NURTURE 1996-2000
ANALYSIS HOMEWORK

Instructions. The following is a list of 79 problems. Your Analysis Exam next year, for most part, will be based on these problems. I will lay special emphasis on Radon-Nikodym theory, Riesz Representation Theorems, Fubini and Tonelli theorem.

You are expected to submit the following 10 problems by 1/11/98:
26, 27, 28, 29, 30, 31, 33, 34, 35 and 43;
and the following 10 by 1/2/99:
60, 61, 62, 63, 64, 65, 66, 67, 68 and 60;
and the following 10 by 1/5/99:
70, 71, 72, 73, 74, 75, 76, 77, 78 and 79.

Please do not submit more problems than the 10 assigned in a given period, for I may not have time to correct them. You may quote results from other problems (including the ones not assigned) provided the quoted result occurs in a problem before the one you are trying to solve. You are free to discuss the remaining 49 problems with me over E-mail or usual mail. I will try to respond.

Notations. The notations of last year's assignment will continue to be in force. In addition we will have the following notations and conventions.

Let X be a set. The symbol 2^X will denote the *power set* of X, i.e. the set of all subsets of X. An *increasing sequence* $\{A_i\}$ in 2^X means a sequence of subsets $A_1 \subset A_2 \subset A_3 \ldots \subset A_n \ldots$. The symbol

$$A_i \uparrow A$$

will be used to denote the fact that $\{A_i\}$ is increasing and $\cup_i A_i = A$. We often write $\lim_i A_i$ to denote $\cup_i A_i$. Similarly if $A_1 \supset A_2 \ldots \supset A_n \supset \ldots$ then $\{A_i\}$ is said to be a *decreasing sequence* (in 2^X). If $A = \cap A_i$, then we write $\lim_i A_i = A$ and we use the notation

$$A_i \downarrow A$$

to indicate that $\{A_i\}$ decreases to A.

$\bar{\mathbb{R}}$ will denote the extended real line. A *function* on a set X will always mean a function from X to $\bar{\mathbb{R}}$, unless otherwise specified. If $(X, \mathcal{F})$ is a measurable space, then by a *function on* $(X, \mathcal{F})$ we mean a function on X which is measurable with respect to $\mathcal{F}$.

If $(X, \mathcal{F})$ and $(Y, \mathcal{G})$ are measurable spaces, then a map $f : X \to Y$ is said to be *measurable with respect to the pair* $(\mathcal{F}, \mathcal{G})$ if $f^{-1}(E) \in \mathcal{F}$ for every $E \in \mathcal{G}$. We use the shorthand $f : (X, \mathcal{F}) \to (Y, \mathcal{G})$ to denote the fact that f is measurable with respect to $(\mathcal{F}, \mathcal{G})$. If the context is clear, we simply say f is measurable for f as above.

Note that a measurable *function* on $(X, \mathcal{F})$ is a map $(X, \mathcal{F}) \to (\bar{\mathbb{R}}, \bar{\mathcal{B}})$ where $\bar{\mathcal{B}}$ is the σ-algebra of *Borel* sets on $\bar{\mathbb{R}}$, i.e. the smallest σ-algebra containing the sets $\{x | \, x > a\}$, $a \in \mathbb{R}$.

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MCT and DCT will be shorthands for the *Monotone Convergence Theorem* and the *Dominated Convergence Theorem.*

If $(X, \mathcal{F})$ is a measurable space, and $E \in \mathcal{F}$ then $\mathcal{F} \cap E$ will denote the subset of 2^E given by all $B \in \mathcal{F}$ such that $B \subset E$. It is easy to verify that $\mathcal{F} \cap E$ is a σ-algebra on E. If μ is a measure on $(X, \mathcal{F})$, then $\mu|_E$ will denote the restriction of μ to $\mathcal{F} \cap E$. Note that $\mu|_E$ is a measure on $(E, \mathcal{F} \cap E)$. If the context is clear, we will write μ for $\mu|_E$.

Convex Functions

(1) Let $a < x \leq x' < y' < b$ and $a < x < y \leq y' < b$. Let φ be a convex function on (a, b). Show that the chord (of the graph of φ) over (x', y') has larger slope than the chord over (x, y); that is
$$\frac{\varphi(y) - \varphi(x)}{y - x} \leq \frac{\varphi(y') - \varphi(x')}{y' - x'}.$$

(2) Suppose φ is convex on (a, b). Show that φ is *absolutely continuous* on every closed interval $[c, d] \subset (a, b)$, that is, if $[c, d]$ is such an interval, then given $\epsilon > 0$, there is a $\delta > 0$ such that
$$\sum_{i=1}^{n} |\varphi(x_i') - \varphi(x_i)| < \epsilon$$
for every finite collection $\{(x_i, x_i')\}$ of non-overlapping intervals in $[c, d]$ with
$$\sum_{i=1}^{n} |x_i - x_i'| < \delta.$$

Lebesgue Measure

Define
$$m^* : 2^{\mathbb{R}} \to [0, \infty]$$
by
$$m^*(A) = \inf\{\sum |I_n| : A \subset \cup_{n \geq 1} I_n^0\}.$$
The function m^* is called the (Lebesgue) *outer measure* on $\mathbb{R}$. A set $E \subset \mathbb{R}$ is said to be *measurable* if for every $A \subset \mathbb{R}$,
$$m^*(A) = m^*(A \cap E) + m^*(A \setminus E)$$
This definition of measurability of a set is due to *Caratheodory.* Let $\mathcal{M} \subset 2^{\mathbb{R}}$ be given by
$$\mathcal{M} = \{E \in 2^{\mathbb{R}} | E \,\text{measurable}\}.$$

(3) (Monotonicity) Show that if $A \subset B$, $m^*(A) \leq m^*(B)$.

(4) Show
  (a) $m^*(\emptyset) = 0$,
  (b) $m^*(A) = 0$ for a countable set A (we include finite sets as countable sets),
  (c) $m^*(\mathbb{R}) = \infty$.

(5) (Countable subadditivity) Show $m^*(\cup_{j \geq 1} A_j) \leq \sum_{j \geq 1} m^*(A_j)$

(6) Show that $m^*(I) = b - a$, where $I = [a, b]$.

(7) Show that $m^*(I^O) = |I|$.

(8) Show that $\mathcal{M}$ is a σ-algebra. Show also that if $m : \mathcal{M} \to [0, \infty]$ is the restriction of m^* to $\mathcal{M}$ then m is a measure - *the Lebesgue measure !* [*Hint*: It is easier to prove both parts together, rather than separately. It is easy to prove that $\emptyset \in \mathcal{M}$ and that $\mathcal{M}$ is closed under complements. Break up the rest of the proof into the following steps. *Step 1* Show that $\mathcal{M}$ is closed under finite unions. *Step 2* Show that if $E_1, \ldots, E_n$ is a finite collection of pairwise disjoint subsets of $\mathbb{R}$ with each $E_j \in \mathcal{M}$ then $m^*(A \cap (\cup_1^n E_j)) = \sum_{j=1}^n m^*(A \cap E_j)$ for every $A \subset \mathbb{R}$. *Step 3* If $\{E_j\}_{j\geq 1}$ is a countable collection of disjoint sets in $\mathcal{M}$, then $m(\cup_{j\geq 1} E_j) = \sum_{j=1}^\infty m(E_j)$. *Step 4* Show that $\mathcal{M}$ is closed under countable unions.]

(9) Let $\mathcal{B}$ be the smallest σ-algebra containing the open subsets of $\mathbb{R}$. $\mathcal{B}$ is called the *Borel* σ-algebra. Show that $\mathcal{B} \subset \mathcal{M}$.

(10) Show that $\mathcal{B}$ contains G_δ and F_σ.

(11) Let $A \subset \mathbb{R}$. Show
   (a) For every $\epsilon > 0$, there exists an open set $G_\epsilon \supset A$ such that $m(G_\epsilon) \leq m^*(A) + \epsilon$.
   (b) There exists a G_δ set H such that $H \supset A$ such that $m(H) = m^*(A)$.
   (c) Any measurable set in $\mathcal{M}$ is of the form $E = K \cup N$, where $K \in F_\sigma$ and $m(N) = 0$.

(12) Show that the following are equivalent:
   (a) $E \in \mathcal{M}$.
   (b) For every $\epsilon > 0$ there exists an open set G_ϵ containing E such that $m^*(G_\epsilon \setminus E) \leq \epsilon$.
   (c) There exists $H \supset E$, H a G_δ set, such that $m^*(H \setminus E) = 0$.
   (d) For every $\epsilon > 0$, there exists $F_\epsilon \subset E$, F_ϵ a closed set, such that $m^*(E \setminus F_\epsilon) \leq \epsilon$.
   (e) There exists $K \subset E$ such that $K \in F_\sigma$ and $m^*(E \setminus K) = 0$.

(13) Let $E \in \mathcal{M}$, $x_o \in \mathbb{R}$. Show
   (a) $E + x_o \in \mathcal{M}$.
   (b) $m(E + x_o) = m(E)$.
   [Hint: Write $E = K \cup N$ where K is F_σ and N is of zero measure. Then consider appropriate tranlates].

(14) Suppose μ is a measure on $(\mathbb{R}, \mathcal{B}$ such that $\mu([0, 1]) < \infty$, and such that for $E \in \mathcal{B}$, and $x_o \in \mathbb{R}$, $\mu(E + x_o) = \mu(E)$ (i.e. μ is *translation invariant*). Then show that there exists a real number $\alpha \geq 0$, such that $\mu = \alpha m$, where m is the Lebesgue measure on $(\mathbb{R}, \mathcal{B})$. (From this one can conclude that μ extends to $\mathcal{M}$).

(15) Define an equivalence relation $\sim$ on I_o as follows: $x \sim y$ if $x - y$ is rational. Let $I_o = \cup_{\alpha\in\Gamma} K_\alpha$ be the corresponding decomposition of I_o into equivalence classes (so that $K_\alpha \cap K_\beta = \emptyset$ if $\alpha \neq \beta$). Let $T : \Gamma \to I_o$ be a map such that such that $T(\alpha) \in K_\alpha$ for every $\alpha \in \Gamma$. By the Axiom of Choice such a T exists. Let

$$E = \{T(\alpha) | \alpha \in \Gamma\}.$$

Show that E is not Lebesgue measurable, i.e. show that E does not lie in $\mathcal{M}$.

(16) Let $E \in \mathcal{M}$, with $m(E) < \infty$, and let $\alpha \in (0, m(E))$. Show that there exists a Lebegue measurable set F in E such that $m(F) = \alpha$.

(17) Suppose $f : \mathbb{R} \to \mathbb{R}$ is $\mathcal{M}$-measurable and $g : \mathbb{R} \to \mathbb{R}$ is $\mathcal{B}$-measurable. Show that $g \circ f$ is $\mathcal{M}$-measurable.

Measure Theory

A measure space $(X, \mathcal{F}, \mu)$ is said to be *complete* if $\mathcal{F}$ contains all the subsets of sets of measure zero, that is, if $E \in \mathcal{F}$, $\mu(E) = 0$, and $D \subset E$, then $D \in \mathcal{F}$.

(18) Let $(X, \mathcal{F}, \mu)$ be a measure space. Show that

$$\mu(E_1) + \mu(E_2) = \mu(E_1 \cup E_2) + \mu(E_1 \cap E_2) \qquad E_i \in \mathcal{F}.$$

(19) (a) Show that the Lebesgue measure space $(\mathbb{R}, \mathcal{M}, m)$ is complete (see definition above).
(b) Show that $(\mathbb{R}, \mathcal{B}, m)$ is *not* complete.

(20) If $(X, \mathcal{F}, \mu)$ is a measure space, show that we can find a unique complete measure space $(X, \widehat{\mathcal{F}}, \widehat{\mu})$ such that
(a) $\mathcal{F} \subset \widehat{\mathcal{F}}$;
(b) $\widehat{\mu}|\mathcal{F} = \mu$;
(c) $E \in \widehat{\mathcal{F}} \iff E = A \cup B$ where $B \in \mathcal{F}$ and $A \subset C$, $C \in \mathcal{F}$, $\mu(C) = 0$.
$(X, \widehat{\mathcal{F}}, \widehat{\mu})$ is called the *completion* of $X, \mathcal{F}, \mu)$.

(21) Let Z be a set. A *monotone class* $\mathfrak{M}$ is a collection of subsets such that if $\{A_i\}$ is an increasing sequence of sets in $\mathfrak{M}$, and $\{B_i\}$ is a decreasing sequence of sets in $\mathfrak{M}$, then $\cup_i A_i$ and $\cap_i B_i$ belong to $\mathfrak{M}$. Show that if $\mathcal{E}$ is a collection of sets in Z, then there is a unique monotone class $\mathfrak{M}$ containing $\mathcal{E}$, and contained in every monotone class containing $\mathcal{E}$. $\mathfrak{M}$ is called the *smallest monotone class containing* $\mathcal{E}$. It is also called the *monotone class generated by* $\mathcal{E}$.

(22) Let $\mathcal{A}$ be an *algebra* of subsets of a non-empty set X, i.e. $\mathcal{A}$ is closed under finite unions, complementations, and $\emptyset, X \in \mathcal{A}$. Show
(a) $\mathcal{A}$ is closed under finite intersections.
(b) By a measure μ on $\mathcal{A}$ we mean a map $\mu : \mathcal{A} \to [0, \infty]$ such that $\mu(\emptyset) = 0$ and if $\{E_i\}$ are a countable collection of disjoint sets in $\mathcal{A}$ such that $\cup_i E_i \in \mathcal{A}$, then μ satisfies the formula for countable additivity. Let $\mathcal{F}$ be the σ-algebra generated by $\mathcal{A}$. Show that μ extends to a unique measure, also denoted μ, on $\mathcal{F}$. [Hint: Imitate the construction of Lebesgue measure in exercises on the Lebesgue measure.]

(23) Let $(X, \mathcal{F})$ and $(Y, \mathcal{G})$ be two measure spaces. A *measurable rectangle* in $X \times Y$ is a set of the form $A \times B$ with $A \in \mathcal{F}$ and $B \in \mathcal{G}$. The *product* of $\mathcal{F}$ and $\mathcal{G}$ is defined to be the smallest σ-algebra containing measurable rectangles. For $E \subset X \times Y$ and $x \in X$, define

$$E_x = \{y | \, (x, y) \in E\}.$$

Similarly for $y \in Y$, define

$$E^y = \{x | \, (x, y) \in E\}.$$

E_x is called the x-section of E and E^y the y-section of E. Show that if $E \in \mathcal{F} \times \mathcal{G}$, then $E_x \in \mathcal{G}$, and $E^y \in \mathcal{F}$ for every $x \in X$ and $y \in Y$. [Hint: Enough to show $E_x \in \mathcal{G}$. Let Ω be the class of all sets $E \in \mathcal{F} \times \mathcal{G}$ such that

$E_x \in \mathcal{G}$ for every $x \in X$. Show that Ω contains measurable rectangles and that Ω is a σ-algebra. Conclude that $\Omega = \mathcal{F} \times \mathcal{G}$.]

(24) Let $(X, \mathcal{F})$ and $(Y, \mathcal{G})$ be as in the previous problem. Let $\mathcal{E}$ be the *class of elementary sets*, i.e. sets of the form $Q = R_1 \cup \ldots \cup R_n$, where each R_i is a measurable reactangle, and $R_i \cap R_j \neq \emptyset$. Show that $\mathcal{F} \times \mathcal{G}$ is the smallest monotone class containing $\mathcal{E}$.

(25) Let $(X, \mathcal{F})$ and $(Y, \mathcal{G})$ be as above. For a function f on $X \times Y$, and an element $x \in X$, define a function f_x on Y by $f_x(y) = f(x, y)$. Similarly define f^y on X by $f^y(x) = f(x, y)$. Show that if f is $\mathcal{F} \times \mathcal{G}$ measurable, then
   (a) For each $x \in X$, f_x is $\mathcal{G}$-measurable.
   (b) For each $y \in Y$, f^y is $\mathcal{F}$-measurable.
   [Hint: Consider $Q = f^{-1}(V)$, where V is an open set. By Exercise 23, $Q_x \in \mathcal{G}$. Conclude that f_x is measurable.]

Integration

(26) Show that if $\{f_n\}$ is a sequence of nonnegative functions on a measurable space $(X, \mathcal{F})$, then

$$\int_X \sum_{n\geq 1} f_n \, d\mu = \sum_{n\geq 1} \int_X f_n \, d\mu.$$

(27) (Fatou's Lemma) Suppose $\{f_n\}$ is a sequence of non-negative functions on $(X, \mathcal{F})$. Show that

$$\int_X \liminf f_n \, d\mu \leq \liminf \int_X f_n \, d\mu.$$

(28) Let $(X, \mathcal{F}, \mu)$ be a measure space and let $E \in \mathcal{F}$. For g a function on $(X, \mathcal{F})$ show that

$$\int_X g\chi_E \, d\mu = \int_E (g|_E) \; d\,(\mu|_E)$$

where the above has the following meaning:

If either side converges, then both sides converge and are equal, and if either side diverges properly (i.e. does *not converge* and is *not* of the form $\infty - \infty$), then both sides diverge properly and to the same value (either $+\infty$ or $-\infty$). We will, in subsequent problems, write $\int_E g \, d\mu$ to denote $\int_X g\chi_E \, d\mu$.

(29) Let $f : (X, \mathcal{F}) \longrightarrow (Y, \mathcal{G})$ be a map between measurable spaces (i.e. f is measurable). Let μ be a measure on $(X, \mathcal{F})$ and define a measure $f_*\mu$ on $(Y, \mathcal{G})$ by

$$(f_*\mu)\,(E) = \mu\left(f^{-1}(E)\right).$$

Show the following *abstract change of variables formula*:

$$\int_{f^{-1}(E)} g \circ f \, d\mu = \int_E g \, df_*\mu,$$

where g is a measurable function on Y, $E \in \mathcal{G}$ and the above has the following meaning - if either side is finite then both sides are finite and equal, and if either side diverges properly (i.e. is $+\infty$ or $-\infty$) then both sides diverge properly and to the same value.

(30) Let $(X, \mathcal{F}, \mu)$ be a measure space, and $f \geq 0$ a function on $(X, \mathcal{F})$. For $E \in \mathcal{F}$, set

$$\nu(E) = \int_E f\, d\mu.$$

Show

(a) $\nu : \mathcal{F} \to [0, \infty]$ is a measure on $(X, \mathcal{F})$.

(b) If g is another a function on $(X, \mathcal{F})$, show

$$\int_X g\, d\nu = \int_X gf\, d\nu$$

where as usual the meaning assigned to the above is that if either side converges then both sides converge and are equal and if either side diverges properly, then both sides diverge properly and to the same infinite value. [Hint: First prove it for $g \geq 0$ by using simple functions and the MCT. Then do the usual trick of breaking up g into its positive and negative parts.]

(31) Let $f \in C(I)$, $I = [a, b]$. Show that

$$\int_a^b f\, dx = \int_I f\, dm$$

where the left side is the Riemann integral and the on the right side m denotes the Lebesgue measure.[Hint: As usual first assume $f \geq 0$, and apply MCT to carefully chosen sequence of simple functions $s_n \uparrow f$. Then do the usual little tricks.]

(32) Show that there exists a function f on $(\mathbb{R}, \mathcal{M})$ such that $\int_{\mathbb{R}} |f|\, dm < \infty$ and such that f is *essentially unbounded*, i.e.

$$m\{x \in I |\, f(x) \geq n\} > 0$$

for all intervals I and for all $n \in \mathbb{N}$. [Hint: Consider $F = g\chi_{[-1,1]}$ where $g(x) = |x|^{\frac{1}{2}}$ for $x \neq 0$ and $g(0) = 125$. Next consider

$$f(x) = \sum_{n \geq 1} \frac{1}{2^n} F(x - r_n),$$

where r_n is the n-th rational (in some enumeration of $\mathbb{Q}$). Show that $\|F\|_1 < \infty$ and hence show that $\|f\|_1 < \infty$. Prove that f has the required properties.]

(33) Suppose $(X, \mathcal{F}, \mu)$ is a measure space and f is a function on $(X, \mathcal{F})$ such that $f \geq 0$ almost everywhere. Suppose further that

$$\int_X f\, d\mu = 0.$$

Show that $f = 0$ almost everywhere.

(34) Let $f \in L^1(\mathbb{R})$ be such that

$$\int_I f\, dm = 0$$

for all intervals I. Show that $f = 0$ almost everywhere.

(35) Let $(X, \mathcal{F}, \mu)$ be a measure space. Let f be a non-negative function on $(X, \mathcal{F})$ which is integrable over X. Show that given $\epsilon > 0$, there exists a $\delta > 0$ such that for every measurable set $A \subset X$, with $\mu(A) < \delta$, we have
$$\int_A f \, d\mu < \epsilon.$$

(36) Recall that $f : I \longrightarrow \mathbb{R}$ is said to be *absolutely continuous* if given $\epsilon > 0$ there is a $\delta > 0$ such that $\sum_{i=1}^{n} |f(x_i') - f(x_i)| < \epsilon$ for every finite collection $\{(x_i, x_i')\}$ of non-overlapping intervals in I with $\sum_{i=1}^{n} |x_i' - x_i| < \delta$. Show that if $f : I \longrightarrow \mathbb{R}$ is an indefinite integral then it is absolutely continuous, where by an indefinite integral we mean $f(x) = f(a) + \int_{[a,x]} g \, dm$ for some integrable function g on $I = [a, b]$. [Hint: Use Exercise 35.]

(37) Show that if f is absolutely continuous on I, then it is of bounded variation (I a finite length interval).

(38) Let f be absolutely continuous on I. Show that f has a derivative a.e. Show also that if $f'(x) = 0$ a.e., then f is a constant.

(39) Let f be absolutely continuous on $[a, b]$, $a, b \in \mathbb{R}$. Show that f is an indefinite integral. In fact show that $f(x) = f(a) + \int_{[a,\, x]} f' \, dm$, for all $x \in [a, b]$.

(40) (Feijer) Let $f : \mathbb{R} \to \mathbb{R}$ be bounded and measurable and satisfying $f(x+1) = f(x)$ for all $x \in \mathbb{R}$. Let $f_n : \mathbb{R} \to \mathbb{R}$ be given by $f_n(x) = f(nx)$, $n \in \mathbb{N}$. Show that
$$\int_{[a,b]} f \, dm \to (b-a) \int_{[0,1]} f \, dm.$$

(41) With notations as in Exercise 40, suppose there is a subsequence of $\{f_n\}$ which converges pointwise. Show that f is a constant.

(42) This is an analogue of Exercise 46 of last year's Analysis assignment. Let $(X, \mathcal{F}, \mu)$ be a measure space. Let $1 \le p \le \infty$ and let q be such that $1/p + 1/q = 1$ (so that $q = 1$ when $p = \infty$ and $q = \infty$ when $p = 1$. Note that $1 \le q \le \infty$).
  (a) For f, g measurable prove *Hölder's inequality*:
  $$\|fg\|_1 \le \|f\|_p \|g\|_q.$$
  Conclude that if $f \in L^p(\mu)$ and $g \in L^q(\mu)$ then $fg \in L^1(\mu)$. For $p = 2$ this gives the *Cauchy-Schwarz* inequality for the inner product (actually Hilbert) space $L^2(\mu)$.
  (b) Prove *Minkowski's inequality*, i.e.
  $$\|f + g\|_p = \|f\|_p + \|g\|_p$$
  for any two measurable functions f and g. Conclude that $L^p(\mu)$ is a vector space with obvious notion of addition and scalar multiplication and that $\|\cdot\|_p$ defines a norm on $L^p(\mu)$. [Hint: Consider the function $x \mapsto x^p$. Show that it is convex. Apply Jensen's inequality] [Remark: The assertion in this problem was not true for $L^p(h)$ of last year's assignment. Our apologies.]

(43) Let $(X, \mathcal{F}, \mu)$ be a measure space and let $1 \le p_1 \le p_2 \le \infty$.
  (a) If $\mu(X) < \infty$, show that $L^{p_2}(\mu) \subset L^{p_1}(\mu)$.
  (b) If $\mu(X) = \infty$, show that there is in general no inclusion relationship between $L^{p_1}(\mu)$ and $L^{p_2}(\mu)$. In other words, give an example to show

that $L^{p_1} \setminus L^{p_2}(\mu)$ could be non-empty and an example to show that $L^{p_2} \setminus L^{p_1} \neq \emptyset$.

(44) With above notations, if r is a number between p_1 and p_2, and $p_2 < \infty$, show that

$$L^{p_1}(\mu) \cap L^{p_2}(\mu) \subset L^r(\mu).$$

(45) Let $f_n \to f$ in $L^p(\mu)$.
  (a) Show there exists a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ which converges pointwise to f.
  (b) Give an example to show that it is possible for $\{f_n(x)\}$ not to converge at any point $x \in X$.

Fubini's Theorem

The following exercises are meant to prove the Fubini Theorem. Exercise 48 (and sometimes Exercise 25) are referred to as the Tonelli Theorem, and Exercises 49 and 50 together form the Fubini Theorem. Sometimes the collection of all these statements (Tonelli and Fubini) are clumped together as the Fubini Theorem. In general, any statement asserting that the "iterated integrals" are equal and each equals the "integral", is called a Fubini Theorem.

(46) Let $(X, \mathcal{F}, \mu)$ and $(Y, \mathcal{G}, \nu)$ be σ-finite measure spaces. Let Ω be the class of all $Q \in \mathcal{F} \times \mathcal{G}$ such that
  (a) The functions φ and ψ on X and Y respectively given by:

$$\varphi(x) = \nu(Q_x)$$
$$\psi(x) = \mu(Q^y)$$

  are $\mathcal{F}$ and $\mathcal{G}$ measurable respectively.
  (b) $\int_X \varphi \, d\mu = \int_Y \psi \, d\nu$.

Show that Ω has the following four properties
  (a) Every measurable rectangle belongs to Ω.
  (b) If $Q_i \uparrow Q$, $Q_i \in \Omega$, $i \in \mathbb{N}$, then $Q \in \Omega$. [Hint: Use MCT.]
  (c) If $\{Q_i\}$ are a countable disjoint collection, $Q_i \in \Omega$, then $\cup_i Q_i \in \Omega$.
  (d) If $Q_i \downarrow Q$, each $Q_i \subset A \times B$, $A \in \mathcal{F}$, $B \in \mathcal{G}$, $\mu(A) < \infty$, $\nu(B) < \infty$, then $Q \in \Omega$. [Hint: Use DCT.]

(47) Let the notations be as in Exercise 46. Since μ and ν are σ-finite, we can write $X = \cup_{n \in \mathbb{N}} X_n$ and $Y = \cup_{n \in \mathbb{N}} Y_n$ where the $\{X_n\}$ are disjoint, as are the $\{Y_n\}$, and $\mu(X_n), \nu(Y_n) < \infty$ for all n. For $Q \in \mathcal{F} \times \mathcal{G}$, let

$$Q_{m,n} = Q \cap (X_n \times Y_m) \qquad (m, n) \in \mathbb{N} \times \mathbb{N}.$$

Let $\mathfrak{M}$ be the class of all $Q \in \mathcal{F} \times \mathcal{G}$ such that $Q_{m,n} \in \Omega$ for all choices of m and n. Show that
  (a) $\mathfrak{M}$ is a monotone class. Hence conclude $\mathfrak{M} = \mathcal{F} \times \mathcal{G}$.
  (b) $\Omega = \mathfrak{M}$.
  (c)

$$\int_X \nu(Q_x) \, d\mu(x) = \int_Y \mu(Q^y) \, d\nu(y)$$

  for every $Q \in \mathcal{F} \times \mathcal{G}$.

(d) Show that the map $\mu \times \nu : \mathcal{F} \times \mathcal{G} \to [0, \infty]$ given by

$$\mu \times \nu(Q) = \int_X \nu(Q_x)\, d\mu(x) \left(= \int_Y \mu(Q^y)\, d\nu(y) \right)$$

is a σ-finite measure on $(X \times Y, \mathcal{F} \times \mathcal{G})$. $\mu \times \nu$ is called the *product measure of* μ and ν.

(48) Let notations be as in Exercise 47. Let f be a non-negative function on $(X \times Y, \mathcal{F} \times \mathcal{G})$. Show that $x \mapsto \int_Y f_x\, d\nu$ and $y \mapsto \int_X f^y\, d\mu$ are $\mathcal{F}$ and $\mathcal{G}$ measurable respectively, and

$$\begin{aligned}\int_X \left(\int_Y f_x(y)\, d\nu(y) \right) d\mu(x) &= \int_{X\times Y} f\, d\,(\mu \times \nu) \\ &= \int_Y \left(\int_X f^y(x)\, d\mu(x) \right) d\nu(y).\end{aligned}$$

[Hint: Use characteristic functions, simple functions, $\dots$.]

(49) With notations as in Exercise 48, let f be a function on $(X \times Y, \mathcal{F} \times \mathcal{G})$. Show that if $|f|_x \in L^1(\nu)$ and $\int_X (|f(x,y)|)\, d\mu(x) < \infty$, then $f \in L^1(\mu \times \nu)$. [Hint: Use Exercise 48 on $|f|$].

(50) If $f \in L^1(\mu \times \nu)$, then show $f_x \in L^1(\nu)$ for almost all $x \in X$ and $f^y \in L^1(\mu)$ for almost all $y \in Y$. Show also that the functions $x \mapsto \int_Y f_x\, d\nu$ and $y \mapsto \int_X f^y\, d\mu$ are $\mathcal{F}$ and $\mathcal{G}$ measurable respectively, and

$$\begin{aligned}\int_X \left(\int_Y f_x(y)\, d\nu(y) \right) d\mu(x) &= \int_{X\times Y} f\, d\,(\mu \times \nu) \\ &= \int_Y \left(\int_X f^y(x)\, d\mu(x) \right) d\nu(y).\end{aligned}$$

[Hint: Break up f into f^+ and f^- and use Exercise 48.]

(51) Let $X = Y = [0, 1]$, $\mu = \nu = m$, where, as usual m is the Lebesgue measure. Let $0 \le \delta_n \downarrow 1$. with $\delta_1 = 0$. Let g_n be a real continous function with $\operatorname{Supp} g_n \subset (\delta_n,\, \delta_{n+1})$, and such that $\int_{[0,1]} g_n\, dm = 1$, $n \in \mathbb{N}$. Set

$$f(x, y) = \sum_{n=1}^{\infty} [g_n(x) - g_{n+1}(x)]\, g_n(y).$$

Note that f is pointwise convergent. Show that the two iterated integrals of f are not equal. Why is Fubini inapplicable ?

(52) Let $X = Y = [0,\, 1]$, $\mathcal{F} = \mathcal{M}$ (the Lebesgue σ-algebra), $\mathcal{G} = 2^X$, $\mu = m$, and $\nu =$ the counting measure on $(Y, \mathcal{G})$. Consider the characteristic function χ_Δ of the diagonal $\Delta = \{(x, x) |\, x \in [0,\, 1]\}$. Show that the two iterated integrals of χ_Δ (with respect to μ and ν) are not equal. Why is Fubini inapplicable ?

(53) Suppose $(X,\, \mathcal{F},\, \mu)$, $(Y,\, \mathcal{G},\, \nu)$ and $(Z,\, \mathcal{H},\, \lambda)$ are three measure spaces. Show

(a) $(\mathcal{F} \times \mathcal{G}) \times \mathcal{H} = \mathcal{F} \times (\mathcal{G} \times \mathcal{H})$ on $X \times Y \times Z$.

(b) Assume μ, ν, λ are σ-finite. Show that $(\mu \times \nu) \times \lambda = \mu \times (\nu \times \lambda)$. Note that this means we can talk about an arbitrary number of products of σ-algebras and measures, without putting parenthesis.

(c) Let $(X_i,\, \mathcal{F}_i,\, \mu_i)$, $i = 1, \dots , n$ be n σ-finite measure spaces. As usual, let $(X_i,\, \widehat{\mathcal{F}_i},\, \widehat{\mu_i})$ denote the completion of $(X_i,\, \mathcal{F}_i,\, \mu_i)$, $i = 1, \dots , n$. For any set S contained in $\{1, 2, \dots , n\}$, let $\mathcal{F}_S = \mathcal{G}_1 \times \ldots \times \mathcal{G}_n$ where

$\mathcal{G}_i = \mathcal{F}_i$ if $i \notin S$ and $\mathcal{G}_i = \widehat{\mathcal{F}_i}$ if $i \in S$. Similarly let μ_S be the measure on $\mathcal{F}_S$ given by $\mu_S = \nu_1 \times \ldots \times \nu_n$, where $\nu_i = \mu_i$ if $i \notin S$ and $\nu_i = \widehat{\mu_i}$ if $i \in S$. Show that for subsets S, P of $\{1, \ldots, n\}$, the completion of $(\prod_{i=1}^n X_i,\, \mathcal{F}_S,\, \mu_S)$ is the same as the completion of $(\prod_{i=1}^n X_i,\, \mathcal{F}_P,\, \mu_P)$.

Remarks: Let $(X, \mathcal{F}, \mu)$ and $(Y, \mathcal{G}, \mu)$ be *complete* σ-finite spaces. Let $(X \times Y,\, (\mathcal{F} \times \mathcal{G})^*,\, (\mu \times \nu)^*)$ be the completion of $(X \times Y,\, \mathcal{F} \times \mathcal{G},\, \mu \times \nu)$. Then one can show that all the conclusions of Exercises 48, 49 and 50 are true, the only difference being as follows:
The $\mathcal{G}$ measurability of f_x and the $\mathcal{F}$ measurability of f^y can only be asserted for almost all $x \in X$ and almost all $y \in Y$ respectively, and so $x \mapsto \int_Y f_x \, d\nu$ can only be defined a.e.-μ. A similar statement holds for f^y and $y \mapsto \int_X f^Y \, d\mu$.

Advanced Several Variables Calculus

Let $(\mathbb{R}^d, \mathcal{M}_d, m_d)$ be the completion of $(\mathbb{R}^d, \prod_{i=1}^d \mathcal{M}, \prod_{i=1}^d m)$ where $\mathcal{M}$ is the Lebesgue σ-algebra on $\mathbb{R}$ and m is the Lebesgue measure on $(\mathbb{R}, \mathcal{M})$. Here we are viewing $\mathbb{R}^d$ as the product of d copies of $\mathbb{R}$. The σ-algebra $\mathcal{M}_d$ is called the *d-dimensional Lebesgue σ-algebra* and the measure m_d is called the *d-dimensional Lebesue measure*. Note that in view of Exercise ex:prod, the measure space $(\mathbb{R}, \mathcal{M}_d, m_d)$ has many other descriptions.

If the context is clear, we will drop the subscript d from $\mathcal{M}_d$ and m_d.

(54) A *half-open cube in $\mathbb{R}^d$ of side b* is a set of the form $\{(x_1, \ldots, x_d) |\, a_i \leq x_i < a_i + b\}$, where $(a_1, \ldots, a_d) \in \mathbb{R}^d$. The *centre* of this cube is defined to be the point $(a_1 + b/2, \ldots, a_d + b/2)$. Show that any open set in $\mathbb{R}^d$ is a disjoint, countable union of half open cubes.

(55) Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation. Show that $m(T(C)) = |\det T|$, where C is the unit cube in $\mathbb{R}^n$. Conclude that $m(T(B)) = |\det T| m(B)$ for every $B \in \mathcal{B}$.

(56) Let $\mathcal{B}_d$ be the σ-algebra on $\mathbb{R}^d$ generated by open sets. $\mathcal{B}_d$ is called the d-dimensional Borel σ-algebra. If the context is clear, we write $\mathcal{B}$ for $\mathcal{B}$.
   (a) Show that $\mathcal{B}_d = \mathcal{B} \times \ldots \times \mathcal{B}$, where the product is taken d–times.
   (b) Show that $\mathcal{B}_d \subset \mathcal{M}_d$.
   (c) Show that the completion of $(\mathbb{R}^d, \mathcal{B}_d, m_d)$ is the Lebesgue measure space.

(57) Show that the d-dimensional Lebesgue measure is translation invariant.

(58) For a vector $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$, let $\|x\|_\infty = max_{1 \leq i \leq d} |x_i|$. Let U be an open subset of $\mathbb{R}^d$, $F : U \to \mathbb{R}^d$ a C^1 function. Let x_o and x_1 be points in U such that the entire line segment joining these two points is in U and such that there is a constant C for which $\|F'(x)\| \leq C$ holds for all points x in the line segment joining x_o and x_1. Here $\| \; \|$ denotes the *operator norm.* Show that

$$\|F(x_1) - F(x_o)\|)_\infty \leq C \|x_1 - x_o\|_\infty.$$

(59) Let U be an open subset of $\mathbb{R}^d$, and $G : U \to \mathbb{R}^d$ a differentiable function. Let $\epsilon < 0$ be given and suppose C is a closed cube in U such that

$\|G'(x) - I_d\| \le \epsilon$ for all $x \in C$. Here I_d is the identity linear transformation on $\mathbb{R}^d$. Show that

$$m^*\left(G(C)\right) \le (1+\epsilon)^d m(C)$$

where m^* is the outer Lebesgue measure on $\mathbb{R}^d$ defined in a manner analogous to its definition in the 1-dimensional case. [Hint: Let x_o be the centre of C, and b the length of any side of C. Apply Exercise 58 to $G - I_d$. Conclude that $G(C)$ is contained in a cube of side $(1+\epsilon)\cdot b$ centred at at $G(x_o)$.]

(60) Let U, V, T be as in the *Change of Variables Theorem* for functions on $\mathbb{R}^d$. Let $a > 0$, B a Borel set contained in U such that $|J_T| \le a$ on B. Show that

$$m(TB) \le a \cdot m(B).$$

[Hint: First assume that B is open and its closure $\bar{B}$ is compact and contained in U. In this special case, show (using compactness) that (a) there is an $M > 0$ such that $\|\left(T^{-1}\right)'(Tx)\| \le M$ for all $x \in \bar{B}$, and (b) for every $\epsilon > 0$, there is a $\delta > 0$ such that $\|T'(x) - T'(x_o)\| \le \epsilon/M$ for every x, x_o in $\bar{B}$. Next write B as the disjoint union of countable family of half open cubes $\{C - i\}$. Choose each C_i so small that each has edge length $\le 2\delta$. Let C be one of these cubes, and x_o its centre. Define $G : U \to \mathbb{R}^d$ by $G = S \circ T(x)$ where S is the linear transformation $\left(T'(x_o)\right)^{-1}$. Show that $\|G'(x_o) - I_d\| \le \epsilon$ for all $x \in C$. Apply Exercise 59 to conclude $m(G(C)) \le (1+\epsilon)^d m(C)$. ¿From this conclude that $m(T(C) \le a \cdot (1+\epsilon) m(C)$ and hence $m(T(B)) \le a \cdot (1+\epsilon) m(B)$. Next show that the assertion is true for arbitrary open sets B in U by using a limiting process. ¿From here to Borel sets should be easy.]

Hilbert Spaces

Assume all spaces are over $\mathbb{R}$, if you are not comfortable with complex inner product spaces. Throughout H is a Hilbert space.

(61) (a) Show that for all $x, y \in H$

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2.$$

(b) If E is a non-empty convex subset of H show that

$$\|x-y\|^2 \le 2\|x\|^2 + 2\|y\|^2 - 4\left\{d(E,0)\right\}^2$$

for $x, y \in E$. Here $d(E,0)$ is the distance of E from 0.[Hint: Consider the average of x and y.]

(c) If E is above and is *closed* then show that E has a unique element of smallest norm.[Hint: Let $z_n \in E$ be such that $\|z_n\| \longrightarrow d(E,0)$. Use above inequality to show $\{z_n\}$ converges.]

(62) Let M be a closed subspace of H. Let

$$M^\perp = \{x \in H \mid < x,\, y >= 0,\, y \in M\}$$

Show that $M^\perp$ is a closed subspace of H.

(63) Let M be a closed subspace of H. Show that there exists a unique pair of linear mappings P, Q:

$$P : H \to M$$
$$Q : H \to M^{\perp}$$

such that $P + Q = I$, the identity map, and such that
(a) $P|_M = I_M$, $P|_{M^{\perp}} = 0$, $Q|_M = 0$ and $Q|M^{\perp} = I_{M^{\perp}}$.
(b) $\|x - Px\| = d(x, M)$.
(c) $\|x\|^2 = \|Px\|^2 + \|Qx\|^2$.
The map P is called the *orthogonal projection* of H onto M. Clearly, then, Q is the orthogonal projection of H onto $M^{\perp}$. Note that the last property ensures that P and Q are bounded. [Hint: For any $x \in H$, let Qx be the unique element of smallest norm in $M + x$. Define $Px = x - Qx$.]

(64) (Riesz Representation for Hilbert Spaces) Let $H \longrightarrow \mathbb{K}$ be a continuous linear functional on H ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, depending on whether H is a real or a complex Hilbert space). Show that there is a unique $y_L \in H$ such that

$$Lx = < x, y_L > \qquad (x \in H).$$

Show also that $\|L\| = \|y_L\|$. [Note that if H is a real Hilbert space this establishes an *isometric isomorphism* $H^* \xrightarrow{\sim} H$, viz. $L \mapsto y_L$. In other words, the dual of a real Hilbert space can be completely identified with itself. The general philosophy of the various Riesz Representation Theorems is to "find" a concrete avatar of the dual space of a given Banach Space. The complex version of this theorem says that the association $L \mapsto y_L$ is *conjugate linear*, but it is a isometric onto map from H^* to H.]

The Radon-Nikodym Derivative

Equalities involving the Radon-Nikodym derivative must be regarded as true almost everywhere (with respect to a suitable measure, clear from the context).

In what follows we will use a freer concept of a measurable function on the measure space $(X, \mathcal{F}, \mu)$, viz., f is said to be measurable with respect to $\mathcal{F}$ if there is a set $E \in \mathcal{F}$ of μ-measure zero on which f is possible undefined, and on $X \setminus E$ is real-valued (or complex valued) and measurable. This is, strictly speaking, not a generalisation of the older concept of a measurable function, but is so if f is in $L^p(\mu)$ or $L^p_{\mathbb{C}}(\mu)$.

Two measures μ and ν on $(X, \mathcal{F})$ are said to be *mutually singular* (written $\mu \perp \nu$) if there are disjoint sets A and B in $\mathcal{F}$ such that $A \cup B = X$ and $\nu(A) = \mu(B) = 0$. We sometimes say that ν is singular with respect to μ.

In the case of signed or complex measures we say $\nu \ll \mu$ if $|\nu| \ll |\mu|$ and $\nu \perp \mu$ if $|\nu| \perp |\mu|$.

(65) Let μ, ν and λ be three measures on $(X, \mathcal{F})$.
(a) Suppose $d\nu/d\mu$ and $d\mu/d\lambda$ exist. Show that $d\nu/d\lambda$ exists and

$$\frac{d\nu}{d\lambda} = \frac{d\nu}{d\mu} \cdot \frac{d\mu}{d\lambda}.$$

(b) Suppose $d\nu/d\mu$ exists and is non-zero $a.e.-[\mu]$. Then show that $d\mu/d\nu$ exists and
$$\frac{d\mu}{d\nu} = \frac{1}{d\nu/d\mu}.$$
(c) Suppose $d\mu/d\lambda$ and $d\nu/d\lambda$ exist and $d\mu/d\lambda > 0$ a.e.-[λ]. Show that $d\nu/d\mu$ exists and
$$\frac{d\nu}{d\mu} = \frac{d\nu/d\lambda}{d\mu/d\lambda} \qquad \text{a.e.-}[\mu].$$
[Hint: For the first two parts use Exercise 30. Remember you cannot use the Radon-Nikodym Theorem since we are not assuming σ-finiteness of any of the measures involved.]

(66) Suppose μ and ν are finite measures on $(X, \mathcal{F})$ and $\lambda = \mu + nu$. Let $g = d\nu/d\lambda$ (which clearly exists since μ, ν are finite and $\mu \ll \lambda$).
(a) Show that $0 \leq g \leq 1$ $a.e. - [\lambda]$.
(b) Show that $1 - g = \frac{d\mu}{d\lambda}$.
(c) Show that if $f \in L^2(\lambda)$, then
$$\int_X f(1-g)\, d\nu = \int_X fg\, d\mu.$$

(67) Let μ, ν, g etc., be as in Exercise 66, and assume without loss of generality that $0 \leq g \leq 1$. Let $A = \{g < 1\}$, $B = \{g = 1\}$ so that $A \cup B = X$ and $A \cap B = \emptyset$. Set
$$\nu_a(E) = \nu(A \cap E), \quad \nu_s(E) = \nu(B \cap E) \qquad (E \in \mathcal{F}).$$
Note that ν_a and ν_s are measures and $\nu = \nu_a + \nu_s$.
(a) Show that $\mu(B) = 0$, $\nu_s(E) = \nu_s(B \cap E)$, i.e. $\mu \perp \nu_s$.
(b) Show that there is a unique $h \in L^1(\mu)$ such that
$$\nu_a(E) = \int_E h\, d\mu \qquad (E \in \mathcal{F}).$$
[Hint: One way of doing the last part is to consider $f = 1+g+\ldots+g^n$, apply the last part of Exercise 66, and then let $n \uparrow \infty$.]
(c) Show that if $\nu \ll \mu$, then $\nu(B) = 0$ and hence $\nu_a = \nu$.

The decomposition $\nu = \nu_a + \nu_s$ is called the *Lebesgue decomposition* of ν with respect to μ. The measure ν_s is called the *singular part* of ν (with respect to μ) and ν_a is called the *absolutely continuous* part of ν (with respect to μ).

(68) Show that the Lebesgue decomposition (see notes at the end of Exercise 67 is unique.

(69) Show that the Lebesgue decomposition and the Radon-Nikodym theorem hold if ν is a complex measure and μ is σ-finite. In other words we can write
$$\nu = \nu_a + \nu_s$$
such that $\nu_s \perp \mu$, $\nu_a \ll \mu$ and there exists a unique $h \in L^1(\mu)$ such that $d\nu_a = h\, d\mu$. In particular, if $\nu \ll \mu$, $\nu_s = 0$ and $d\nu/d\mu$ exits. [Hint: Break up ν into real and imaginary parts, and break up each of these into positive and negative parts].

(70) Suppose μ is a measure and ν is a complex measure. Show that the following are equivalent

(a) $\nu \ll \mu$.
(b) To every $\epsilon > 0$ there corresponds a $\delta > 0$ such that if $E \in \mathcal{F}$ saistifes $\mu(E) < \delta$, then $|\nu(E)| < \epsilon$.

(71) Let μ be a finite measure on $(X, \mathcal{F})$ and $f \in L^1_{\mathbb{C}}(\mu)$. Suppose $S \subset \mathbb{C}$ is a closed subset and the averages

$$A_E(f) = \frac{1}{\mu(E)} \int_E f \, d\mu$$

lie in S for every $E \in \mathcal{F}$ such that $\mu(E) > 0$. Show that $f(x) \in S$ for almost all $x \in X$.

(72) Let μ be a complex measure. Show that there is a $\mathbb{C}$-valued function h on X (the ambient space of μ) such that $|h| = 1$ and such that $d\mu = h \, d|\mu|$. [Hint: Use the Radon-Nikodym Theorem and Exercise 71.]

(73) Suppose μ is a measure on $(X, \mathcal{F})$, $g \in L^1_{\mathbb{C}}(\mu)$. Let λ be the complex measure on $\mathcal{F}$ given by

$$\lambda(E) = \int_E g \, d\mu \qquad (E \in \mathcal{F}).$$

Show that

$$|\lambda|(E) = \int_E |g| \, d\mu.$$

[Remark: This is a generalisation of Exercise 35(b) of your last year's assigment.]

(74) Let μ be a σ-finite signed measure (and hence a real-valued complex measure) on $(X, \mathcal{F})$. Show that there exist $A, B \in \mathcal{F}$ such that $A \cup B = X$, $A \cap B = \emptyset$ and

$$\mu^+(E) = \mu(A \cap E), \quad \mu^-(E) = -\mu(B \cap E) \qquad (E \in \mathcal{F}).$$

[Hint: First assume μ is bounded so that it is a complex measure. Let $h = d\mu/d|\mu|$. Set $A = \{h = 1\}$ and $B = \{h = -1\}$.] This decomposition is called the *Hahn decomposition*, and is valid for any signed measure (not necessarily σ-finite).

(75) Show that the decomposition $\mu = \mu^+ - \mu^-$ is minimal in the following sense. Let μ be a σ-finite signed measure and suppose $\mu = \lambda_1 - \lambda_2$ where λ_1 and λ_2 are measures. Show that $\lambda_1 \geq \mu^+$ and $\lambda_2 \geq \mu^-$.

1. Riesz Representation

The exercises in this section include those involving duals of L^p spaces.

(76) Suppose $1 \leq p \leq \infty$ and q is the exponent conjugate to p. Suppose μ is a σ-finite measure and g is a complex valued measurable function such that $fg \in L^1_{\mathbb{C}}(\mu)$ for every $f \in L^p_{\mathbb{C}}(\mu)$. Prove that $g \in L^q_{\mathbb{C}}(\mu)$.

(77) Suppose $X = \{a, b\}$, and on $(X, 2^X)$ we define a measure μ given by $\mu(\{a\}) = 1$ and $\mu(\{b\}) = \mu(X) = \infty$. Show that $L^\infty(\mu)$ is not the dual of $L^1(\mu)$. Why does our Theorem on duals of L^p spaces fail here ?

(78) Suppose $1 < p < \infty$, and suppose q is the exponent conjugate to p. Show that the dual of $L^p_{\mathbb{C}}(\mu)$ is $L^q_{\mathbb{C}}(\mu)$ even if μ is not σ-finite.

(79) Let X be a compact metric space such that $C(X)$ is reflexive (i.e. the imbedding of $C(X)$ into its double dual is surjective. Recall that the embedding is an isometry). Show that X must be a finite set.

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