{F} \cap E = \{A \in \mathcal{F} |\, A \subset E\}.$$

The resriction of μ to $\mathcal{F} \cap E$ is denoted $\mu|_E$. Clearly $\mu|_E$ is a measure on $(E,\, \mathcal{F} \cap E)$. If $S \geq 0$ is a simple function on X, then the symbol $\int_E S\, d\mu$ will be used as a shorthand for $\int_E (S|_E)\; d\,(\mu|_E)$.

Remark 8.1. Let $S \geq 0$ be simple. The integral clearly enjoys the following properties.

- $\int_X c \cdot S\, d\mu = c \int_X S\, d\mu$, $c \geq 0$.
- $\int_E S\, d\mu = \int_X S \cdot \chi_E\, d\mu$ for $E \in \mathcal{F}$
- If $\nu : \mathcal{F} \to [0,\, \infty]$ is the fuction given by

  $$\nu(E) = \int_E S\, d\mu$$

  then ν is a measure on $(X,\, \mathcal{F})$.
- Suppose $T \geq 0$ is another simple function on $(X,\, \mathcal{F})$, then

  $$\int_X (S + T)\; d\mu = \int_X S\, d\mu + \int_X T\, d\mu.$$

  Note that if $S = \sum_j \alpha_j \chi_{A_j}$ and $T = \sum_i \beta_i \chi_{B_i}$, then

  $$S + T = \sum_{i,j} (\alpha_j + \beta_i) \chi_{A_j \cap B_i}.$$

Definition 8.3. Let $f \geq 0$ be a measurable function on X. The *integral of f over X with respect to μ*, denoted $\int_X f\, d\mu$ is,

$$\int_X f\, d\mu = \sup_{0 \leq S \leq f} \int_X S\, d\mu$$

where the supremum is taken over simple S. We say f is *integrable* if its integral is finite.

As usual, if $E \in \mathcal{F}$, then $\int_E f\, d\mu$ is shorthand for $\int_E (f|_E)\, d(\mu|_E)$. The following properties are obvious:

- $\int_X c \cdot f\, d\mu = c \int_X f\, d\mu$ for $c \geq 0$.
- $\int_E f\, d\mu = \int_X f \chi_E\, d\mu$, $E \in \mathcal{F}$.
- If $0 \leq g \leq f$, then $\int_X g\, d\mu \leq \int_X f\, d\mu$.
- If $E \subset F$, $E, F \in \mathcal{F}$, then $\int_E f\, d\mu \leq \int_F f\, d\mu$. (Recall that f is assumed to be non-negative).
- If $f = 0$ then $\int_X f\, d\mu = 0$.
- If $E \in \mathcal{F}$ and $\mu(E) = 0$, then for $f \geq 0$, $\int_E f\, d\mu = 0$.

9. The Monotone Convergence Theorem

In order to prove linearity of the integral it is useful to have the Monotone Convergence Theorem—a theorem important in its own right.

THEOREM 9.1. (The Monotone Convergence Theorem). *Let $f_n \uparrow f$ pointwise, where $\{f_n\}$ is a sequence of non-negative measurable functions on X. Then*

$$\int_X f_n \, d\mu = \int_X f \, d\mu.$$

Note that such a statement is not true for Riemann integrals.

PROOF. The sequence of extended real numbers $\{\int_X f_n \, d\mu\}$ is a monotone sequence, and hence $\alpha = \lim_n \int_X f_n \, d\mu$ exists as an extended real number. Clearly

$$\alpha \leq \int_X f \, d\mu.$$

Let $0 \leq S \leq f$ on X, S simple, and let $0 < c < 1$ be a constant. Then we claim that

$$\alpha \geq \int_X c \cdot S \, d\mu.$$

Let $E_n = \{f_n \geq c \cdot S\}$. Then each $E_n \in \mathcal{F}$ and $\{E_n\}$ is an increasing sequence of sets. Since $0 < c < 1$, therefore $E_n \uparrow X$. Let ν be the measure on $(X, \mathcal{F})$ given by

$$\nu(E) = \int_E S \, d\mu.$$

Since ν is a measure, therefore $\nu(E_n) \uparrow \nu(X)$. Hence

$$\int_{E_n} c \cdot S \, d\mu \uparrow \int_X S \, d\mu.$$

But $cS \leq f_n$ on E_n. This gives

$$\int_{E_n} cS \, d\mu \leq \int_{E_n} f_n \, d\mu \leq int_X f_n \, d\mu.$$

The last inequality follows from the fact that $E_n \subset X$. This shows that $\int_X cS \, d\mu \leq \lim_n \int_X f_n \, d\mu = \alpha$. Thus the claim is proved. Now let $c \uparrow 1$. We get

$$\int_X S \, d\mu \leq \alpha$$

for all simple S such that $0 \leq S \leq f$. By definition of $\int_X f \, d\mu$, we get $\int_X f \, d\mu \leq \alpha$. This gives the desired result. □

COROLLARY 9.1. *Let f, g be non-negative measurable functions on X. Then*

$$\int_X (f+g) d\mu = \int_X f \, d\mu + \int_X g \, d\mu$$

PROOF. Let $S_n \uparrow f$ and $S'_n \uparrow g$ as $n \uparrow \infty$. Here S_n and S'_n are simple non-negative functions. Apply the Monotone Convergence Theorem to $\{S_n\}$, $\{S'_n\}$ and $\{S_n + S'_n\}$. Since we already have linearity of the integral for simple functions, we are done. □

10. Integration

Now let f be an arbitrary measurable function on X. Define $f^+ = \max\{f, 0\}$ and $f^- = \max\{-f, 0\}$. One can check that f^+ and f^- are measurable. Clearly they are non-negative functions. Note that $f = f^+ - f^-$ and $|f| = f^+ + f^-$. Note that $f^+ \leq |f|$ and $f^- \leq |f|$ and hence if $\int_X |f|\,d\mu < \infty$ then $\int_X f^+\,d\mu < \infty$ and $\int_X f^-\,d\mu < \infty$.

DEFINITION 10.1. The 1-*norm* of a measurable function f on X is

$$\|f\|_1 = \int_X |f|\,d\mu.$$

A measurable function f on X is said to be *integrable with respect to* μ if $\|f\|_1 < \infty$. In this case, we define the *integral of f with respect to* μ as

$$\int_X f\,d\mu = \int_X f^+\,d\mu - \int_X f^-\,d\mu.$$

Note that if $\mu(E) = 0$ then $\int_E f\,d\mu = 0$, where, as before the integral on the right is a shorthand for $\int_E f|_E\,d(\mu|_E)$.

DEFINITION 10.2. A property $\wp$ for points in X is said to hold *almost everywhere [μ]* (abbreviated to $a.e.$-[μ]) if the set of points for which $\wp$ fails is contained in a set of measure zero.

On the space of measurable functions on X we have an equivalence relation $\sim$ defined by $f \sim g$ if $f = g a.e.$-[μ]. If $f \sim g$, then f is integrable if and only if g is, and in this case $\int_X f\,d\mu = \int_X g\,d\mu$ (why ?).

Note that if f is integrable, then f takes the values ∞, and $-\infty$ on a set of measure zero.

DEFINITION 10.3. The space $L^1(\mu)$ is the set of equivalence classes (under $\sim$) of integrable functions.

REMARK 10.1. Note that the integral with respect to μ is well defined on $L^1(\mu)$ since two integrable functions which are $\sim$–equivalent have the same integral. Moreover, since an integrable function takes on the two infinite values on sets of measure zero, therefore, $L^1(\mu)$ is a vector space over $\mathbb{R}$. The crucial point to prove is that addition can be defined. While two integrable functions may not be added for fear of introducing undefined values like $\infty - \infty$, their equivalence classes can be, for the lacunary set if of measure zero, allowing for arbitrary changes in values there. For an integrable function, it is conventional to write $f \in L^1(\mu)$, when what is meant is that the equivalence class of f is in $L^1(\mu)$.

DEFINITION 10.4. The map $L^1(\mu) \to \mathbb{R}$ induced by the integral $\int_X (\cdot)\,d\mu$ on integrable functions (see Remark above) will also be called the *integral over X with respect to* μ and will be denoted $\int_X (\cdot)\,d\mu$. The 1-*norm* on $L^1(\mu)$, is the map $L^1(\mu) \to \mathbb{R}$ induced by the 1-norm on integrable functions (see Remark above). We continue to use the symbol $\|\cdot\|_1$ for this map on $L^1(\mu)$.

THEOREM 10.1. *With above notations and conventions we have*

1. $(L^1(\mu), \|\cdot\|_1)$ *is a normed linear space.*
2. $|\int_X f\,d\mu| \leq \int_X |f|\,d\mu = \|f\|_1$
3. $\int_X (\cdot)\,d\mu$ *is a linear functional on* $L^1(\mu)$.

PROOF. The second assertion is easy. For the first, the only non-trivial statement to be proved is that if $\|f\|_1 = 0$ then $f = 0$ in $L^1(\mu)$ (i.e. $f = 0, a.e. - [\mu]$). Without loss of genrality, we may assume $f \geq 0$ (by replacing f by $|f|$ if necessary). Then

$$\int_X f \, d\mu = 0$$

so that

$$\int_{\{f \neq 0\}} f \, d\mu = 0.$$

But

$$\{f \neq 0\} = \cup_n \left\{ \frac{1}{n-1} \geq f > \frac{1}{n} \right\}$$

and hence

$$\begin{aligned} 0 &= \int_{\{f \neq 0\}} f \, d\mu \\ &= \sum_n \int_{\left\{\frac{1}{n-1} \geq f \geq \frac{1}{n}\right\}} f \, d\mu \\ &\geq \sum_n \frac{1}{n} \mu \left(\left\{ \frac{1}{n-1} > f \geq \frac{1}{n} \right\} \right). \end{aligned}$$

This implies that

$$\mu \left(\left\{ \frac{1}{n-1} \geq f > \frac{1}{n} \right\} \right) = 0$$

Summing over n, we get $\mu(\{f \neq 0\}) = 0$. Thus $f = 0 \, a.e. - [\mu]$.

The last part is not as straightforward as it seems. It is *not* true in general that

$$(f+g)^+ = f^+ + g^+$$

or that

$$(f+g)^- = f^- + g^-.$$

Let $f + g = h$. Then $h \in L^1(\mu)$. Then

$$h^+ - h^- = h = f + g = f^+ - f^- + g^+ - g^- \qquad a.e. - [\mu].$$

Since f, g, h are finite $a.e. - [\mu]$, we get

$$h^+ + f^- + g^- = f^+ + g^+ + h^- \qquad a.e. - [\mu].$$

Take integrals of both sides, transpose appropriate terms and get

$$\int_X h \, d\mu = \int_X f \, d\mu + \int_X g \, d\mu.$$

□

REMARK 10.2. Suppose $f_n \to f$ in $L^1(\mu)$, in other words, $\|f_n - f\|_1 \to 0$ as $n \to \infty$. Since $|\int_X (f_n - f) \, d\mu| \leq \|f_n - f\|_1$, therefore it follows that $\int_X f_n \, d\mu \to \int_X f \, d\mu$ as $n \to \infty$.

Lecture 4

11. The Dominated Convergence Theorem

We have seen that under certain conditions, $\lim_n$ and $\int_X$ commute. More precisely, if the conditions of the Monotone Convergence Theorem (MCT) are satisfied then this is true. But the MCT is for increasing sequences. The (Lebesgue) Dominated Convergence Theorem (abbreviated DCT) is applicable to other situations.

THEOREM 11.1. [The Dominated Convergence Theorem] *Let $\{f_n\}$ be a sequence in $L^1(\mu)$ such that $f_n \to f$ pointwise a.e. $-[\mu]$, Suppose there exists $\phi \in L^1(\mu)$ such that*

$$|f_n| \le \phi \qquad (n \in \mathbb{N}).$$

Then $f_n \to f$ in $L^1(\mu)$.

PROOF. Let

$$g_n = 2\phi - |f_n - f|.$$

Then $g_n \ge 0$. Apply Fatou's Lemma (Exercise 27 of your homework assignment). Get

$$\int_X \liminf g_n \, d\mu \le \liminf \int_X g_n \, d\mu$$

i.e.

$$\int_X 2\phi \, d\mu \le \int_X 2\phi \, d\mu + \liminf \left\{ - \int_X |f_n - f| \, d\mu \right\}$$

or

$$0 \ge \limsup \int_X |f_n - f| \, d\mu$$

This gives,

$$0 \ge \limsup \int_X |f_n - f| \, d\mu \ge \liminf \int_X |f_n - f| \, d\mu \ge 0.$$

Thus

$$\lim_n \int_X |f_n - f| \, d\mu = 0$$

and hence the result.

□

EXAMPLE 11.1. The DCT is only a sufficient condition for the limit of integrals to be the integral of the limit. We will give a sequence of integrable functions $\{f_n\}$ on $(\mathbb{R}, \mathcal{M}, m)$ such that $\{f_n\}$ is *not* dominated by an L^1 function, and nevertheless the conclusion of the DCT holds. Define f_n by

$$f_n(x) = \frac{1}{x} \chi_{[1/(n+1), 1/n]}(x) \ge 0.$$

Then

(i) $f_n \to f := 0$ pointwise.

(ii) $\int_{\mathbb{R}} |f_n - f|\, dm = \int_{\frac{1}{n+1}}^{\frac{1}{n}} \dfrac{dx}{x} = \log \dfrac{n+1}{n} \to 0$ as $n \to \infty$.

However, if there was a $\phi \in L^1(m)$ such that $f_n \leq \phi$ for all n, then $\phi(x) \geq 1/x$ for $x > 0$ and hence $\phi \notin L^1(m)$.

12. Applications of DCT

THEOREM 12.1. *Let $f : I \to \mathbb{R}$ be such that f' exists and is bounded on I. Then*

$$f(b) - f(a) = \int_I f'\, dm$$

REMARK 12.1. Note that f' must necessarily be measurable. Thinking of f as a function on all of $\mathbb{R}$, for each $a \in \mathbb{R}$, define the *translate of f by a* as the function $T_a f$ given by $T_a f(x) = f(x+a)$. Clearly $T_a f$ is also measurable for every $a \in \mathbb{R}$. Now consider

$$g_n = n\left[T_{1/n} f - f\right].$$

Clearly g_n is also measurable. Let $n \uparrow \infty$ and see what happens. How would you take care of the right end point ?

PROOF. Since $|f'| \leq M$, therefore $\int_I |f'|\, dm \leq M \cdot m(I) < \infty$. Thus $f' \in L^1(m)$. Let g_n be as above. Then, for each x and each n we have $\xi_{x,n} \in [x, x+1/n]$ such that

$$|g_n| \leq |f'(\xi_{x,n})| \leq M.$$

By DCT, we get

$$\int_I g_n\, dm \longrightarrow \int_I f'\, dm.$$

On the other hand,

$$\begin{aligned}
\int_I g_n\, dm &= n\left\{\int_I f(x+1/n)\, dm(x) - \int_I f(x)\, dm(x)\right\} \\
&= n\left\{\int_{a+\frac{1}{n}}^{b+\frac{1}{n}} f\, dm - \int_a^b f\, dm\right\} \\
&= n\left\{\int_b^{b+\frac{1}{n}} f\, dm - \int_a^{a+\frac{1}{n}} f\, dm\right\} \\
&\longrightarrow f(b) - f(a) \quad \text{as } n \to \infty \qquad (\text{since } f \text{ is continuous})
\end{aligned}$$

□

Another application of the DCT is in proving that $L^1(\mu)$ is complete under the metric induced by its norm, i.e. $L^1(\mu)$ is a *Banach space*. In order to show this, we need the following Proposition.

PROPOSITION 12.1. *Let $(X, \mathcal{F}, \mu)$ be a measure space. Let $\{f_n\}$ be a sequence in $L^1(\mu)$ such that*

$$\sum_{n\geq 1} \|f_n\|_1 < \infty.$$

Then $g = \sum_{n\geq 1} f_n$ *converges a.e.*$-[\mu]$ *and* $g \in L^1(\mu)$*. Moreover* $g_n = \sum_{k=1}^{n} f_k \to g$ *in* $L^1(\mu)$ *as* $n \to \infty$ *and hence, in particular, we have*

$$\int_X \sum_{n\geq 1} f_n \, d\mu = \sum_{n\geq 1} \int_X f_n \, d\mu.$$

REMARK 12.2. Note that it will also follow that $\sum |f_n| < \infty$, *a.e.* $-[\mu]$ since $\||f_n|\|_1 = \|f_n\|_1$.

PROOF. Let $\phi = \sum_{n\geq 1} |f_n|$. Then by MCT,

$$\int_X \phi \, d\mu = \sum_{n\geq 1} \|f_n\|_1 < \infty.$$

In other words, $\phi < \infty$ *a.e.* $-[\mu]$ and $\phi \in L^1(\mu)$. Thus the series $\sum_{n\geq 1} |f_n|$ converges *a.e.* $-[\mu]$ and hence it follows that $\sum_{n\geq 1} f_n$ converges *a.e.* $-[\mu]$. Let $g = \sum_{n\geq 1} f_n$ and $g_N = \sum_{n=1}^{N} f_n$. Then $|g_N| \leq \phi \in L^1(\mu)$. By the DCT, we get that $\int_X g_N \, d\mu \to \int_X g \, d\mu$. In other words,

$$\sum_{n=1}^{N} \int_X f_n \, d\mu \longrightarrow \int_X \sum_{n\geq 1} f_n \, d\mu.$$

This gives the required result. □

THEOREM 12.2. $L^1(\mu)$ *is a Banach space with* $\|\cdot\|_1$ *as norm.*

PROOF. Let $\{h_n\}$ be a Cauchy sequence in $L^1(\mu)$. Let $n_1 < n_2 < \ldots < n_k < \ldots$ be a sequence of positive integers such that

$$\|h_n - h_m\|_1 \leq \frac{1}{2^k} \qquad (n, m \geq n_k)$$

Let

$$f_1 = h_{n_1} f_k = h_{n_k} - h_{n_{k-1}} \qquad (k \geq 2).$$

Then

$$\sum_{k\geq 1} \|f_k\|_1 \leq \infty.$$

By the Proposition above, $h = \sum_{k\geq 1} f_k$ is finite *a.e.* $-[\mu]$, and $h \in L^1(\mu)$. One checks easily that $\|h_n - h\|_1 \to 0$ as $n \to \infty$. □

13. L^p spaces

Fix a measure space $(X, \mathcal{F}, \mu)$.

DEFINITION 13.1. Let $1 \leq p < \infty$. The *p-norm* of a measurable function f is

$$\|f\|_p := \left\{ \int_X |f|^p \, d\mu \right\}^{\frac{1}{p}}.$$

A measurable function f on X is said to be *p-integrable with respect to* μ if $\int_X |f|^p \, d\mu < \infty$, i.e. if $\|f\|_p < \infty$. Note that f is 1-integrable preciesely when it is integrable.

REMARK 13.1. Suppose f is p-integrable, and $f = g$ $a.e. - [\mu]$, for some other measurable function g on X. Then clearly $|f|^p = |g|^p$ $a.e. - [\mu]$, whence we conclude that g is also p-integrable. Moreover, in this case

$$\int_X |f|^p \, d\mu = \int_X |g|^p \, d\mu.$$

Further, note that if f is p-integrable, then as in the case $p = 1$, $\int_X |f|^p \, d\mu = 0$ if and only if $f = 0$ $a.e. - [\mu]$

DEFINITION 13.2. Let $1 \geq p < \mu$. Then $L^p(\mu)$ is the set of all equivalence classes of p-integrable functions on X with respect to μ, under the equivalence relation of "equality $a.e. - [\mu]$". For f a p-integrable function, let $[f] \in L^p(\mu)$, be its equivalence class. Define

$$\|[f]\|_p = \left\{ \int_X |f|^p \, d\mu \right\}^{\frac{1}{p}}.$$

By the above remarks this is well defined. We call it the L^p norm of $[f]$ or the p-norm of f.

REMARK 13.2. It is convention to write $f \in L^p(\mu)$ when what is meant is that $[f] \in L^p(\mu)$. In the same vein, for a p-integrable function f, we write $\|f\|_p$ for $\|[f]\|_p$. If it is clearly kept in mind that equalities are only almost everywhere $[\mu$, then no confusion arises from these conventions. We will follow them. Finally, note that the p-norm makes sense as an exteneded real number for any measurable function f, and it is p integrable precisely when $\|f\|_p < \infty$.

DEFINITION 13.3. Let f be a measurable function on X. We say f is *essentially bounded* if the essential supremum of f,

$$\|f\|_\infty := \inf\{y > 0 | \, \mu\{|f| > 0\} = 0\}$$

is finite.

REMARK 13.3. As usual, one can see that the essential supremum of two measurable functions which are equal $a.e. - [\mu]$ agree.

DEFINITION 13.4. $L^\infty(\mu)$ is the set of equivalence classes of essentially bounded measurable functions under the relationship of "equality $a.e. - [\mu]$". For $[f] \in L^\infty(\mu)$, $\|[f]\|_\infty := \|f\|_\infty$. This is well-defined by the above remark. As is conventional, we write $f \in L^\infty(\mu)$ when we mean $[f] \in L^p(\mu)$.

EXAMPLE 13.1.
(1) Consider $(I, \mathcal{M} \cap I, m|_I)$. Then for a continuous function on I, the essential supremum agrees with its maximum.
(2) Let $\{r_n\}$ be some enumeration of the rational numers $\mathbb{Q}$. Set

$$f(x) = \begin{cases} n & \text{if } x = r_n \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

Then f is unbounded everywhere (in the ordinary sense), but $\|f\|_\infty = 0$.

THEOREM 13.1. *Let $(X, \mathcal{F}, \mu)$ be a measure space with $0 < \mu(X) < \infty$. Then*

$$\|f\|_p \longrightarrow \|f\|_\infty \qquad \text{as } p \to \infty.$$

PROOF. We will show that

$$\limsup_{p\to\infty} \|f\|_p \leq \|f\|_\infty \leq \liminf_{p\to\infty} \|f\|_p.$$

Now,

$$\int |f|^p \, d\mu \leq \|f\|_\infty^p \mu(X).$$

This gives

$$\|f\|_p \leq \|f\|_\infty \mu(X)^{\frac{1}{p}},$$

from which we conclude that

$$\limsup_{p\to\infty} \|f\|_p \leq \|f\|_\infty.$$

To show the second equality, we may assume that $\|f\|_\infty < \infty$. Let $0 < \lambda < \|f\|_\infty$, and let $E = \{|f| > 0\}$. Then $0 < \mu(E) < \infty$ and

$$\mu(E) = \int_E 1 \, d\mu \leq \frac{1}{\lambda^p} \int_E |f|^p \, d\mu \leq \frac{1}{\lambda^p} \int_X |f| \, d\mu.$$

Thus

$$\lambda^p \mu(E) \leq \|f\|_p^p,$$

i.e.

$$\lambda \cdot \mu(E)^{\frac{1}{p}} \leq \|f\|_p.$$

Thus,

$$\lambda \leq \liminf_{p\to\infty} \|f\|_p.$$

Let $\lambda \uparrow \|f\|_\infty$. We get that

$$\|f\|_\infty \leq \liminf_{p\to\infty} \|f\|_p$$

as required. $\square$

Lecture 5

14. Jensen's Inequality

THEOREM 14.1. [Jensen's Inequality] *Let* $(X, \mathcal{F}, \mu)$ *be a measure space with* $\mu(X) = 1$. *If* $f : X \to \mathbb{R}$ *is an integrable function, taking values in the interval* (a, b) *with* $a = -\infty$ *and* $b = \infty$ *allowed, and if* $\varphi : (a, b) \to \mathbb{R}$ *is convex, then*

$$\varphi\left(\int_X f\, d\mu\right) \leq \int_X (\varphi \circ f)\, d\mu.$$

PROOF. Let $t = \int_X f\, d\mu$. Then for $a < s < t < u < b$ we have, by Exercise 1 of the Homework assignment

$$\frac{\varphi(t) - \varphi(s)}{t - s} \leq \frac{\varphi(u) - \varphi(t)}{u - t}.$$

Let β be the supremum of the quotients on the left. Then for $s < t$

$$\frac{\varphi(t) - \varphi(s)}{t - s} \leq \beta$$

and

$$\frac{\varphi(u) - \varphi(t)}{u - t} \geq \beta$$

The two inequalities can be combined to give

$$\frac{\varphi(t) - \varphi(s)}{t - s} \leq \beta \qquad (a < s < b).$$

This means

$$\varphi(s) \geq \varphi(t) + \beta(t - s) \qquad (a < s < b).$$

Hence

$$\varphi(f(x)) - \varphi(t) - \beta(f(x) - t) \geq 0$$

for every $x \in X$. Since φ is continuous (in fact it is absolutely continuous. See Exercise 2 of your Homework assignment) therefore $\varphi \circ f$ is measurable. Integrating the above inequality, and using the fact that for any constant $\alpha \in \mathbb{R}$, $\int_X \alpha\, d\mu = \alpha$ (for $\mu(X) = 1$), we get the required inequality (remember that t and $\varphi(t)$ are constants). □

EXAMPLE 14.1. Let $b_1, \ldots, b_n \geq 0$ and $a_j = \log b_j$ for $j = 1, \ldots n$. Let $X = \{1, 2, \ldots, n\}$ and consder the measur space $(X, \mathcal{F}, \mu)$ where $\mathcal{F} = 2^X$ and $\mu = 1/n \cdot \#$, where $\#$ is the *counting measure on* $(X, \mathcal{F})$. Then $\mu(X) = 1$. Jensen's inequality applies to the convex function $\varphi(x) = \exp(x)$ and the function f on X given by $f(j) = a_j$. We immediately get

$$\exp \frac{1}{n} \sum_{j=1}^{n} a_j \leq \frac{1}{n} \sum_{j=1}^{n} \exp a_j.$$

In other words,

$$\exp \frac{1}{n} \log b_1 \ldots b_n \leq \frac{1}{n}(b_1 + \ldots + b_n).$$

This gives the well-known inequality

$$(b_1 \ldots b_n)^{\frac{1}{n}} \leq \frac{b_1 + \ldots + b_n}{n}.$$

15. The Riesz-Fischer Theorem

Fix a measure space $(X, \mathcal{F}, \mu)$.

THEOREM 15.1. *Let $1 \leq p \leq \infty$, and let q be such that $1/p + 1/q = 1$, where we use the convention that $q = \infty$ of $p = 1$ and $q = 1$ if $p = \infty$. Let f, g be measurable functions on X. Then*

(1)
$$\int_X |fg| \, d\mu \leq \|f\|_p \|g\|_q \qquad \textit{(Hölder's inequality)}$$

(2)
$$\|f + g\|_p \leq \|f\|_p + \|g\|_p \qquad \textit{(Minkowski's inequality)}$$

(3) *$L^p(\mu)$ is a normed linear space.*

PROOF. This is Exercise 42 of your Homework assignment. □

THEOREM 15.2. [The Riesz-Fischer Theorem] *$L^p(\mu)$ is a Banach space.*

PROOF. We will leave the case $p = \infty$ as an (easy!) exercise. So assume $1 \leq p < \infty$. Let f_n be meaurable functions such that $\sum_{n=1}^{\infty} \|f_n\|_p < \infty$. Let $\phi = \sum_{n=1}^{\infty} |f_n|$. Then $\|\phi\|_p \leq \sum_n \|f_n\|_p < \infty$. Hence $\phi < \infty$ *a.e.* $-$ $[\mu]$. Let $g_n = \sum_{j=1}^{n} f_j$ and $g = \sum_{j=1}^{\infty} f_n$. Then $\|g\|_p \leq \|\phi\|_p < \infty$ and hence $g \in L^p(\mu)$. Moreover,

$$\|g_n - g\|_p \leq \| \sum_{j=n+1}^{\infty} f_j \|_p \leq \sum_{j=n+1}^{\infty} \|f_j\|_p \longrightarrow 0 \quad \text{as } n \to \infty.$$

Thus $g_n \to g$ in $L^p(\mu)$. Now imitate the proof of the completness of $L^1(\mu)$ to get the result. □

16. Radon-Nikodym Theorem

DEFINITION 16.1. Suppose μ and ν are measures on $(X, \mathcal{F})$ such that the follwing relationship holds,

$$\nu(E) = \int_E f \, d\mu \qquad (E \in \mathcal{F}),$$

for some measurable function $f \geq 0$. Then we say that f is the *Radon-Nikodym derivative of ν with resoect to μ* and we write $f = d\nu/d\mu$.

Note that the Radon-Nikodym derivative, if it exists, is unique up to equality *a.e.* $-$ $[\mu]$, for, if a measurable function h is such that $\int_E h \, d\mu = 0$ for every $E \in \mathcal{F}$, then $h = 0$, *a.e.* $-$ $[\mu]$. Indeed, in this case, $\int_X h^+ \, d\mu = \int_{\{h>0\}} h \, d\mu$ and hence $h^+ = 0$ *a.e.* $-$ $[\mu]$. Similarly $h^- = 0$, *a.e.* $-$ $[\mu]$.

Not that if $d\nu/d\nu$ exists, then $\nu(E) = 0$ for every $E \in \mathcal{F}$ such that $\mu(E) = 0$. This motivates the following definition:

Definition 16.2. Let μ, ν be measures on $(X, \mathcal{F})$. We say ν *is absolutely continuous with respect to* μ (written $\nu \ll \mu$) if for every $E \in \mathcal{F}$ with $\mu(\mathcal{F}) = 0$, we have $\nu(E) = 0$.

Clearly, if $d\nu/d\mu$ exists, then $\nu \ll \mu$. We would like to give a converse to this. However, we will prove a converse under special conditions, namely, when μ and ν are σ-finite, a notion which we now define.

Definition 16.3. A measure μ on $(X, \mathcal{F})$ is said to be σ-finite if $X = \bigcup_{n=1}^{\infty} E_n$ with each E_n measurable and of finite μ-measure.

Theorem 16.1. [The Radon-Nikodym Theorem] *Let μ and ν be σ-finite measures on $(X, \mathcal{F})$ and suppose $\mu \ll \nu$. Then the Radon-Nikodym derivative $d\nu/d\mu$ exists.*

Proof. Write $X = \bigcup_n E_n$ where the E_n have finite μ and ν measure. We may assume, without loss of generality, that the E_n are disjoint. Clearly it is enough to prove the existence of the derivative on each E_n. In other words, it is enough to assume that $\mu(X) < \infty$ and $\nu(X) < \infty$.

We will first prove a special case, viz., the case where $\nu \leq \mu$. Let $\|\cdot\|_{\mu,p}$ and $\|\cdot\|_{\nu,p}$ be the norms of $L^p(\mu)$ and $L^p(\nu)$ $(p \geq 1)$. Then one checks easily that $L^p(\mu) \subset L^p(\nu)$. It is clearly enough to show that

$$\int_X f\, d\nu \leq \int_X f\, d\mu \qquad (f \geq 0 \text{ measurable}).$$

But this inequality is clearly true for simple $f \geq 0$ and hence by MCT for all measurable $f \geq 0$. Moreover, by Exercise 42, $L^p(\nu) \subset L^1(\nu)$ for $p \geq 1$. Thus we have a linear functional $\Lambda : L^2(\mu) \to \mathbb{R}$ given by

$$\Lambda f = \int_X f\, d\nu.$$

Since $|\Lambda f| \leq \|f\|_{\mu,2}\sqrt{\mu(X)} \leq \|f\|_{\nu,2}\sqrt{\mu(X)}$ (the first inequality follows from Hölder's inequality with $g = 1$), therefore Λ is a continous linear functional. But $L^2(\mu)$ is a Hilbert space (the Riesz-Fischer theorem gives completeness), therefore by the Riesz Representation Theorem for Hilbert spaces, there exists a unique $g \in L^2(\mu)$ such that $\Lambda f = \int_X fg\, d\mu$ for every $f \in L^2(\mu)$. Taking $f = \chi_E$, $E \in \mathcal{F}$ we get

$$\nu(E) = \int_E g\, d\mu.$$

By definition of the Radon-Nikodym derivative $d\nu/d\mu = g$ *a.e.* $- [\mu]$.

If $\nu \not\leq \mu$, then consider $\lambda = \mu + \nu$. By the above arguments, $d\mu/d\lambda$ and $d\nu/d\lambda$ exist. Let $E = \{d\mu/d\lambda = 0\}$. Then $\mu(E) = \int_E d\mu/d\lambda\, d\lambda = 0$. This means that $\nu(E) = 0$ and hence $\lambda(E) = 0$. In other words $d\mu/d\lambda > 0$ *a.e.* $- [\lambda]$. By the homework exercises on Radon-Nikodym theorem, it follows that $d\nu/d\mu$ exists and is given by

$$\frac{d\nu}{d\mu} = \frac{d\nu/d\lambda}{d\mu/d\lambda} \qquad a.e. - [\mu].$$

□

17. Change of Variables

Let U, V be open sets in $\mathbb{R}^n$ and $T : U \to V$ a one-to-one onto map such that T and T^{-1} are C^1, i.e., T and T^{-1} are differentiable and their derivatives are continuous. We wish to see how the form of an integral $\int_V f\,dm$ changes under the substitution $x \mapsto Tx$. The precise form of our theorem is

THEOREM 17.1. [Change of Variables Formula]. *Let U, V, T be as above. Let μ be the measure on $(U, \mathcal{B})$ given by*

$$\mu(B) = m(TB), \qquad B \subset U, B \in \mathcal{B}.$$

Let $J_T : U \to \mathbb{R}$ be the "Jacobian" of T, i.e., $J_T = \det\left(T'(x)\right)$. Then

(1) *$\mu \ll m$, and $d\mu/dm = |J_T|$. In other words,*

$$m(TB) = \int_B |J_T|\,dm \qquad (B \in \mathcal{B}).$$

(2) *For $f : V \to \mathbb{R}$,*

$$\int_V f\,dm = \int_U (f \circ T)\,|J_T|\,dm,$$

where the equality above has to be interpreted in the sense that if either of the integrals in this formula exists, then both exist and are equal, and if either diverges properly then both do and to the same infinite value.

PROOF. The second part follows from the first part and the "abstract of change of variables formula" (Exercise 29 of your Homework assignment). Indeed if $\varphi = T^{-1}$ then,

$$\begin{aligned}\int_U (f \circ T)\,|J_T|\,dm &= \int_U (f \circ T)\;d\mu \\ &= \int_U (f \circ T)\,d\varphi_* m \\ &= \int_{\varphi^{-1}(U)} (f \circ T \circ \varphi)\;dm \\ &= \int_V f\,dm\end{aligned}$$

as required.

To prove the first part, let $B_{n,k} = \{(k-1)/n \le |J_T| < k/n\} \cap B$, for n, k positve integers. Then B is the disjoint union of $B_{n,k}$ over k. By Exercise 60 of your Homework assignment, we have

$$m\left(T(B_{n,k})\right) \le \frac{k}{n} m(B_{n,k}).$$

We also have $|J_\varphi(y)| \le n/(k-1)$ for all $y \in T(B_{n,k})$. Hence, again using Exercise 59, and the first inequality, we get

$$\frac{k-1}{n} m(B_{n,k}) \le m\left(T(B_{n,k})\right) \le \frac{k}{n} m(B_{n,k}).$$

for every $k \in \mathbb{N}$. On the other hand, clearly

$$\frac{k-1}{n} m(B_{n,k}) \le \int_{B_{n,k}} |J_T|\,dm \le \frac{k}{n} m(B_{n,k})$$

for $k \in \mathbb{N}$. Thus,

$$\left| m\left(T(B_{n,k})\right) - \int_{B_{n,k}} |J_T| \, dm \right| \leq \frac{1}{n} m(B_{n,k}).$$

Since $B = \bigcup_k B_{n,k}$ and the union is disjoint, we get

$$\left| m\left(T(B)\right) - \int_{B} |J_T| \, dm \right| \leq \frac{1}{n} m(B).$$

Since n is arbitrary,

$$m\left(T(B)\right) = \int_{B} |J_T| \, dm.$$

Thus we are done when $m(B) < \infty$. The general case is proved by writing B as the increasing limit of Borel sets of finite measure and invoking MCT. $\square$

Lecture 6

18. Complex Measures

Much the way that a function of bounded variation on a closed bounded interval is dominated by its total variation function, it is possible to find a measure λ which dominates a given complex measure μ on a measurable space $(X, \mathcal{F})$ in the sense that $|\mu(E)| \leq \lambda(E)$ for every $E \in \mathcal{F}$. Moreover we would like to find a minimal such λ. The obvious candidiate is $\lambda = |\mu|$, where $|\mu| : \mathcal{F} \longrightarrow [0, \infty]$ is given by

$$|\mu|(E) = \sup_{\{E_i\}} \sum_{i=1}^{\infty} |\mu(E_i)|$$

the supremum being taken over all *partitions* $\{E_i\}$ of E. Here we are using the term "partition" in a special sense, viz., a collection of sets $\{E_i\}$ is said to be a partition of $E \in \mathcal{F}$ if (a) the collection is countable, (b) is pairwise disjoint and (c) each $E_i \in \mathcal{F}$. It is not clear a-priori that $|\mu|$ is a measure (we will prove this later).

Definition 18.1. The set function $|\mu|$ is called the *total variation measure*[1] of μ, and $|\mu|(X)$ is called the *total variation of* μ. We write

$$\|\mu\| = |\mu|(X).$$

One checks (easily) that the total variation is anorm on the $\mathbb{C}$-vector space of complex measures on $(X, \mathcal{F})$.

Theorem 18.1. *The total variation measure* $|\mu|$ *of a complex measure* μ *on* $\mathcal{F}$ *is a measure on* $\mathcal{F}$.

Proof. The only point that needs checking is countable additivity. To that end first note that if $E \in \mathcal{F}$ is such that $|\mu|(E) = 0$, then (as a simple consequence of the definitions), $|\mu(E)| = 0$, and further, if $A \subset E$ is a measurable subset then $|\mu|(A) = 0$. Its follows that we have countable additivity for any $|\mu|$-null set $E \in \mathcal{F}$.

Now suppose $|\mu|(E) > 0$. Let $\{E_i\}$ be a partition of E. Let B_i be the sets in $\{A_i\}$ (in some enumeration) which satisfy $|\mu|(A_i) = 0$. Let C_i be the remaining members of $\{A_i\}$ (again enumerated in some manner). Pick real numbers t_i so that $0 < t_i < |\mu|(C_i)$. By definition of $|\mu|(C_i)$, there exists a partition $\{D_{ij}\}$ of $\{C_i\}$ such that $\sum_{j=1}^{\infty} |\mu(D_{ij})| > t_i$. It follows that

$$\sum_{i=1}^{\infty} |\mu(B_i)| + \sum_i \sum_j |\mu(D_{ij})| > \sum_{i=1}^{\infty} t_i.$$

[1] Even though we have not yet proved that $|\mu|$ is actually a measure on $(X, \mathcal{F})$.

But $\{B_i\}$ and $\{D_{ij}\}_{ij}$ together form a partition of E. Thus, $|\mu|(E) > \sum_{i=1}^{\infty} t_i$. Let $t_i \uparrow |\mu|(C_i)$. We see that

$$|\mu|(E) \geq \sum_{i=1}^{\infty} |\mu|(A_i). \tag{*}$$

In particular if $\{A_{ij}\}$ is a partition of $\{A_i\}$ then

$$\sum_{i=1}^{\infty} |\mu|(A_i) \geq \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} |\mu|(A_{ij}). \tag{†}$$

Let $\{E_i\}$ be a partition of E, and $A_{ij} = A_i \cap E_j$. Then,

$$\begin{aligned}\sum_j |\mu(E_j)| &\leq \sum_j \sum_i |\mu(A_{ij})| \\ &\leq \sum_i \sum_j |\mu|(A_{ij}) \\ &\leq \sum_i |\mu|(A_i) \qquad \text{(by (†))}.\end{aligned}$$

In particular $|\mu|(E) \leq \sum_i |\mu|(A_i)$. This together with $(*)$ proves that $|\mu|$ is a measure. □

THEOREM 18.2. *Let μ be a complex measure on $(X, \mathcal{F})$. Then $|\mu|(X) < \infty$, i.e. $|\mu|$ is a finite measure.*

PROOF. We claim that if $E \in \mathcal{F}$ is such that $|\mu|(E) = \infty$, then there exist $A, B \in \mathcal{F}$ such that $E = A \cup B$, $A \cap B = \emptyset$, $|\mu(X)| > 1$ and $|\mu|(B) = \infty$. To see this, let $t = 6\,(1 + |\mu(E)|)$. Then there exists a partition $\{E_j\}$ of E such that $\sum_j |\mu(E_j)| > t$. In particular, for some n,

$$\sum_{j=1}^{n} |\mu(E_j)| > t.$$

Now, given any set of complex numbers $\{z_1, \dots, z_n\}$, there exists a subset S of the indices $\{1, \dots, n\}$ such that $|\sum_{j \in S} z_j|$ is at least $\frac{1}{6}\sum_{j=1}^{n} |z_j|$.[2] If we put $z_j = \mu(E_j)$, and use the just mentioned fact, we see that

$$\left|\sum_{j \in S} z_j\right| > 1 + |\mu(E)|.$$

Let $C = \bigcup_{j \in S} E_j$ and $D = X \setminus C$. Then $C \subset E$ and

$$|\mu(C) > 1 + |\mu(E)|.$$

Hence,

$$|\mu(D)| > |\mu(C)| - |\mu(E)| > 1.$$

[2]This can be seen as follows. If $s = \sum_{j=1}^{n} |z_j|$ then of the four quadrants bounded by the diagonal lines $y = \pm x$, at least one, call it Q, is such that the sum of $|z_j|$ of the z_j which lie in Q is at least a quarter of s. By multiplying all the z_j by a a complex number c with $|c| = 1$, if necessary, we may assume that Q is the quadrant given by $|y| \leq x$. Let $S = \{j |\, z_j \in Q\}$. The claim follows by using the fact that $\operatorname{Re} z_j > |z_j|/\sqrt{2}$ for $j \in S$ and the fact that $4\sqrt{2} \leq 6$.

Thus both $|\mu(C)|$ and $|\mu(D)|$ have absolute value greater than 1. Since $\{C, D\}$ is a partition of E, at least one of C or D (if not both) have infinite $|\mu|$ measure. Pick such a set from $\{C, D\}$ and call it A and call the remaining one as B. Clearly this proves the claim.

Now put $B_0 = X$. Suppose we have chosen $B_0 \supset B_1 \supset \ldots \supset B_n$, with $|\mu|(B_n) = \infty$. Decompose B_n into $B_n = A_{n+1} \cup B_{n+1}$, with $|\mu(A_{n+1})| > 1$ and $|\mu|(B_{n+1}) = \infty$. Then we get $A_1, A_2, \ldots, A_n, \ldots$ which are mutually disjoint and $|\mu(A_i)| > 1$. This implies that $\sum_i (A_i)$ does not converge absolutely, contradicting the defintion of complex measure. Hence $|\mu|(X) < \infty$. □

19. Signed Measures, Positive and Negative Variations

If μ is a signed measure on a measurable space $(X, \mathcal{F})$, then, by following the steps we sused for a complex measure, we can define the total variation measure $|\mu|$ on $(X, \mathcal{F})$. However, we cannot conclude that $|\mu|$ is a finite measure, for the proof of Theorem 18.2 needs the fact that μ is a complex measure in a crucial way. A signed measure need not be a complex measure—it is so, if and only if it does not take (positive or negative) infinite values, i.e. if and only if it takes real values. For a signed measure μ, we continue to write $\|\mu\|$ for $|\mu|(X)$, with the understanding that possibly $\|\mu\| = \infty$. Define

$$\mu^+ = \frac{1}{2}\left(|\mu| + \mu\right),$$
$$\mu^- = \frac{1}{2}\left(|\mu| - \mu\right).$$

μ^+ and μ^- are measures on $\mathcal{F}$, and

$$\mu = \mu^+ - \mu^-$$
$$|\mu| = \mu^+ + \mu^-.$$

The measure μ^+ is called the *positive variation* of μ and μ^- is called the *negative variation* of μ.

A signed measure which is real valued (i.e. is a complex measure) is called a *bounded signed* measure, for it its range must lie in the bounded interval $[-\|\mu\|, \|\mu\|]$. If μ is a bounded signed measure, then μ^+ and μ^- are finite measures (and clearly the converse is also true). The above decomposition is often called the *Jordan decomposition* or the *Hahn-Jordan decomposition.* A signed measure μ is said to be *σ-finite* if $|\mu|$ is σ-finite. This is equivalent to saying that X can be written as a union of countable number of measurable sets $\{E_n\}$, on each of which μ is bounded. We may, as usual, assume the E_n are disjoint, or at the other extreme, assume that they are an increasing sequence.

20. Measurability and Integration Revisited

Let $(X, \mathcal{F}, \mu)$ be a measure space. For this and the rest of the lectures we use a slightly freer concept of measurability (which needs μ). A function which is undefined on a set E with $\mu(E) = 0$, and which is measurable on $X \setminus E$ will be said to be measurable on X (or strictly speaking on the measure space $(X, \mathcal{F}, \mu)$). A *measurable complex valued function* on X (or simply a *complex function on* $(X, \mathcal{F}, \mu)$) is a measurable function $f : (X \setminus E, \mathcal{F}) \longrightarrow (\mathbb{C}, \mathcal{B}_{\mathbb{C}})$ where $E \in \mathcal{F}$ and $\mu(E) = 0$. Here $\mathcal{B}_{\mathbb{C}}$ is the Borel σ-algebra on $\mathbb{C}$.

Let $p \geq 1$ be a real number. A complex function f on $(X, \mathcal{F}, \mu)$ is said to be *p-integrable* if $|f|$ is p-integrable. We say f is *integrable*, if it is 1-integrable. If f is integrable, and $\operatorname{Re} f = u$ and $\operatorname{Im} f = v$ (so that $f = u + iv$), then note that both u and v are integrable (in the real sense) since $|u|$ and $|v|$ are both less that $|f|$. In this case we define the integral of f to be

$$\int_X f\, d\mu := \int_X u\, d\mu + i \int_X v\, d\mu.$$

As in the real case we can define L^p-spaces as follows: $L^p_{\mathbb{C}}(\mu)$ is the space of equivalence classes of p-integrable complex functions under the equivalence relation of "equality *a.e.* $- [\mu]$". Note that $L^p(\mu) \subset L^p_{\mathbb{C}}(\mu)$. $L^p_{\mathbb{C}}(\mu)$ is a $\mathbb{C}$-vector space. The Hölder, Minkowski inequalities hold for this space, as does the Riesz-Fischer Theorem (so that $L^p_{\mathbb{C}}(\mu)$ is a complex Banach space). Moreover $L^2_{\mathbb{C}}(\mu)$ is a Hilbert space with inner product $(f, g) = \int_X f\bar{g}\, d\mu$. The proofs of all these statements are exactly the same as in the real case. To make sense of the Hölder inequality for the case $p = 1$, we have to define $\|\cdot\|_\infty$ for complex measurable functions, and this is done exactly as in the real case. We leave the exact defintion of the space $L^\infty_{\mathbb{C}}(\mu)$ to the reader.

Now suppose μ is a *complex measure.* According to Exercise 72 of your Homework assigment, $d\mu = h\, d|\mu|$. We say that a complex function f *integrable* with respect to μ if fh is integrable with respect to $|\mu|$ and in this case we define

$$\int_X f\, d\mu = \int_X fh\, d|\mu|.$$

21. $L^p_{\mathbb{C}}(\mu)$ and its Dual

Let $p \in [1, \infty]$. Then the *conjugate exponent* of p is defined to be the unique number $q \in [1, \infty]$ which satisfies $1/p + 1/q = 1$, with the understanding that $q = \infty$ if $p = 1$, and $q = 1$ if $p = \infty$.

Our goal in this section is to identify the (complex) dual of $L^p_{\mathbb{C}}(\mu)$ for $p \in [1, \infty)$ (note that we do not allow p to be ∞). It turns out to be $L^q_{\mathbb{C}}(\mu)$, where q is the exponent conjugate to p, in a sense made precise below. As a corollary, one easily gets that the real dual of the real Banach space $L^p(\mu)$ is $L^q(\mu)$. This theorem are also often called *Riesz Representation Theorem*, along with certain other theorems which identify the dual of a well-known Banach space. Strictly speaking the name is reserved for the theorem which identifies the dual of continuous functions (complex or real valued) on a locally compact Hausdorff space, the theme of our next lecture (but we concentrate there on compact metric spaces). However, the Riesz Representation Theorems can be viewed as a philosophy, and so one can bring many theorems under its umbrella. Recall that a bounded linear transformation is the same as a continuous linear transformation. You are expected to remember the definition of the norm of a bounded linear operator, in particular of a bounded linear functional.

THEOREM 21.1. *Suppose and μ is a σ-finite measure on $(X, \mathcal{F})$, and p is an exponent with $1 \leq p < \infty$ and q its conjugate exponent. Then the (Banach space) dual of $L^p_{\mathbb{C}}(\mu)$ is $L^q_{\mathbb{C}}(\mu)$ in the following sense: If Λ is a bounded linear functional* *on $L^p_{\mathbb{C}}(\mu)$ then there is a unique $g_\Lambda \in L^q_{\mathbb{C}}(\mu)$ such that*

$$\Lambda f = \int_X f g_\Lambda \, d\mu \qquad (f \in L^p_{\mathbb{C}}(\mu)). \tag{1}$$

Moreover,

$$\|\Lambda\| = \|g_\Lambda\|_q. \tag{2}$$

PROOF. The uniqueness of g_Λ is an easy exercise best left to the reader. Note that, if (1) holds then by Hölder's inequality, we must have that

$$\|\Lambda\| \le \|g_\Lambda\|_q. \tag{*}$$

First assume that μ is a finite measure so that $\chi_E \in L^p(\mu)$ for $E \in \mathcal{F}$. Define $\nu(E) = \Lambda(\chi_E)$. One checks easily that if $E_n \uparrow E$ ($E_n \in \mathcal{F}$), then $\|\chi_{E_n} - \chi_E\|_p to 0$. This shows that ν is a complex measure. Clearly $\nu \ll \mu$. Hence by the Radon-Nikodym Theorem (see Exercise 69 of your Homework assigment), there exists a $g \in L^1_{\mathbb{C}}(\mu)$ such that

$$\Lambda(\chi_E) = \int_X \chi_E g \, d\mu.$$

It follows that

$$\Lambda f = \int_X f g \, d\mu \qquad (f \in L^\infty_{\mathbb{C}}(\mu))$$

for every $f \in L^\infty_{\mathbb{C}}(\mu)$ is the *uniform limit* of simple functions[3]. Note that we cannot use our usual MCT arguments because g is not non-negative (in fact g is complex).

If $p = 1$,

$$\left| \int_E g \, d\mu \right| \le \|\Lambda\| \cdot \|\chi_E\|_1 = \|\Lambda\| \cdot \mu(E) \qquad (E \in \mathcal{F}).$$

By Exercise 71 of your Homework assignment, $|g(x)| \le \|\Lambda\|$, and hence $g \in L^\infty_{\mathbb{C}}(\mu)$ and $\|g\|_\infty \le \|\Lambda\| < \infty$. This aalong with (*) shows that $\|g\|_\infty = \|\Lambda\|$.

Now suppose $1 < p < \infty$. We can always find a measurable function α, $|\alpha| = 1$, such that $\alpha g = |g|$. Indeed, on the set where g is zero, we take α to be 1, and outside this set we take $\alpha = |g|/g$. Let $E_n = \{|g| \le n\}$ and let $f = \chi_{E_n} |g|^{q-1} \alpha$. Then, on E_n, we have $|f|^p = |g|^q$. Moreover, since $f \in L^\infty_{\mathbb{C}}(\mu)$, therefore,

$$\begin{aligned} \int_{E_n} |g|^q \, d\mu &= \int_X f g \, d\mu \\ & \Lambda f \\ &\le \|\Lambda\| \cdot \|f\|_p \\ &= \|\Lambda\| \left\{ \int_{E_n} |g|^q \right\}^{\frac{1}{p}}. \end{aligned}$$

Since $1 - 1/p = 1/q$, we see from the above inequalities that

$$\left\{ \int_{E_n} |g|^q \, d\mu \right\}^{\frac{1}{q}} \le \|\Lambda\|,$$

[3]The point needs elaboration. We may assume that f is bounded, say $|f(x)| < M < \infty$ for all $x \in X$. Let $A_{nj} = \{f \in [-M + (j-1)M/n, -M + jM/n)\}$. Then, setting $S_n = \sum_{j=1}^{2n} (-M + (j-1)M/n)\chi_{A_{nj}} + M\chi_{\{f=M\}}$ we see that S_n converges to f uniformly. Moreover, $\|S_n - f\|_p \le \sup_{x \in X} |S_n(x) - f(x)| \mu(X)^{\frac{1}{p}}$, and hence $S_n \to f$ in $L^p(\mu)$ as $n \to \infty$.

i.e.

$$\int_X \chi_{E_n} |g|^q \, d\mu \le \|\Lambda\|^q.$$

Apply MCT, and get that $g \in L^q_{\mathbb{C}}(\mu)$ and moreover, $\|g\|_q \le \|\Lambda\|$. Now if Φ_g is defined on $L^p_{\mathbb{C}}(\mu)$ by $f \mapsto \int_X fg \, d\mu$, we see that Φ_g is a bounded linear functional, which agrees with Λ on $L^\infty_{\mathbb{C}}(\mu)$. Now $L^\infty_{\mathbb{C}}(\mu)$ is dense in $L^p_{\mathbb{C}}(\mu)$ (why ?) and hence $\Phi_g = \Lambda$. Set $g_\Lambda = g$. We have just seen that $\|g_\Lambda\|_q \le \|\Lambda\|$. Using this and (*), we see that we are through when μ is a finite measure.

Suppose now that $\mu(X) = \infty$. Since μ is σ-finite, we may write $X = \bigcup_n E_n$, where the $E_n \in \mathcal{F}$ are mutually disjoint, and such that $\mu(E_n) < \infty$. Define a function $h : X \longrightarrow (0, \infty)$ by $h = n^{-2}\mu(E_n)^{-1}\chi_{E_n}$. Then h is measurable, in fact $h \in L^1(\mu)$. Let ν be the measure given by the "indefinite integral" of h (i.e. $\nu(E) = \int_E h \, d\mu$, $E \in \mathcal{F}$). One checks that $F \mapsto h^{1/p}F$ is a linear isometry of $L^p_{\mathbb{C}}(\nu)$ onto $L^p_{\mathbb{C}}(\mu)$. Let L be the bounded linear functional on $L^p_{\mathbb{C}}(\nu)$ corresponding to Λ under the above isometry. By the first part of the proof, there is a $G_L \in L^q_{\mathbb{C}}(\nu)$ which "represents" L (i.e. $LF = \int_X FG_L \, d\nu$, for $F \in L^p_{\mathbb{C}}(\nu)$). One checks, readily that g_Λ may be taken to $h^{1/q}G_L$ (with the understanding that $1/q = 0$ if $p = 1$). ☐

COROLLARY 21.1. *The dual of the real Banach space $L^p(\mu)$ is $L^q(\mu)$ in the sense that if Λ is a bounded linear functional on $L^p(\mu)$, then there exists a unique $g_\Lambda \in L^q(\mu)$ such that Λ is given by $f \mapsto \int_X fg_\Lambda \, d\mu$ and $\|\Lambda\| = \|g_\Lambda\|_q$.*

PROOF. Clearly Λ can be extended in an obvious way to $L^p_{\mathbb{C}}(\mu)$ as a bounded (complex) functional, and moreover the norm of Λ is preserved under this extension. By the Theorem, we have a g_Λ representing this extended Λ. A little thought shows that g_Λ must necessarily be real-valued. The rest is easy. ☐

Lecture 7

This and the next lecture will concentrate on proving that the dual of the Banach space of continuous complex-valued functions on a compact metric space X is the space of complex measures on $(X, \mathcal{B}_X)$ (with the total variation as norm). The σ-algebra $\mathcal{B}_X$ is the "Borel σ-algebra" on X, i.e. the smallest σ-algebra containing the open sets of X. The real analogue is easily deducible, viz., the dual of the real Banach space of continuous real valued functions on X is the space of bounded signed measures on $(X, \mathcal{B}_X)$.

The approach is a disguised form of the Daniell approach, however I balked at having to reproduce the entire formalism of the Daniell integral, for something which is essentially down to earth (at least on metric spaces, or any space with a good Urysohn type Theorem). I have painted with broad strokes for lack of time prevented me from giving all the details, but there is nothing that cannot be filled in by a motivated and intelligent reader.

Usually, the Riesz Representation Theorem is proved for locally compact spaces, and the compact version becomes a corollary.

22. MCT Revisited

We will need the following strengthened form of the MCT

THEOREM 22.1. [MCT] *Let $(X, \mathcal{F}, \mu)$ be a measure space, and let $\{f_n\}$ be a sequence in $L^1(\mu)$ such that $f_n \uparrow f$ pointwise as $n \to \infty$. Then $f \in L^1(\mu)$ if and only if $\lim_n \int_X f_n \, d\mu < \infty$ and in this case $\int_X f \, d\mu = \lim_n \int_X f_n \, d\mu$.*

PROOF. Let $g_n = f_n - f_1$, $g = f - f_1$. Then $0 \le g_n \uparrow g$. Apply MCT. $\square$

23. The Borel σ-algebra $\mathcal{B}_X$

Let X be a metric space, and $\mathcal{B}_X$ its Borel σ-algebra. Now on a metric space, every closed set is G_δ. Indeed, if F is closed then $F = \bigcap_{n=1}^{\infty} \{x | \, d(x, F) < 1/n\}$. As a consequence, every open set on X is F_σ. Now let G be an open set in X. We claim that there is a continuous function f on X, with $0 \le f \le 1$, and such that $G = \{f > 0\}$. To see this, write $G = \bigcup_n F_n$, with each F_n closed. By Urysohn's lemma there are continuous functions f_n, $0 \le f_n \le 1$ such that $F_n = \{f_n = 1\}$ and $G = \{f_n > 0\}$. A little thought shows that the function $f = \sum_{n=1}^{\infty} 2^{-n} f_n$ is the required one.

Let $\mathcal{A}$ be the smallest σ-algebra such that every member of $C(X)$ is measurable ($C(X)$ is the Banach space of real-valued continuous functions on X). In other words $\mathcal{A}$ is the smallest σ-algebra containing sets of the form $f^{-1}(B)$, $f \in C(X)$, $B \in \mathcal{B}_{\mathbb{R}}$. Clearly $\mathcal{A} \subset \mathcal{B}_X$. On the other hand the argument in the previous paragraph shows that every open set G is in $\mathcal{A}$. Therefore, we have proved:

THEOREM 23.1. *$\mathcal{B}_X$ is the smallest σ-algebra on X such that every continous function $f : X \longrightarrow \mathbb{R}$ is measurable.*

REMARK 23.1. The above need not work for an arbitrary Hausdorff topological space. What is needed is a Urysohn Lemma *and* the fact that every open set is F_σ. While the Riesz Representation Theorem makes sense for locally arbitrary compact Hausdorff spaces, the statement is a complicated one, which takes into account the failure of the above Theorem for these more general spaces.

24. Positive Functionals

Let X be a compact metric space. As usual, let $\mathcal{B}_X$ denote its Borel σ-algebra. $C(X)$ will denote continuous real-valued functions on X.

DEFINITION 24.1. A linear functional

$$\Lambda : C(X) \longrightarrow \mathbb{R}$$

is said to be *positive* if $\Lambda f \geq 0$ for every *non-negative* continuous function f on X.

REMARK 24.1. If Λ is a positive linear functional on $C(X)$, then clearly $\Lambda f \geq \Lambda g$ if $f \geq g$. This also shows that Λ is bounded, and in fact $\|\Lambda\| = \Lambda(1)$.

For the rest of this section, fix a postive linear functional Λ on $C(X)$. We will construct a measure μ_Λ on $\mathcal{B}_X$ such that $\Lambda f = \int_X f \, d\mu_\Lambda$ for $f \in C(X)$. Later we will show that Borel measure with this property is unique. Actually, the process of constructing μ_Λ is such that we actually construct it on a σ-algebra which is larger than $\mathcal{B}_X$, and μ_Λ turns out to be complete with respect to this σ-algebra.

We use the symbol $f \vee g$ to denote $\max\{f, g\}$ and $f \wedge g$ for $\min\{f, g\}$ for any two functions on X. Note that if $f, g \in C(X)$, then $f \vee g$ and $f \wedge g$ are both in $C(X)$. Let $S_\uparrow$ be the set of *increasing sequences* $\{\phi_n\}$ *of continuous functions.* If $f : X \to \bar{\mathbb{R}}$ is an extended real valued function then $S_{\uparrow f}$ will denote the subset of $S_\uparrow$ whose limit is f. Let

$$L'_u = \{f : X \to \bar{\mathbb{R}} | \, S_{\uparrow f} \neq \emptyset\}.$$

Note that if $f, g \in L'_u$ then $f \vee g$ and $f \wedge g$ are in L'_u. The following observations are immediate for L'_u :

- If $\{\phi_n\} \in S_{\uparrow f}$, then $\lim_n \Lambda\phi_n$ exists as an extended real number.
- If $\{\phi_n\}, \{\psi_n\} \in S_{\uparrow f}$ then for each $m \in \mathbb{N}$, $\{\phi_m \vee \psi_n\}_n \in S_{\uparrow f}$ and $\phi_m \leq \phi_m \vee \psi_n$ for every $n \in \mathbb{N}$. Thus
$$\lim_{n\to\infty} \Lambda\phi_n \leq \lim_{n\to\infty} \Lambda(\phi_m \vee \psi_n).$$
- For each $m \in \mathbb{N}$, $(\phi_m \vee \psi_n - \psi_n) \downarrow 0$ as $n \to \infty$. Since we are dealing with continuous functions, the convergence is actually *uniform*[4]. Hence
$$\lim_{n\to\infty} \Lambda\phi_m \vee \psi_n = \lim_{n\to\infty} \Lambda\psi_n.$$
- It follows then that $\lim_{n\to\infty} \Lambda\phi_n \leq \lim_{n\to\infty} \Lambda\psi_n$. Since $\{\phi_n\}$ and $\{\psi_n\}$ were arbitrary members of $S_{\uparrow f}$ therefore
$$\lim_{n\to\infty} \Lambda\phi_n = \lim_{n\to\infty} \Lambda\psi_n.$$

[4]If $\{g_n\}$ is a sequence of continuous functions on the compact metric space X and $g_n \downarrow 0$ pointwise, then one checks easily that $\|g_n\|_\infty \downarrow 0$.

In view of the above, we can define

$$\Lambda : L'_u \longrightarrow \bar{\mathbb{R}}$$

in a well-defined way by the formula $\Lambda f = \lim_n \Lambda \phi_n$, where $\{\phi_n\} \in S_{\uparrow f}$.

Now define

$$L_u = \{f \in L'_u | \, \Lambda f < \infty\}.$$

Note that L_u is closed under sums (whenever the sum is defined), and under multiplication by positive real numbers. Note also that if $f, g \in L_u$ then $f \vee g$ and $f \wedge g$ are also in L_u. Clearly, Λ is real-valued on L_u.

We are trying to get to the largest possible linear space to which Λ can be extended as a positive linear functional. Consider functions $f :\longrightarrow \bar{\mathbb{R}}$ such that

$$\inf\{\Lambda g | \, g \geq f, g \in L_u\} = -\inf\{\Lambda h | \, h \geq -f, h \in L_u\}.$$

Denote the collection of such functions L_1. Clearly Λ extends to $f \in L_1$ as the above common value. A little thought shows that $\Lambda f < \infty$ for $f \in L_1$ and that $L_u \subset L_1$. Moreover, one checks that L_1 is a vector space and if $f, g \in L_1$, then so do $f \vee g$ and $f \wedge g$. Note that this means that $|f| \in L_1$ if $f \in L_1$ (for $f^+ = f \vee 0$ and $f^- = (-f) \wedge 0$). Λ is a positive linear functional on L_1, i.e. $\Lambda f \geq 0$ whenever $f \geq 0$ and $f \in L_1$.

One also has the following analogoues of the MCT (in the newer form given in this lecture) and DCT. We leave the proofs to the reader.

PROPOSITION 24.1. *With notations as above, we have*

(1) *Let $\{f_n\}$ be a sequence of functions in L_1, ans suppose $f_n \uparrow f$ as $n \uparrow \infty$. Then $f \in L_1$ if and only if $\lim_n \Lambda f_n < \infty$ and in this case $\Lambda f = \lim_n \Lambda f_n$.*
(2) *Let $\{f_n\}$ be a sequence of fucntions in L_1 such that $f_n \to f$ pointwise. Suppose there exists $g \in L_1$ such that $|f_n| \leq g$ for all $n \in \mathbb{N}$. Then $f \in L_1$ and $\Lambda f = \lim_n \Lambda f_n$.*

Note that since $C(X)$ consists of measurable functions, therefore so does L_u (being the limit of such functions. How about L_1 ? We will show that any function in L_1 is "close" to some measurable function. To that end, for a function (taking values in $\bar{\mathbb{R}}$) g on X, let $S_{g\downarrow}$ consist of sequences $\{\phi_n\}$ of functions in L_u such that $\phi_n \downarrow g$ and $\lim_n \phi_n > -\infty$. Note that if $S_{g\downarrow}$ is non-empty, then any sequence $\{\phi_n\}$ in L_u decreasing to g must necessarily be in $S_{g\downarrow}$. Moreover, if $\{\phi_n\}, \{\psi_n\} \in S_{g\downarrow}$ then $\lim_n \Lambda\phi_n = \lim_n \Lambda\psi_n$ and both are real-numbers. Let

$$L_{ul} = \{g : X \longrightarrow \bar{\mathbb{R}} | \, S_{g\downarrow} \neq \emptyset\}.$$

Then $L_{ul} \subset L_1$ (by Proposition 24.1) and clearly members of L_{ul} are measurable (being limits of such functions). The subscripts u and ul added to L are meant to evoke "upper" and "upper-lower" limits. The next theorem shows that L_1 and L_{ul} are nearly the same (or at least that Λ cannot distinguish them).

THEOREM 24.1. *Given $f \in L_1$ there exists $g \in L_{ul}$ such that $g \geq f$ and $\Lambda f = \Lambda g$.*

PROOF. By definition of L_1, for each $n \in \mathbb{N}$ there exists $h_n \geq f$ such that $h_n \in L_u$ and $\Lambda h_n \leq \Lambda f + 1/n$. Let $g_n = h_1 \wedge \ldots \wedge h_n$. Then $h_n \geq g_n \geq f$, and g_n is a decreasing sequence of functions in L_u. Let $g = \lim_n g_n$. One checks that g satisfies the conclusion of the theorem. ☐

Let

$$\mathcal{F} = \{A \in 2^X \,|\, \chi_A \wedge f \in L_1 \forall f \in L_1\}.$$

One can show that $\mathcal{F}$ is a σ-algebra on X, and that every $f \in L_1$ is $\mathcal{F}$-measurable. In particular every continuous function is $\mathcal{F}$-measurable. By Theorem 23.1, it follows that $\mathcal{F} \supset \mathcal{B}_X$. Since $\chi_A = \chi_A \wedge 1$ and $1 \in L_1$, therefore, $\chi_A \in L_1$ for every $A \in \mathcal{F}$. Define

$$\mu_\Lambda(A) = \Lambda(\chi_A).$$

This gives a measure on $\mathcal{F}$. Moreover, by MCT (in its strengthened form), DCT and Proposition 24.1, we see that

$$\Lambda f = \int_X f\, d\mu_\Lambda \qquad (f \in L_1).$$

Since $\mathcal{F} \supset \mathcal{B}_X$ we may consider μ_Λ to be a Borel measure.

Example 24.1. On $C([0,1])$ consider the positive fucntional Λ given by "Riemann integration" i.e. the functional $f \mapsto \int_0^1 f(t)\, dt$. If we follow the above procedure, then $\mathcal{F} = \mathcal{M}$, the Lebesgue σ-algebra, and μ_Λ is the Lebesgue measure m. This gives an alternate way of getting the Lebesgue σ-algebra and the Lebesgue measure.

25. The Riesz Representation Theorem for Positive Functionals

Theorem 25.1. *Let $\Lambda : C(X) \longrightarrow \mathbb{R}$ be a positive linear functional. Then there exists a unique finite Borel measure μ on X such that*

$$\Lambda f = \int_X f\, d\mu \qquad (f \in C(X)). \tag{*}$$

Moreover, if Λ and μ are so related then $\|\Lambda\| = \|\mu\|$.

Proof. We have already shown the existence of a μ satisfying $(*)$. The statement about norms follows by taking $f = 1$ in $(*)$. It remains to show uniqueness. So suppose μ is a Borel measure satisfying $(*)$. We have to show that $\mu = \mu_\Lambda$ on $\mathcal{B}_X$, where μ_Λ is the measure we constructed earlier. As before, extend Λ to L_u', L_u and L_1. By MCT (in the form we have given in this lecture) and the definition of Λ on L_u and L_{ul} we see that

$$\Lambda f = \int_X f\, d\mu \qquad (f \in L_{ul}).$$

Let $B \in \mathcal{B}_X$. Since $\chi_B \in L_1$ (indeed, $B \in \mathcal{F}$ and hence by definition of the σ-algebra $\mathcal{F}$, $\chi_B \wedge 1 \in L_1$. But $\chi_B \wedge 1 = \chi_B$), therefore by Theorem 24.1 we have $g \geq \chi_B$, $g \in L_{ul}$ with $\Lambda g = \Lambda(\chi_B)$. Since $g \in L_{ul}$ therefore this can be rephrased as

$$\Lambda(\chi_B) = \int_X g\, d\mu. \tag{†}$$

Let $\varphi = g - \chi_B$. Then φ is $\mathcal{B}_X$-measurable (since g is, being a member of L_{ul}), $\varphi \geq 0$ and $\varphi \in L_1$ (being the difference of two members in L_1). Now $\Lambda\varphi = 0$. Applying Theorem 24.1 again, there exists a $\psi \in L_{ul}$ with $\psi \geq \varphi$ and $\Lambda\psi = \Lambda\varphi$. Thus

$$0 \leq \int_X \varphi\, d\mu \leq \int_X \psi\, d\mu = \Lambda\psi = \Lambda\varphi = 0 \tag{‡}$$

Putting together (†) and (‡) we see that

$$\mu_\lambda(B) = \Lambda(\chi_B) = \int_X g\,d\mu = \int_X g\,d\mu - \int_X \varphi\,d\mu = \int_X \chi_B\,d\mu = \mu(B).$$

Thus $\mu_\Lambda = \mu$ on $\mathcal{B}_X$. □

Lecture 8

In this lecture we prove the rest of the Riesz Representation Theorem, viz., bounded linear functionals on $C_{\mathbb{C}}(X)$ are represented uniquely (and in a norm preserving way) by complex Borel measures on X, where X is a compact metric space. Here $C_{\mathbb{C}}(X)$ is the complex Banach space (norm being the supremum norm) of continuous complex valued functions on X.

26. Some Lemmas

By a complex valued simple function on a measurable space $(X, \mathcal{F})$ we mean complex valued measurable functions s such that $s(X)$ is a finite set in $\mathbb{C}$.

LEMMA 26.1. *Let $(X, \mathcal{F}, \mu)$ be a measure space. Then the class S of all complex valued simple functions on X which are non-zero on a set of finite measure is dense in $L^p_{\mathbb{C}}(\mu)$ for every $p \in [1, \infty)$.*

PROOF. Clearly $S \subset L^p(\mu)$. Suppose $f \geq 0$ is p-integrable. Let s be a simple function such that $0 \leq s \leq f$. Then $s \in L^p(\mu)$ and so $s \in S$. Moreover, in this case $|f - s|^p \leq f^p$. We can find a sequnce of simple functions $\{s_n\}$, $0 \leq s_n \leq s_{n+1} \leq f$ such that $s_n \uparrow f$. Since $|f - s_n|^p \leq f^p$ therefore by the DCT, $\|f - s_n\|_p \to 0$ as $n \to \infty$. If $f \in L^p_{\mathbb{C}}(\mu)$, then breaking f into its real and imaginary parts, and then breaking each into its positive and negative parts, we can find a sequence in S approaching f in $L^p_{\mathbb{C}}(\mu)$. □

LEMMA 26.2. *Let X be a compact metric space*[5]*. Let μ be a measure on $(X, \mathcal{B}_X)$. Then $C_{\mathbb{C}}(X)$ is dense in $L^p_{\mathbb{C}}(\mu)$.*

PROOF. Let S be as above. Lusin's theorem can clearly be extended to show that given $s \in S$ and $\epsilon > 0$, one can find a $g \in C_{\mathbb{C}}(X)$ which agrees with s everywhere except possible a set of μ-measure less than ϵ, and such that $\|g\|_\infty \leq \|s\|_\infty$. Hence

$$\|g - s\|_p \leq 2\epsilon^{\frac{1}{p}} \|s\|_\infty.$$

Thus the closure of $C_{\mathbb{C}}(X)$ in $L^p_{\mathbb{C}}(\mu)$ contains S. But by Lemma 26.1, S is dense in $L^p_{\mathbb{C}}(\mu)$. Hence we are done. □

Recall that λ is a complex measure, then there is a $h \in L^1_{\mathbb{C}}(|\lambda|)$, with $|h| = 1$ such that $\lambda(E) = \int_E h\, d|\lambda|$ (see Exercise 72 of your Homework assignment). Recall that integration with repsect to λ is defined as by $\int_X f d\lambda := \int_X f h\, d|\lambda|$.

[5]the requirement of being compact or being a metric space are not strictly necessary

Lemma 26.3. *Let X be a compact metric space, and λ a complex measure on $(X, \mathcal{B}_X)$ such that*

$$\int_X f\, d\lambda = 0 \qquad (f \in C_{\mathbb{C}}(X)).$$

Then $\lambda = 0$.

Proof. By Exercise 72 of your Homework assignment, $d\lambda = h\, d|\lambda|$, where h is Borel measurable with $|h| = 1$. We claim that

$$|\lambda|(X) \leq \|\bar{h} - f\|_1 \qquad (f \in C_{\mathbb{C}}(X)) \tag{*}$$

where the norm on the right is that of $L^1_{\mathbb{C}}(|\lambda|)$. Suppose $(*)$ is true. By Lemma 26.2, we can make the quantity $\|\bar{h} - f\|_1$ as small as we wish. This implies that $|\lambda|(X) = 0$. Hence $\lambda = 0$. The relation $(*)$ is shown as follows. Let $f \in C_{\mathbb{C}}(X)$. Then

$$\int_X fh\, d|\lambda| = \int_X f\, d\lambda = 0$$

by our hypothesis on λ. Therefore

$$|\lambda|(X) = \int_X \bar{h}h\, d|\lambda| = \int_X (\bar{h}h - fh)\, d|\lambda| \leq \int_X |\bar{h} - f| d|\lambda| = \|\bar{h} - f\|_1.$$

Hence we are done. □

27. The Riesz Representation Theorem

Theorem 27.1. *The dual of the complex Banach space $C_{\mathbb{C}}(X)$ is the space of complex measures on $(X, \mathcal{B}_X)$. More precisely, given a bounded linear functional*

$$\Lambda : C_{\mathbb{C}}(X) \longrightarrow \mathbb{C}$$

there exists a unique complex measure λ on $(X, \mathcal{B}_X)$ such that $\|\Lambda\| = \|\lambda\|$ and

$$\Lambda f = \int_X f\, d\lambda \qquad (f \in C_{\mathbb{C}}(X)). \tag{$*_\Lambda$}$$

Proof. Uniqueness of a λ satisfying $(*)_\Lambda$ follows from Lemma 26.3. Thus we only have to show existence of a λ satisfying $(*)_\Lambda$ and such that $\|\Lambda\| = \|\lambda\|$. We may assume that $\Lambda \neq 0$. In this case, by multipying by $\|\Lambda\|^{-1}$ if necessary, we may assume, without loss of generality that $\|\Lambda\| = 1$.

Let $C^+(X)$ be the set of nonnegative continuous functions on X. Let $L : C^+(X) \longrightarrow \mathbb{R}$ be given by

$$Lf = \sup\{|\Lambda h| \mid h \in C_{\mathbb{C}}(X), |h| \leq f\}.$$

Then

- $Lf \geq 0$.
- $0 \leq Lf_1 \leq Lf_2$ if $0 \leq f_1 \leq f_2$.
- If $c > 0$ is a constant, then $Lcf = cLf$ for $f \in C^+(X)$.
- $|\Lambda f| \leq L(|f|)$.

We will now show that for $f, g \in C^+(X)$, $L(f + g) = Lf + Lg$. Given $\epsilon > 0$, by definition of L, there exist f' and g' such that $|f'| \leq f$, $|g'| \leq g$ and $Lf \leq |\Lambda f'| + \epsilon$, $Lg \leq |\Lambda g'| + \epsilon$. Let a, b be complex numbers of modulus 1 such that $a\Lambda f' = |\Lambda g'|$ and $b\Lambda f' = |\Lambda f'|$. Then

$$Lf + Lg \leq |\Lambda f'| + |\Lambda g'| + 2\epsilon = \Lambda(af' + bg') + 2\epsilon \leq L(|f'| + |g'|) + 2\epsilon \leq L(f + g) + 2\epsilon.$$

In other words

$$L(f+g) \le Lf + Lg. \tag{$\dagger$}$$

On the other hand, suppose $h \in C_{\mathbb{C}}(X)$ is such that $|h| \le f+g$. Let $K = \{f = 0\} \cap \{g = 0\}$, and let V be the complement of K in X. On V define $h_1 = fh/(f+g)$, $h_2 = gh/(f+g)$ and on K define $h_1 = h_2 = 0$. One checks that h_1 and h_2 are continuous on X, $h_1 + h_2 = h$, $|h_1| \le f$ and $|h_2| \le g$. It follows that $|\Lambda h| = |\Lambda h_1 + \Lambda h_2| \le |\Lambda h_1| + \Lambda h_2| \le Lf + Lg$. By definition of L, we then get

$$L(f+g) \le Lf + Lg. \tag{$\ddagger$}$$

Thus, from (†) and (‡) we see that $L(f+g) = Lf + Lg$, for $f, g \in C^+(X)$. Now if $f \in C(X)$, then $f^+, f^- \in C^+(X)$, and so one defines $Lf = Lf^+ - Lf^-$. This way, we extend L to all of $C(X)$. Clearly, L is a positive linear functional. Moreover,

$$\|L\| = L(1) = \sup\{\Lambda f | |f| \le 1\} = \|\Lambda\| = 1.$$

Let μ be the measure representing L ensured by the Riesz Representation for positive functionals. Then

$$\mu(X) = \|\mu\| = \|L\| = 1.$$

For $1 \le p \le \infty$, let $\|\cdot\|_p$ denote the norm in $L^p_{\mathbb{C}}(\mu)$. We then have

$$|\Lambda f| \le L(|f|) = \|f\|_1 \qquad (f \in C_{\mathbb{C}}(X)).$$

In other words, Λ, when thought of as a functional on the normed linear space $(C_{\mathbb{C}}(X), \|\cdot\|_1)$ (this is different from the usual norm on $C_{\mathbb{C}}(X)$) is a bounded linear functional with norm less than or equal to 1.[6]. Now $(C_{\mathbb{C}}(X), \|\cdot\|_1)$ is dense in $L^1_{\mathbb{C}}(\mu)$ (see Lemma 26.2), and therefore there is a norm preserving extension of Λ to $L^1_{\mathbb{C}}(\mu)$. By Riesz Representation for $L^1_{\mathbb{C}}(\mu)$, the extended functional is represented by $g \in L^\infty_{\mathbb{C}}(\mu)$, with $|g| \le 1$ (we are using the fact that the norm of Λ as a functional on $L^1_{\mathbb{C}}(\mu)$ is at most 1). In other words,

$$\Lambda f = \int_X fg \, d\mu \qquad (f \in L^1_{\mathbb{C}}(\mu)).$$

We thus get a *complex measure* λ given by,

$$\lambda(B) = \int_B g \, d\mu \qquad (B \in \mathcal{B}_X),$$

and clearly λ satisfies $(*_\Lambda)$. It only remains to show that $\|\lambda\| = 1$.

Let $f \in C_{\mathbb{C}}(X)$ with $\|f\|_\infty \le 1$. Then,

$$\int_X |g| \, d\mu \ge |\int_X fg \, d\mu| = |\Lambda f|.$$

This gives,

$$\int_X |g| \, d\mu \ge \|\Lambda\| = 1.$$

But $|g| \le 1$, and $\|\mu\| = 1$. The only way the above three facts can be reconciled is to have $|g| = 1$ *a.e.* $-[\mu]$. It follows by Exercise 73 of your Homework assignment, that $|\lambda| = \mu$ and hence $\|\lambda\| = \|\mu\| = 1$. □

The following corollary is obvious.

[6] $C_{\mathbb{C}}(X)$ is complete with respect to its usual norm, viz., $\|\cdot\|_\infty$, but need not be complete with respect to $\|\cdot\|_1$ as we can see from Lemma 26.2

Corollary 27.1. *The dual of the real Banach space $C(X)$ is the space of bounded signed measures on $(X, \mathcal{B}_X)$ in the sense that given a bounded linear functional Λ on $C(X)$, there exits a unique bounded Borel signed measure μ such that $\|\mu\| = \|\Lambda\|$ and*

$$\Lambda f = \int_X f \, d\mu \qquad (f \in C(X)).$$

Example 27.1. Let $I = [a, b]$ be a closed bounded interval in $\mathbb{R}$. Then a bounded signed measure on $(I, \mathcal{B}_I)$ gives rise to a function of bounded variation on I in the following way. Let σ be a bounded signed measure on our measurable space. Define

$$g_\sigma(x) = \sigma([a, x)) \qquad (x \in I).$$

One checks that $g_\sigma(a) = 0$, g_σ is *left continuous* and is a function of bounded variation. Moreover, for any continuous function f on I one checks easily that

$$\int_I f \, d\sigma = \int_a^b f \, dg_\sigma .$$

If we have a function $g \in BV(I)$ such that g is left continuous, then we can construct a bounded signed measure by first defining it for the algebra of intervals given by $[\alpha, \beta)$ and $[\alpha, b]$ for $\alpha, \beta \in I$. On such intervals, we define it by

$$\sigma_g[\alpha, \beta) = g(\beta) - g(\alpha).$$

and

$$\sigma_g[\alpha, b] = g(b) - g(\alpha).$$

A canonical procedure now allows us to extend σ_g to all of $\mathcal{B}_X$. Moreover, two left continuous functions of bounded variation will give rise to the same signed measure if and only if they differ by a constant. So if we concentrate on left continuous functions $g \in BV(I)$ such that $g(a) = 0$, then we get a one-to-one correspondence with bounded signed measures. The two processes mentioned are inverses of each other. And under this correspondence the total variation of σ is the same as the variation of g_σ. Thus the dual of $C(I)$ can be identified with left continuous fucntions of bounded variation on I, which are zero at a.