NURTURE 1996-2000
ANALYSIS HOMEWORK

Instructions. Submit at least 20 problems by 1/4/98 and at least 20 more by 1/6/98. You can bring the remaining solutions with you for the contact programme.

Notations. The symbol I will be used to denote a closed and bounded interval in the real line. Its lower limit will be denoted a and its upper limit b, so that $I = [a, b]$. The interior of I will be denoted I^o. In other words $I^o = (a, b)$. The special interval $[0, 1]$ will be written I_o. The length of an interval I will be denoted $|I|$. The space of continuous real valued functions on I will be denoted $C(I)$. Real numbers and complex numbers will be written (as usual) $\mathbb{R}$ and $\mathbb{C}$ respectively.

Basic Topology

(1) Let K be a compact subset of $\mathbb{R}$ and F a closed subset of $\mathbb{R}$. Show that there exists $x_1 \in K$ and $x_2 \in F$ such that

$$|x_1 - x_2| = \inf_{x' \in K, x'' \in F} \{|x' - x''|\}.$$

(2) Let E be a subset of $\mathbb{R}$. Show that E contains a countable subset D such that D is dense in E (i.e. the closure of D in E is E. Here we are thinking of E as a metric space in its own right with metric inherited from $\mathbb{R}$).

(3) Given $0 < \alpha < 1$, construct a closed, nowhere dense, perfect set $C_\alpha \subset I_o$ such that

$$I_o \setminus C_\alpha = \cup_{j \geq 1} I_j^\circ \qquad \text{and} \qquad \sum |I_j| = \alpha.$$

Here I_j, $j \in \mathbb{N}$ are intervals.

(4) Let H be a subset of $\mathbb{R}$ which is the intersection of a countable number of open subsets of $\mathbb{R}$. Show that if H is dense in $\mathbb{R}$, then it is uncountable.

Sequences and Series

In the problems in this section discover and prove a relationship between

(5) $\limsup(a_n + b_n)$ and $\limsup a_n + \limsup b_n$.
[Hint: Compare $\{a_n\}$ and $\{b_n\}$ given by $a_n = (-1)^{n+1}$ and $b_n = (-1)^n$.]

(6) $\liminf(a_n - b_n)$ and $\liminf a_n - \limsup b_n$.
[Hint: Compare $\{a_n\}$ and $\{b_n\}$ given by:

$$a_n = \begin{cases} -1 & \text{if } n \text{ odd} \\ 0 & \text{otherwise} \end{cases}$$

and

$$b_n = \begin{cases} 1 & \text{if } n \text{ odd} \\ 2 & \text{otherwise}]. \end{cases}$$

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(7) $\limsup(a_n^{b_n})$ and $\limsup(a_n)^{\liminf b_n}$.

Sequences and Series of Functions

(8) Let $f_n \in C(I)$, $n \in \mathbb{N}$ and let f_n converge to f pointwise on I. Suppose that for every Cauchy sequence $\{x_n\}$ in I we have

$$f_n(x_n) \longrightarrow f(\lim x_n).$$

Show that $f_n \longrightarrow f$ uniformly on I.

(9) Let $f \in C(I_o)$. Define $f_n \in C(I_o)$ by

$$f_n(x) = f(x^n) \qquad x \in I.$$

Show that $\{f_n\}$ is equicontinuous if and only if f is a constant. [Hint: Arzela-Ascoli].

(10) A real-valued function f on I_o is said to be *Hölder continuous* of order α if there is a constant C such that $|f(x) - f(y)| \leq C|x - y|^\alpha$. Define

$$||f||_\alpha = \max|f(x)| + \sup \frac{|f(x) - f(y)|}{|x - y|^\alpha}.$$

Show that for $0 < \alpha \leq 1$, the set of functions with $||f||_\alpha \leq 1$ is a compact subset of $C(I_o)$. [Hint: Arzela-Ascoli].

(11) Consider

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{1 + n^2 x}.$$

- (a) For what values of x does the series converge absolutely ?
- (b) On what intervals does it converge uniformly ?
- (c) On what intervals does it fail to converge uniformly ?
- (d) Is f continuous wherever the series converges ?
- (e) Is f bounded ?

(12) (Helly's selection theorem) Let $f_n : I \to \mathbb{R}$ be a sequence of non-decreasing functions on I and let M be a non-negative real number such that

$$|f_n(x)| \leq M.$$

Show that there exists a subsequence $\{f_{n_k}\}$ which converges pointwise on I to a non-decreasing function f. Show that if f is continuous, then $f_{n_k} \longrightarrow f$ uniformly on I.

(13) Let f be continuous periodic real-valued function on $\mathbb{R}$ with period 2π; that is, $f(x + \pi) = f(x)$. Show that, given $\epsilon > 0$, there is a finite series φ given by

$$\varphi = a_0 + \sum_{n=1}^{N} (a_n \cos nx + b_n \sin nx)$$

such that $|\varphi(x) - f(x)| < \epsilon$ for all x. [Hint: Note that periodic functions of period 2π are really functions on the unit circle in the complex plane. Apply Stone-Weierstrass (Thm. 7.32 of Rudin) to an appropriate algebra of functions on the unit circle. After doing the problem, it might be worth pondering over its implications to the theory of Fourier Series].

(14) Let X and Y be compact metric spaces. Show that for each continuous real valued function f on $X \times Y$ and each $\epsilon > 0$, we can find continuous real valued functions $g_1, \dots, g_n$ on X and $h_1, \dots, h_n$ on Y such that for each $(x, y) \in X \times Y$ we have

$$|f(x, y) - \sum_{i+1}^{n} g_i(x)h_i(y)| < \epsilon.$$

[Hint: Stone-Weierstrass.]

General

(15) Let E be a bounded subset of $\mathbb{R}$ and suppose every continuous real valued function on E is uniformly continuous on E. Show that E is compact.

(16) Let (X, d) be a metric space and endow $X \times X$ with the metric D given by the formula

$$D\left((x_1, x_2), (x_1', x_2')\right) = d(x_1, x_1') + d(x_2, x_2').$$

Show that there is no continuous, one-to-one and onto function $f : X \times X \to \mathbb{R}$ such that f^{-1} is also continuous. [Hint: Suppose such an f existed. Then show
 (a) X is path connected.
 (b) For any point $(x, y) \in X$, show that $X \times X \setminus \{(x, y)\}$ is path connected.
 (c) $\mathbb{R} \setminus \{0\}$ is *not* path connected.
Deduce a contradiction.]

Functions of Several Variables

In this section, all vector spaces and linear transformations are assumed to be over the real number $\mathbb{R}$. A *linear functional* on $\mathbb{R}^n$ is a linear transformation from $\mathbb{R}^n \to \mathbb{R}$. Recall that the *norm* of a linear transformation T is the supremum of Tx as x varies over the unit sphere (or equivalently, the closed unit ball), and is denoted $||T||$ (see Rudin).

(17) Prove that for every linear functional A on $\mathbb{R}^n$, there exists a unique element $y_A \in \mathbb{R}^n$ such that $Ax = x \cdot y_A$ for every $x \in \mathbb{R}^n$. Prove also that $||A|| = y_A$. [Remark: Such theorems are called *representation theorems*. The above can be considered as a special case of the *Riesz Representation Theorem* for Hilbert spaces. There are other Riesz Representation theorems for other spaces – and they all identify the dual space of a well-known Banach space with another concrete well known Banach space.]

(18) Suppose $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is given by

$$f(x_1, \dots, x_n; y_1, \dots, y_n) = x_1y_1 + \dots + x_ny_n.$$

 (a) Is f differentiable at every point of $\mathbb{R}^n \times \mathbb{R}^n$?
 (b) Write the matrix form of f' at the points where f' exists.

(19) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x, y) = \begin{cases} (x^2 + y^2)\sin\frac{1}{\sqrt{x^2+y^2}} & (x, y) \neq 0 \\ 0 & (x, y) = 0 \end{cases}$$

 (a) Show that f is differentiable at $(0, 0)$.

(b) Is the derivative continuous ?

(20) (Euler's equation) Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is a differentiable function satisfying

$$f(tx) = t^m f(x) \tag{*}$$

for all $t \in \mathbb{R}$ and all $x \in \mathbb{R}^n$. Show that

$$\sum_{i=1}^{n} x_i D_i f(x) = m f(x).$$

[Remark: Functions satisfying the identity $(*)$ for all x are called *homogenous of degree* m.]

(21) Consider $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \begin{cases} \frac{x}{2} + x^2 \sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0. \end{cases}$$

Can you find a local inverse of f in a neighbourhood of 0 ? [Hint: Read the hypotheses of the Inverse Function Theorem. Does f satisfy them ?]

(22) Let $f : \mathbb{R} \to \mathbb{R}^3$ be differentiable and suppose $|f(t)| = 1$ for every t. Show that

$$f'(t) \cdot f(t) = 0.$$

[Remark: This is a way of saying that if a particle is moving on the unit sphere in 3-space, then its velocity vector is tangential to the sphere.]

(23) Let $f = (f_1, f_2)$ be the mapping of $\mathbb{R}^2$ into $\mathbb{R}^2$ given by

$$f_1(x, y) = e^x \cos y, \quad f_2(x, y) = e^x \sin y.$$

(a) What is the range of f ?
(b) Show that the Jacobian of f is not zero at any point of $\mathbb{R}^2$.
(c) Conclude that f has local inverses. Is f one-to-one on $\mathbb{R}^2$?
(d) Find an explicit formula for the differentiable inverse g of f in a neighbourhood of $b = (1/2, \sqrt{3}/2) \in \mathbb{R}^2$ with $g(b) = (0, \pi/3)$. (Note that such local inverses are guaranteed by part (c)). You are also required to produce such a neighbourhood (on which the inverse g exists).

Notations. Let E be an open subset of $\mathbb{R}^n$. Define the classes $C^{(k)}(E)$ of functions as follows: $C^{(1)}(E)$ is the class of functions whose partial derivatives exist and are continuous. By recursion, define $C^{(k)}(E)$ to be the class of functions f such that the partial derivatives $D_1 f, \ldots, D_n f$ exist and belong to $C^{(k-1)}(E)$. If $f \in C^{(k)}(E)$, denote $D_{i_k} D_{i_{k-1}} \ldots D_{i_1} f$ by $D_{i_1 \ldots i_k} f$.

(24) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be in $C^{(2)}(\mathbb{R}^2)$. For each $x \in \mathbb{R}$ define $g_x : \mathbb{R} \to \mathbb{R}$ by $g_x(y) = f(x, y)$. Suppose that for each x, the equation

$$g_x'(y) = 0$$

has a unique solution $y = c(x)$. If $D_{2,2} f(x, y) \neq 0$ for all (x, y), show that c is differentiable and

$$c'(x) = -\frac{D_{2,1} f(x, c(x))}{D_{2,2} f(x, c(x))}.$$

(25) Let $f \in C^{(k)}(E)$. Show that the k-th order derivative $D_{i_1 \ldots i_k} f$ is unchanged if the subscripts $i_1, \ldots, i_k$ are permuted.

Functions of Bounded Variation

There is a natural generalisation of the notion of monotone functions on an interval I, viz., the notion of functions of *bounded variation.*

Definition 0.1. Let $\Delta : a = x_0 < x_1 < \ldots < x_n = b$ be a partition of $I = [a, b]$. For a function $f : I \to \mathbb{R}$ define

$$V(f, \Delta) = \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})|$$
$$V(f, I) = \sup_{\Delta} V(f, \Delta).$$

The function f is said to be of *bounded variation on* I if $V(f, I) < \infty$. We denote the space of functions of bounded variation on I by $BV(I)$.

(26) Show that $BV(I)$ is a vector space over $\mathbb{R}$ with the obvious notion of addition and scalar multiplication.
(27) Let $J_1 = [a, b]$ and $J_2 = [b, c]$ and $J = [a, c]$. Show that $V(f, J) = V(f, J_1) + V(f, J_2)$ for every function $f : J \to \mathbb{R}$.
(28) Let $f \in BV(I)$. Show that f can be written as the difference of two monotone functions. Conclude that the set of discontinuities of f is at most countable.
(29) Let $f_n \longrightarrow f$ pointwise on I. Show that

$$V(f, I) \leq \liminf V(f_n, I).$$

(30) Show that there exist $f, g \in BV(I_o)$ with $g(I_o) \subset I_o$ such that $f \circ g$ is not in $BV(I_o)$.
(31) Prove the following :-
A function f lies in $BV(I)$ if and only if there exists a non-decreasing function $\varphi : I \to \mathbb{R}$ such that

$$|f(x'') - f(x')| \leq \varphi(x'') - \varphi(x')$$

for every pair of elements $x', x'' \in I$ with $x' \leq x''$.
(32) If $f \in BV(I)$, then show that the function $m : I \to \mathbb{R}$ given by

$$m(x) = V(f, [a, x])$$

is continuous if and only if f is continuous.

The Riemann-Stieltjes Integral

Notation. Let X be any set and $E \subset X$ a subset. Then χ_E will denote the real valued function on E given by $\chi_E(x) = 1$ if $x \in E$ and $\chi_E(x) = 0$ if $x \notin E$. You are expected to have read the relevant chapter of Rudin. In other words you should be familiar with the concept of $\int_a^b f d\alpha$ for α a monotone function on I. In the following exercises we develop a slightly more general integral, and deduce its first properties.

(33) Mimic the definition if Riemann-Stieltjes integral over I with respect to a monotone function and come up with a definition of a Riemann-Stieltjes integral with respect to a function of bounded variation on I.

(34) Let $f \in C(I)$, $g \in BV(I)$ and define

$$F(x) = \int_a^x f dg \qquad a \leq x \leq b.$$

Show that F need not be continuous. If g is continuous at x_0, show that so is F.

(35) Let f be a real valued function on I.

(a) If f is monotone on I, show that

$$V(f, I) = |f(b) - f(a)|.$$

(b) If f is differentiable[1] on I and its derivative f' is continuous on I, then

$$V(f, I) = \int_I |f'(x)| dx$$

(36) Let $f : I \to \mathbb{R}$, $g \in BV(I)$ be such that

$$\limsup_{x<x_0} |f(x) - f(x_0)| > 0$$

and

$$\limsup_{x<x_0} |g(x) - g(x_0)| > 0.$$

Show that $\int_I f dg$ does not exist. [Remark: The same proof will also show that $\int_I f dg$ does not exist whenever both

$$\limsup_{x>x_0} |f(x) - f(x_0)| > 0$$

$$\limsup_{x>x_0} |g(x) - g(x_0)| > 0$$

for some $x_0 \in I$.]

(37) Let $f : I \to \mathbb{R}$ be such that $\int_I f dg$ exists for all $g \in BV(I)$. Show that f is continuous. [Hint: Use Problem 36.]

(38) Suppose $f \in C(I)$ and $g \in BV(I)$. Show that

$$|\int_I f dg| \leq ||f||_\infty V(g, I).$$

[Remark: The existence of $\int_I f dg$ is part of the assertion. The symbol $||f||_\infty$ denotes the supremum of $|f|$ on I.]

(39) Show that if $f_n \in C(I)$ and $f_n \longrightarrow f$ uniformly on I, then

$$\lim_{n\to\infty} \int_I f_n dg = \int_I f dg.$$

(40) Consider the function on [0, 1] given by

$$f_n(x) = \begin{cases} 4n^2 x & 0 \leq x \leq \frac{1}{2n} \\ 4n - 4n^2 x & \frac{1}{2n} \leq x \leq \frac{1}{n} \\ 0 & x \geq \frac{1}{n}. \end{cases}$$

[1] By this we mean that the right derivative of f at a exists, the left derivative of f at b exists and that f is differentiable on I^o. The *derivative* of f thus makes sense on all of I, even though I is a closed interval

Note that f_n is continuous. Is the statement

$$\lim_{n\to\infty}\int_0^1 f_n(x)dx = \int_0^1 \lim_{n\to\infty} f_n(x)dx$$

true ? [Hint: It may be helpful to graph f_n.]

(41) Let $f \in C(I)$ and $g_n \in BV(I)$ for $n \in \mathbb{N}$. Suppose
(a) $g_n \longrightarrow g$ pointwise on I.
(b) There exists $K < \infty$ such that $V(g_n, I) \le K$ for every $n \in \mathbb{N}$.
Show that $V(g, I) \le K$ and that $\int_I f dg_n \longrightarrow \int_I f dg$.

(42) Let $g \in BV(I)$ and assume that g is continuous at $x_0 \in I$. Show that

$$\int_I \chi_{\{x_0\}} dg = 0.$$

(43) Let $f \in C(I)$ and suppose

$$\int_I |f(x)|dx = 0.$$

Show that $f \equiv 0$ on I.

(44) Let $g : [-1, 1] \to \mathbb{R}$. Let $I = [-1, 1]$.
(a) If $g = \chi_{(0,1]}$, show that $\int_I f dg$ exists if and only if f is continuous from the right at 0. In this case show that

$$\int_I f dg = f(0).$$

(b) State and prove a similar result for $g = \chi_{[0,1]}$.
(c) If $g = \frac{1}{2}\chi_{\{0\}} + \chi_{(0,1]}$, show that $\int_I f dg$ exists if and only if f is continuous at 0. In this case what is $\int_I f dg$?

(45) This and the next problem develop the rudiments of L^p theory.
(a) Let $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be a strictly increasing continuous function, and let $a, b > 0$. Show that

$$ab \le \int_0^a \varphi(x)dx + \int_0^b \varphi^{-1}(y)dy.$$

[Hint and Remark: Draw a picture and see what you get. This inequality is called *Young's inequality.*]
(b) Let p and q be positive real numbers such that

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Note that this automatically forces the inequalities $1 < p < \infty$ and $1 < q < \infty$. Use Young's inequality for an appropriate φ to conclude that for $a \ge 0$, $b \ge 0$,

$$ab \le \frac{a^p}{p} + \frac{b^q}{q}.$$

(46) For an *increasing* function h on I and $p > 0$, let $L^p(h)$ denote the class of functions f on I such that $\int_I |f|^p dh < \infty$. Let p, q be as in Problem 45 (i.e. $1/p + 1/q = 1$ and $p, q > 0$).

(a) For $f \in L^p(h)$, $g \in L^q(h)$, show that $\int_I fgdh$ exists and

$$\left|\int_I fgdh\right| \leq \left\{\int_I |f|^p dh\right\}^{\frac{1}{p}} \left\{\int_I |g|^q dh\right\}^{\frac{1}{q}}$$

[Remark: This is called *Hölder's inequality*. For $p = q = 2$,it is called the Cauchy-Schwarz inequality].

(b) For $1 \leq p < \infty$, show that $L^p(h)$ is a vector space with the obvious notion of addition and scalar multiplication.

(c) For $f \in L^p(h)$ $(1 \leq p < \infty)$, define

$$||f||_p = \left\{\int_I |f|^p dh\right\}^{\frac{1}{p}}$$

Show that $d(f, g) = ||f - g||_p$ defines a metric on $L^p(h)$. Is $L^p(h)$, with this metric, complete ?

The Mehta Research Institute, Chhatnag, Jhusi, Allahabad District, U.P., INDIA 211 019

E-mail address: `pramath@mri.ernet.in`