LECTURES 23 AND 24

Dates of Lectures: April 5 and 7, 2022

We fix a ring R throughout these lectures. All modules appearing are R-modules, unless otherwise specified.

The symbol ☡ is for flagging a cautionary comment or a tricky argument. It occurs in the margins and is Knuth's version of Bourbaki's "dangerous bend symbol".

1. **The Horseshoe Lemma**

1.1. **Injective Resolutions.** Let $M \in \mathrm{Mod}_R$. Embed M into an injective module E^0. Let $Z^1 = E^0/M$. Embed Z^1 into an injective module E^1. Then we have an exact sequence $0 \to M \to E^0 \to E^1$. Suppose we have an exact sequence $0 \to E^0 \to E^1 \to \cdots \to E^n$ with $E_1, \ldots, E^n$ injective modules. Let Z^{n+1} be the cokernel of $E^{n-1} \to E^n$, and embed Z^{n+1} into an injective module E^{n+1}. Then clearly $0 \to M \to E^0 \to \cdots \to E^{n+1}$. By induction we see that we have a resolution of M by injective modules,

$$0 \longrightarrow M \longrightarrow E^0 \longrightarrow \ldots \longrightarrow E^n \longrightarrow \ldots$$

Lemma 1.1.1. *Let*

$$0 \to A \longrightarrow B \longrightarrow C \to 0$$

be an exact sequence in Mod_R*, and* $A \to E_A^\bullet$ *and* $C \to E_C^\bullet$ *injective resolutions in* Mod_R*. Then there exists an injective resolution* $B \to E_B^\bullet$ *and an exact sequence of complexes*

$$0 \to E_A^\bullet \longrightarrow E_B^\bullet \longrightarrow E_C^\bullet \to 0. \tag{†}$$

Proof. Write ∂_A^p and ∂_C^p for the p^{th}-coboundary maps in $E_A^\bullet$ and $E_C^\bullet$. Since $E_A^\bullet$ and $E_C^\bullet$ are injective complexes, if $E_B^\bullet$ exists as in the assertion, then necessarily E_B^p is the direct sum $E_A^p \oplus E_C^p$. Therefore set

$$E_B^p = E_A^p \oplus E_C^p \qquad (p \in \mathbf{N}).$$

We have to find maps $\partial_B^p \colon E_B^p \to E_B^{p+1}$ such that the resulting complex $E_B^\bullet$ resolves B and fits into the sequence (†) making it exact. At each level $p \in \mathbf{N}$ we have a split exact sequence

$$0 \to E_A^p \xrightarrow{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]} E_B^p \xrightarrow{[0\ 1]} E_C^p \to 0.$$

Since E_A^0 is an injective object and A is a subobject of B, the map $A \to E_A^0$ extends (in perhaps many ways) to B giving us a map $\varphi \colon B \to E_A^0$. Let $\psi \colon B \to E_C^0$ be the composite $B \to C \to E_C^0$. It is clear that the following diagram with exact rows commutes:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & E_A^0 & \xrightarrow{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]} & E_A^0 \oplus E_C^0 & \xrightarrow{[0\ 1]} & E_C^0 & \longrightarrow & 0 \\
 & & \uparrow & & \left[\begin{smallmatrix}\varphi\\\psi\end{smallmatrix}\right]\uparrow & & \uparrow & & \\
0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0
\end{array}$$

It is easy to check that the middle vertical arrow is injective. Thus we have an exact sequence $0 \to B \to E^0_B$. Let $A^0 = \operatorname{coker} A \to E^0_A$, $B^0 = \operatorname{coker} B \to E^0_B$, and $C^0 = \operatorname{coker} C \to E^0_C$. Then we have a short exact sequence (use the snake lemma on the above commutative diagram with exact rows)

$$0 \to A^0 \longrightarrow B^0 \longrightarrow C^0 \to 0.$$

Repeating the argument we gave earlier, since E^1_A is an injective object, we have a map $\varphi^0 \colon B^0 \to E^1_A$ extending the natural map $A^0 \to E^1_A$ and a map $\psi^0 \colon B^0 \to E^1_C$ which is the composite $B^0 \to C^0 \to E^1_C$. Repeating earlier arguments one notes that

$$B^0 \xrightarrow{\left[\begin{smallmatrix}\varphi^0\\\psi^0\end{smallmatrix}\right]} E^1_B$$

is injective and that the diagram below, whose rows are exact, commutes.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & E_A^1 & \xrightarrow{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]} & E_A^1 \oplus E_C^1 & \xrightarrow{[0\ 1]} & E_C^0 & \longrightarrow & 0 \\
 & & \uparrow & & \left[\begin{smallmatrix}\varphi^0\\\psi^0\end{smallmatrix}\right]\uparrow & & \uparrow & & \\
0 & \longrightarrow & A^0 & \longrightarrow & B^0 & \longrightarrow & C^0 & \longrightarrow & 0
\end{array}$$

Set $\partial^0_B \colon E^0_B \to E^1_B$ to be the composite

$$E^0_C \longrightarrow\!\!\!\!\!\rightarrow B^0 \xrightarrow{\left[\begin{smallmatrix}\varphi^0\\\psi^0\end{smallmatrix}\right]} E^1_B.$$

Then $0 \to B \to E^0_B \to E^1_B$ is exact. The process can be repeated ad infinitum. For example, set A^1, B^1, C^1 to be the cokernels of $A^0 \to E^1_A$, $B^0 \to E^1_B$, and $C^0 \to E^1_C$ respectively. Then $A^1 \hookrightarrow E^2_A$, $C^1 \hookrightarrow E^2_C$ and we can find appropriate $\varphi^1 \colon B^1 \to E^2_A$ and $\psi^1 \colon B^1 \to E^2_C$ and set ∂^1_B to be the composite of $E^1_B \twoheadrightarrow B^1$ followed by $\left[\begin{smallmatrix}\varphi^1\\\psi^1\end{smallmatrix}\right] \colon B^1 \to E^2_B$. One checks that $H^1(0 \to E^0_B \to E^1_B \to E^2_B \to 0) = 0$.

A standard induction argument then gives the result. $\square$

2. **Derived Functors**

2.1. **Injective resolutions.** Recall that $\mathbf{K}^+(R)$ is the subcategory of the homotopy category $\mathbf{K}(R)$ whose objects are bounded below complexes. The following lemma show that when one further restricts the the subcategory of bounded below *injective* complexes, quasi-isomorphisms become isomorphisms. In fancy language this means that the full subcategory of bounded below injective complexes in $\mathbf{K}(R)$ is equivalent to the "derived category" $\mathbf{D}^+(R)$ of bounded below complexes, a very useful theorem in higher homological algebra.

Lemma 2.1.1. *Let $A^\bullet$ be a bounded below complex, and $\varphi \colon A^\bullet \to E^\bullet$ and $\psi \colon A^\bullet \to I^\bullet$ be two injective resolutions (i.e. quasi-isomorphisms, with $E^\bullet$ and $I^\bullet$ complexes* *of injective objects) in $\mathbf{K}(R)$ with $E^\bullet$ and $I^\bullet$ bounded below. Then there is unique isomorphism $\alpha\colon E^\bullet \to I^\bullet$ in $\mathbf{K}(R)$ such that $\alpha \circ \varphi = \psi$.*

Proof. This is an immediate consequence of Proposition 2.3.3 of Lectures 21 and 22. □

2.2. **Additive and half-exact functors.** Let S be a second ring. A covariant functor $F\colon \mathrm{Mod}_R \to \mathrm{Mod}_S$ is said to be an *additive* functor if the natural map $\mathrm{Hom}_R(M, N) \to \mathrm{Hom}_S(F(M), F(N))$ is an abelian group homomorphism for every $M, N \in \mathrm{Mod}_R$. If F is contravariant, then we require that the natural map $\mathrm{Hom}_R(M, N) \to \mathrm{Hom}_S(F(N), F(M))$ is a homomorphism of abelian groups for every pair of R-modules M and N. One consequence is that an additive functor (co or contra) must respect direct sums, i.e. $F(M \oplus N) = F(M) \oplus F(N)$. Here is a quick proof (for the covariant case). Let $i\colon M \to M \oplus N$ and $j\colon N \to M \oplus N$ be the canonical inclusions. In other words, $i = (1, 0)$ and $j = (0, 1)$.[1] Let $\pi_M\colon M \oplus N$ and $\pi_N\colon M \oplus N \to N$ be the canonical projections, i.e. $\pi_M = [1\ 0]$ and $\pi_N = [0\ 1]$. Here, as aways, 1, or when I feel more expansive, $\mathbf{1}$, is a shorthand for the appropriate identity map. Since $\mathbf{1}_{F(M)} = F(\mathbf{1}_M)$, we see that $F(\pi_M) \circ F(i) = \mathbf{1}_{F(M)}$. This means $F(i)$ is injective and $F(\pi_M)$ is surjective. By symmetry, $F(j)$ is injective and $F(\pi_N)$ is surjective. Check that $(F(\pi_M), F(\pi_N))\colon F(M \oplus N) \to F(M) \oplus F(N)$ is an isomorphism. In fact its inverse is the map $[F(i)\ F(j)]$.

A covariant functor $F\colon \mathrm{Mod}_R \to \mathrm{Mod}_S$ is said to be *left exact* if it is additive and $0 \to F(M) \to F(N) \to F(T)$ is exact whenever $0 \to M \to N \to T \to 0$ is an exact sequence. Note that we are not demanding that $F(M) \to F(T)$ is surjective, even though $M \to T$ is. Suppose F is covariant and left exact and $0 \to A \to B \to C$ is an exact sequence. If I is the image of B in C and P the cokernel of $B \to C$, then we have exact sequences $0 \to A \to B \to I \to 0$ and $0 \to I \to C \to P \to 0$. This gives us exact sequences $0 \to F(A) \to F(B) \to F(I)$ and $0 \to F(I) \to F(C) \to F(P)$. It follows that $0 \to F(A) \to F(B) \to F(C)$ is exact. A little thought shows that we have proved that F is left exact if and only if $0 \to F(M) \to F(N) \to F(T)$ is exact whenever $0 \to M \to N \to T$ is exact.

A contravariant functor from Mod_R to Mod_S is said to be *left exact* if it is additive and $0 \to F(T) \to F(N) \to F(M)$ is exact whenever $0 \to M \to N \to T \to 0$ is exact. As in the covariant case, it is enough to check that $0 \to F(T) \to F(N) \to F(M)$ is exact whenever $0 \to M \to N \to T$ is exact.

The definitions of covariant and contravariant right exact functors is left to you. As always, these matters are best done in the world of abelian categories. Then one would say that a functor is right exact if the corresponding functor on the opposite categories is left exact.

Functors which are right exact or left exact are called *half exact*. A functor which is both right exact and left exact is called an *exact functor*.

2.3. **Derived functors.** Let $F\colon \mathrm{Mod}_R \to \mathrm{Mod}_S$ be left exact and covariant. Let $M \in \mathrm{Mod}_R$ and $M \to E^\bullet$ an injective resolution of M. If $M \to I^\bullet$ is a second injective resolution, then we know that in the homotopy category $\mathbf{K}(R)$ we have

[1] Recall that according to our conventions, $(1, 0) = \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$ and $(0, 1) = \left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]$. In particular $(a, b) \neq [a\ b]$, each side being the transpose of the other.

an isomorphism $\theta\colon E^\bullet \xrightarrow{\sim} I^\bullet$ such that

$$\begin{array}{ccc} & & E^\bullet \\ & \nearrow & \\ M & & \Big\downarrow{\scriptstyle \theta}\ \wr \\ & \searrow & \\ & & I^\bullet \end{array} \tag{2.3.1}$$

commutes. It is easy to see that an additive functor preserves homotopies. This means $F(\theta)\colon F(E^\bullet) \to F(I^\bullet)$ is well defined and is an isomorphism in $\mathbf{K}(S)$.

The above discussion shows that up to canonical isomorphisms, the S-modules $\mathrm{H}^i(F(E^\bullet))$, $i \geq 0$, do not depend upon the choice of the injective resolution $E^\bullet$ of M. Set

$$\mathrm{R}^i F(M) := \mathrm{H}^i(F(E^\bullet)), \qquad i \geq 0. \tag{2.3.2}$$

Note that by the left exactness of F we have

$$\mathrm{R}^0 F(M) = F(M). \tag{2.3.3}$$

The $\mathrm{R}^i F$ are functors on Mod_R. Here is a brief sketch of the argument. Suppose $f\colon M \to N$ is an R-map. Fix injective resolutions $M \to E^\bullet_M$ and $N \to E^\bullet_N$. According to Proposition 2.3.3 of Lectures 21 and 22 we have a unique map $\varphi\colon E^\bullet_M \to E^\bullet_N$ in $\mathbf{K}(R)$ such that the following diagram commutes in $\mathbf{K}(R)$.

$$\begin{array}{ccc} M & \longrightarrow & E^\bullet_M \\ {\scriptstyle f}\Big\downarrow & & \Big\downarrow{\scriptstyle \varphi} \\ N & \longrightarrow & E^\bullet_N \end{array} \tag{2.3.4}$$

We therefore get S-maps

$$\mathrm{R}^i(f)\colon \mathrm{R}^i F(M) \longrightarrow \mathrm{R}^i(N) \tag{2.3.5}$$

for $i \geq 0$, with $\mathrm{R}^i(f) := \mathrm{H}^i(F(\varphi))$. It is easy to see that $\mathrm{R}^i(f)$ are additive functors. Note that under the identification (2.3.3), $\mathrm{R}^0(f) = f$. Thus, at the level of functors, one has

$$\mathrm{R}^0 F = F. \tag{2.3.6}$$

The functor $\mathrm{R}^i F$ is called the i^{th} *right derived functor of* F. It can be defined for any additive functor F, not necessarily only for left exact functors. However, in that case, we no longer have the identification (2.3.6). It is sometimes useful to have this more general definition.

Next, if F is contravariant and left exact, one sets

$$\mathrm{R}^i F(M) = \mathrm{H}^i(F(P_\bullet))$$

for $i \geq 0$, where $P_\bullet \to M$ is a projective resolution of M. These are well-defined, are additive functors, and are such that $\mathrm{R}^0 F = F$. Once again, $\mathrm{R}^i F$ are called the i^{th} right derived functors of F. One can redo the proofs, or simply note that projectives are injectives in the opposite category, and that we have a covariant left exact functor $\mathrm{Mod}_R^\circ \to \mathrm{Mod}_S$ induced by F, where Mod_R° is the opposite category of Mod_R. Of course, one has to have developed the basics of abelian categories, which we haven't.

For right exact functors $F\colon \mathrm{Mod}_R \to \mathrm{Mod}_S$ we have *left derived functors* $\mathrm{L}_i F$, $i \geq 0$, with $\mathrm{L}_0 F = F$. For F covariant, this is defined at the level of objects by the formula

$$\mathrm{L}_i F(M) = \mathrm{H}_i(F(P_\bullet)), \quad i \geq 0,$$

where $P_\bullet \to M$ is a projective resolution of F. The proofs are replicas of the left exact situation. Or, ... chant the "opposite category" mantra. If F is contravariant, one must use injective resolutions of M. Once again, the requirement that F be right exact is not really necessary, but if it is dropped (and it sometimes is), then we no longer have $\mathrm{L}_0 F = F$.

2.3.7. One can make a more sophisticated statement. Suppose $\nu\colon F \to G$ is a natural transformation of left exact functors. Applying F and G to the diagram (2.3.4) and then relating the two commutative squares $F((2.3.4))$ and $G((2.3.4))$via ν, we see that we have a natural transformation of functors $\mathrm{R}^i F \to \mathrm{R}^i G$.

2.3.8. The following statement is made for left exact covariant functors. As an exercise, write out what the correct statement should be in the remaining three cases of half exact functors.

Theorem 2.3.9. *Let $F\colon \mathrm{Mod}_R \to \mathrm{Mod}_S$ be a covariant left exact functor and*

$$0 \longrightarrow M \longrightarrow N \longrightarrow T \longrightarrow 0$$

a short exact sequence of R-modules. Then we have a long exact sequence of S-modules,

$$0 \longrightarrow F(M) \longrightarrow F(N) \longrightarrow F(T) \xrightarrow{\delta} \mathrm{R}^1F(M) \longrightarrow \mathrm{R}^1F(N) \longrightarrow \dots$$
$$\dots \longrightarrow R^{n-1}F(T) \xrightarrow{\delta} \mathrm{R}^nF(M) \longrightarrow \mathrm{R}^nF(N) \longrightarrow \mathrm{R}^n(T) \xrightarrow{\delta} \dots$$

where all unlabelled maps arise from the functoriality of $\mathrm{R}^i F$.

Proof. Let $M \to E^\bullet_M$, $T \to E^\bullet_T$. By the Horseshoe Lemma Lemma **??**, we have an injective resolution $N \to E^\bullet_N$ which fits into a short exact sequence of complexes as below:

$$0 \longrightarrow E^\bullet_M \longrightarrow E^\bullet_N \longrightarrow E^\bullet_T \longrightarrow 0$$

The above short exact sequence of complexes is *semi-split*, i.e. for each n, the corresponding sequence $0 \to E^n_M \to E^n_N \to E^n + T \to 0$ is split. This is so because E^n_M is injective. Next, since F is additive, the discussion in the first paragraph of §2.2 shows that $0 \to F(E^n_M) \to F(E^n_M) \to F(E^n_T) \to 0$ is also split exact. In particular, we have a short exact sequence of complexes

$$0 \longrightarrow F(E^\bullet_M) \longrightarrow F(E^\bullet_N) \longrightarrow F(E^\bullet_T) \longrightarrow 0$$

The assertion of the theorem now follows. □

2.3.10. The above Theorem says that $\{\mathrm{R}^i F\}_{i\geq 0}$ is a *delta functor* (or a δ-*functor*), the definition of which is obvious from the statement of the theorem. See §1.5 (and see especially the remark in 1.5.7 of *loc.cit.*) over here for an alternate proof using mapping cones.

3. Ext **and** Tor

In this section we define the two most important derived functors in commutative algebra.

3.1. **The functor $\mathrm{Ext}^i_R(-,\star)$.** There are two equivalent definitions for $\mathrm{Ext}^i_R(M,N)$, for $M,N\in\mathrm{Mod}_R$ and $i\geq 0$. For clarity, we will (temporarily) call one of them $\mathrm{Ext}^i_R(M,N)$ and the other $\mathrm{ext}^i_R(M,N)$, and then show that they agree (up to canonical isomorphism, of course).

Recall that if M and N are R-modules, then $\mathrm{Hom}_R(M,\star)\colon \mathrm{Mod}_R\to\mathrm{Mod}_R$ is a covariant left exact functor and $\mathrm{Hom}_R(-,\,N)\colon \mathrm{Mod}_R\to\mathrm{Mod}_R$ is a contravariant left exact functor.

Definition 3.1.1. Let $M,N\in\mathrm{Mod}_R$. For $i\geq 0$, $\mathrm{Ext}^i_R(M,\star)$ is the i^{th} right derived functor of $\mathrm{Hom}_R(M,\star)$ and $\mathrm{ext}^i(-,\,N)$ is the i^{th} right derived functor of $\mathrm{Hom}_R(-,\,N)$.

From the remark in 2.3.7, we see that $\mathrm{Ext}^i_R(-,\,\star)$ and $\mathrm{ext}^i_R(-,\,\star)$ are bi-functors.

Theorem 3.1.2. *Let M and N be R-modules. For each $i\geq 0$, we have a bifunctorial isomorphism*

$$\mathrm{Ext}^i_R(M,\,N)\xrightarrow{\ \sim\ }\mathrm{ext}^i_R(M,\,N).$$

Proof. We will skip the boring and routine bifunctoriality part of the statement and instead concentrate on producing the isomorphism.

Let $P_\bullet\to M$ be a projective resolution of M and $N\to E^\bullet$ an injective resolution of N. Consider the double complex $A^{\bullet\bullet}=\mathrm{Hom}_R(P_\bullet,E^\bullet)$, where $A^{ij}=\mathrm{Hom}_R(P_i,E^j)$. Since E^j is injective, therefore $\mathrm{Hom}_R(-,E^j)$ is exact and hence $\mathrm{Hom}_R(P_\bullet,E^j)$ is a resolution of $\mathrm{Hom}_R(M,\,E^j)$. In other words, for each j, the complex

$$0\longrightarrow\mathrm{Hom}_R(M,\,E^j)\longrightarrow A^{0,j}\longrightarrow A^{1,j}\longrightarrow\dots\longrightarrow A^{i,j}\longrightarrow A^{i+1,j}\longrightarrow\dots$$

is exact. We can regard $\mathrm{Hom}_R(M,E^\bullet)$ as a double complex $Z^{\bullet\bullet}$ concentrated in the 0^{th} column. And we have a map of double complexes $\varphi\colon Z^{\bullet\bullet}\to A^{\bullet\bullet}$. Applying Theorem 1.5.5 of Lectures 21 and 22 we see that the induced map

$$\mathrm{Tot}(\varphi)\colon\mathrm{Hom}_R(M,\,E^\bullet)\longrightarrow\mathrm{Tot}^\bullet(A^{\bullet\bullet})$$

is a quasi-isomorphism.

In the same way, for each i, we have an exact sequence

$$0\longrightarrow\mathrm{Hom}_R(P_i,\,N)\longrightarrow A^{i,0}\longrightarrow A^{i,1}\longrightarrow\dots A^{i,j}\longrightarrow A^{i,j+1}\longrightarrow\dots$$

is exact, and this time taking $Z^{\bullet\bullet}$ to be the double complex concentrated in the 0^{th} row given by $\mathrm{Hom}_R(P_\bullet,\,N)$, and letting $\psi\colon Z^{\bullet\bullet}\to A^{\bullet\bullet}$ be the resulting map, we see, by again applying Theorem 1.5.5 of *loc.cit.*, that we have a quasi-isomorphism

$$\mathrm{Tot}(\psi)\colon\mathrm{Hom}_R(P_\bullet,\,N)\longrightarrow\mathrm{Tot}^\bullet A^{\bullet\bullet}.$$

It follows that for $i\geq 0$, we have an isomorphism

$$(\#)\qquad \mathrm{Ext}^i_R(M,\,N)\xrightarrow{\ \sim\ }\mathrm{ext}^i_R(M,\,N)$$

such that the following diagram commutes

$$\begin{array}{ccc} \mathrm{Ext}^i_R(M,\,N) & = & \mathrm{H}^i(\mathrm{Hom}_R(M,\,E^\bullet)) \\ & & \wr\Big\downarrow \mathrm{Tot}\,(\varphi) \\ (\#)\ \Big\downarrow\wr & & \mathrm{H}^i(\mathrm{Tot}^\bullet(A^{\bullet\bullet})) \\ & & \wr\Big\uparrow \mathrm{Tot}\,(\psi) \\ \mathrm{ext}^i_R(M,N) & = & \mathrm{H}^i(\mathrm{Hom}_R(P_\bullet,\,N)) \end{array}$$

This completes the proof. ◻

3.1.3. The symbol $\mathrm{ext}^i_R(M,N)$ was temporary. In view of the above result, we will use the symbol $\mathrm{Ext}^i_R(M,N)$ for this functor too.

3.2. **The functor** $\mathrm{Tor}^R_i(-,\,\star)$**.** Let $N \in \mathrm{Mod}_R$. We know that $-\otimes_R N$ is right exact. It is in general not exact. For example, the sequence $0 \to \mathbf{Z}/2\mathbf{Z} \xrightarrow{0} \mathbf{Z}/2\mathbf{Z} \xrightarrow{\mathbf{1}} \mathbf{Z}/2\mathbf{Z} \to 0$ is not exact, and this sequence is obtained by applying $-\otimes_{\mathbf{Z}} (\mathbf{Z}/2\mathbf{Z})$ to the short exact sequence $0 \to Z \xrightarrow{2} \mathbf{Z} \to Z/\mathbf{Z} \to 0$.

Lemma 3.2.1. *Let P be a projective module. Then $-\otimes_R P$ is an exact functor.*

Proof. This is obvious if P is a free module. Since projective modules are direct summands of free modules, the result follows. ◻

Definition 3.2.2. Let $N \in \mathrm{Mod}_R$ and i a non-negative integer. The functor $\mathrm{Tor}^R_i(-,\,N)\colon \mathrm{Mod}_R \to \mathrm{Mod}_R$ is the i^{th} left exact functor of $-\otimes_R N$.

Theorem 3.2.3. *Let M and N be R-modules. Then for every $i \geq 0$, $\mathrm{Tor}^R_i(M,\,N) = \mathrm{Tor}^R_i(N,\,M)$.*

Proof. Let $P_\bullet \to M$ and $Q_\bullet \to N$ be projective resolutions and let $A_{\bullet\bullet} = P_\bullet \otimes_A Q_\bullet$. By Lemma 3.2.1, each row $P_\bullet \otimes_R Q_j$ is a resolution of $M \otimes_R Q_j$, and each column $P_i \otimes_R Q_\bullet$ is a resolution of $P_i \otimes_R N$. The rest of the proof is the same as the one given for Theorem 3.1.2. We need the observation made in 1.5.6 of Lectures 21 and 22. ◻

4. Derived functors through acyclic objects

4.1. **Acyclic objects.** . Let $F\colon \mathrm{Mod}_R \to \mathrm{Mod}_S$ be an additive functor and M an R-module. If F is left exact (respectively right exact), then M is said to be *F-acyclic* if $\mathrm{R}^iF(M) = 0$ (respectively $\mathrm{L}_iF(M) = 0$) for $i \geq 1$. In the definition, we are not specifying the variance of F. It could be covariant or contravariant.

The following result is stated for covariant half exact functors. It is a simple exercise to get the correct statements for contravariant half exact functors.

Lemma 4.1.1. *Let $F\colon \mathrm{Mod}_R \to \mathrm{Mod}_S$ be an additive covariant functor, and suppose*

$$(*) \qquad 0 \longrightarrow M \longrightarrow N \longrightarrow T \longrightarrow 0$$

is an exact sequence of R-modules.

(a) *Let F be left exact.*

(i) *If M is F-acyclic then*

$$0 \longrightarrow F(M) \longrightarrow F(N) \longrightarrow F(T) \longrightarrow 0$$

is exact.

(ii) *If M and N are F-acyclic then T is also F-acyclic.*

(b) *Let F be right exact.*

(i) *If T is F-acyclic then*

$$0 \longrightarrow F(M) \longrightarrow F(N) \longrightarrow F(T) \longrightarrow 0$$

is exact.

(ii) *If N and T are F-acyclic then M is also F-acyclic.*

Proof. We will only prove (a). The proof of (b) is the same *mutatis mutandis*. In fact these statements are valid for half exact functors between abelian categories (as usual we will not use elements in our proofs), and so one can invoke the "opposite category" mantra.

Statement (i) of (a) is obvious by Theorem 2.3.9 applied to $(*)$, and the fact that M is acyclic, so that $\mathrm{R}^1F(M) = 0$. For part (ii) of (a), we have, the exact sequence

$$\mathrm{R}^iF(N) \longrightarrow \mathrm{R}^iF(T) \longrightarrow \mathrm{R}^{i+1}F(M).$$

If $i \geq 1$, the two ends of the above exact sequence vanish, forcing the middle term to also vanish. □

4.1.2. In the same way, if F is half exact and M and T are F-acyclic, then so is N. However, if F is left exact and N and T are$F$- acyclic, we cannot assert that M is F-acyclic. What is true then is that $\mathrm{R}^i(M) = 0$ for $i \geq 2$. Similarly, if F is right exact and M and N are F-acyclic, then we cannot assert that T is also F-acyclic. Without additional information, we can only assert that $\mathrm{L}_i(T) = 0$ for $i \geq 2$.

As usual, the following statement is made for covariant functors. The analogous statement for contravariant functors is proved the same way. The formulation of that statement and its proof is left to you.

Lemma 4.1.3. *Let F be half exact covariant and $C^\bullet$ an F-acyclic exact complex.*[2]

(a) *If F is left exact and $C^\bullet$ is bounded below, then $F(C^\bullet)$ is exact.*

(b) *If F is right exact and $C^\bullet$ is bounded above, then $F(C^\bullet)$ is exact.*

Proof. We will only prove (a), since (a) and (b) are disguised forms of each other. Since $C^\bullet$ is bounded below, by translating $C^\bullet$ if necessary, we assume without loss of generality that $C^n = 0$ for $n < 0$. Let $Z^i = Z^i(C^\bullet)$ for $i \in \mathbf{Z}$. Then $Z^i = 0$ for $i \leq 0$, whence Z^0 is F-acyclic. Suppose Z^i is F-acyclic. Since $C^\bullet$ is exact, the sequence $0 \to Z^i \to C^i \to Z^{i+1} \to 0$ is also exact. By (a)(ii) of Lemma 4.1.1, Z^{i+1} is also F-acyclic. By induction we see that Z^i is F-acyclic for all $i \geq 0$ (and Z^i is acyclic for $i < 0$ since $Z^i = 0$ for such i). By (i) of part (a) of Lemma 4.1.1 we see that

$$0 \longrightarrow F(Z^i) \longrightarrow F(C^i) \longrightarrow F(Z^{i+1}) \longrightarrow 0$$

is exact for all $i \in \mathbf{Z}$ (trivially so, when i is negative). Now the map $F(C^i) \to F(C^{i+1})$ factors as the composite $F(C^i) \to F(Z^{i+1}) \to F(C^{i+1})$ (since F is a functor). The above exact sequence shows that $F(C^i) \to F(Z^{i+1})$ is surjective, and the left exactness of F shows that $F(Z^{i+1}) \to F(C^{i+1})$ is injective. Thus $F(Z^{i+1})$ is the image of $F(C^i) \to F(C^{i+1})$. On the other hand, by the left exactness of F,

[2] $C^\bullet$ being F-acyclic means each C^n is F-acyclic.

it is also the kernel of $F(C^{i+1}) \to F(C^{i+2})$. It follows that $F(C^\bullet)$ is exact at $i+1$. Since $i \in \mathbf{Z}$ is arbitrary, we are done. □

The following lemma has an obvious version for contravariant functors, and as usual you are expected to formulate that. The proofs are identical *mutatis mutandis* in the covariant and contravariant cases.

Lemma 4.1.4. *Let F be a contravariant half exact functor and $\varphi\colon A^\bullet \to B^\bullet$ a quasi-isomorphism between F-acyclic complexes.*

(a) *Let F be left exact, and $A^\bullet$ and $B^\bullet$ bounded below. Then $F(\varphi)\colon F(A^\bullet) \to F(B^\bullet)$ is a quasi-isomorphism.*
(b) *Let F be right exact, and $A^\bullet$ and $B^\bullet$ bounded above. Then $F(\varphi)\colon F(A^\bullet) \to F(B^\bullet)$ is a quasi-isomorphism.*

Proof. As usual, we only prove (a), since the proof of (b) is identical, *mutatis mutandis*. The mapping cone $C^\bullet_\varphi$ of φ is bounded below, exact, and made up of F-acyclic modules. By Lemma 4.1.3, $F(C^\bullet_\varphi)$ is exact. Since F is additive, it is clear that $F(C^\bullet_\varphi) = C^\bullet_{F(\varphi)}$, the mapping cone of $F(\varphi)$. It follows that $F(\varphi)$ is a quasi-isomorphism. □

The following theorem, stated as usual only for covariant functors, is one of the most important tools for calculating derived functors. Do formulate the theorem in the contravariant case (and of course it is true there for the trivial reason that it is true in the covariant case).

Theorem 4.1.5. *Let F be a half exact covariant functor and M an R-module.*

(a) *If F is left exact and $0 \to M \to A^0 \to \cdots \to A^n \to \dots$ is an F-acyclic resolution of M then there are canonical isomorphisms*

$$\mathrm{H}^i(F(A^\bullet)) \xrightarrow{\sim} \mathrm{R}^i F(M)$$

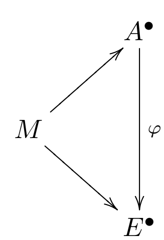
for $i \geq 0$.

(b) *If F is right exact and $\cdots \to A_n \to A_{n-1} \to \cdots \to A_0 \to M \to 0$ is an F-acyclic resolution of M, then there are canonical isomorphisms*

$$\mathrm{H}_i(F(A_\bullet)) \xrightarrow{\sim} \mathrm{L}_i F(M)$$

for $i \geq 0$.

Proof. It suffices to prove (a). Let $M \to E^\bullet$ be an injective resolution of M. By Proposition 2.3.3 of Lectures 21 and 22 we have a quasi-isomorphism $\varphi\colon A^\bullet \to E^\bullet$ such that the diagram



commutes. Now injective modules are clearly F-acylic (if E is injective, then the exact sequence $0 \to E \xrightarrow{\mathbf{1}_E} E \to 0$ is an injective resolution of E). By Lemma 4.1.4, $F(\varphi)\colon F(A^\bullet) \to F(E^\bullet)$ is a quasi-isomporphism. The Theorem follows. □

4.2. **Flat modules and** Tor_i^R**.** An R-module Q is said to be *flat* if $-\otimes_R Q$ is an exact functor.

4.2.1. Examples. Projective modules are flat by Lemma 3.2.1. If S is a multiplicative system, then $S^{-1}R$ is flat over R. Since direct limits of exact sequences of direct systems is exact (easy exercise), and since tensor product commutes with direct limits (by the universal property of tensor products and the $(\operatorname{Hom}, \otimes)$ adjointness), the direct limit of projective modules is flat. The converse is also true, namely, if a module is flat, then it is the direct limit of projective modules.

Suppose Q is a flat R-module. If $N \in \operatorname{Mod}_R$ and $P_\bullet \to N$ is a projective resolution, then as $-\otimes_R Q$ is exact, $P_\bullet \otimes_R Q \to N \otimes_R Q$ is a resolution, i.e.

$$\ldots \longrightarrow P_n \otimes_R Q \longrightarrow P_{n-1} \otimes_R Q \longrightarrow \ldots \longrightarrow P_0 \otimes_R Q \longrightarrow N \otimes_R Q \longrightarrow 0$$

is exact. It is then immediate that $\operatorname{Tor}_i^R(N, Q) = 0$ for $i \geq 1$. By Theorem 3.2.3 it follows that $\operatorname{Tor}_i^R(Q, N) = 0$ for $i \geq 1$.

Proposition 4.2.2. *Let M and N be R-modules and$Q_\bullet \to M$ a flat resolution of M. Then $\operatorname{Tor}_i^R(M, N) = \operatorname{H}_i(Q_\bullet \otimes_R N)$ for $i \geq 0$.*

Proof. We just proved that $\operatorname{Tor}_i^R(Q, N) = 0$ for $i \geq 1$ for every flat module Q. We are therefore done by part (b) Theorem 4.1.5. ☐

4.3. Ext **and** Tor **through Koszul complexes.** For $t \in R$, the *homology Koszul complex* $K_\bullet(t)$ is the complex $0 \to R \xrightarrow{t} R \to 0$, where the R on the left is $K_1(t)$ and the one on the right is $K_0(t)$. The *cohomology Koszul complex* $K^\bullet(t)$ is the same complex, but with the R on the left being $K^0(t)$ and the one on the right being $K^1(t)$. Let $t_1, \ldots, t_d \in R$ and let $\mathbf{t}$ denote the sequence $(t_1, \ldots, t_d)$. The *homology Koszul complex* $K_\bullet(\mathbf{t})$ is the complex $K_\bullet(t_1) \otimes \cdots \otimes K_\bullet(t_d)$, where the convention is that $A_\bullet \otimes B_\bullet$ is shorthand for the total complex of the commuting double complex obtained by tensoring $A_\bullet$ with $B_\bullet$. Similarly, one can define the *cohomology Koszul complex* $K^\bullet(\mathbf{t})$ as the tensor product $K^\bullet(t_1) \otimes \cdots \otimes K^\bullet(t_d)$. It turns out that $K^\bullet(\mathbf{t})$ is isomorphic to $\operatorname{Hom}_R(K_\bullet(\mathbf{t}), R)$ as well as to the translate of the homology Koszul by d units to the right.

If $\mathbf{t}$ is a *regular sequence* (also known as an R-sequence), i.e., if t_1 is a nonzero divisor of R, and for $i \geq 2$, t_i is a nonzero divisor of $R/\langle t_1, \ldots, t_{d-1}\rangle$, then $K_\bullet(\mathbf{t})$ is a free (and hence projective) resolution of R/I where $I = \langle t_1, \ldots, t_d\rangle$. For $d = 1$, this is obvious by the definition of a nonzero divisor. In the general case, one uses induction and Theorem 1.5.5 of Lectures 21 and 22. The details are left to you. It follows that in this case

$$\operatorname{Tor}_i^R(R/I, M) = \operatorname{H}_i(K_\bullet(\mathbf{t} \otimes M)) \quad \text{and} \quad \operatorname{Ext}_R^i(R/I, M) = \operatorname{H}^i(\operatorname{Hom}_R(K_\bullet(\mathbf{t}), M)).$$

(See also `https://www.cmi.ac.in/~pramath/AGI/notes/CechNotes.pdf`.)